František Rublík
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ON THE TWO-SIDED QUALITY CONTROL

FRANTIŠEK RUBLÍK

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1. INTRODUCTION

Statisticians often have to test whether \(100(1 - A)\)% of a population have values of an investigated quantity in a prescribed interval \(<m - \delta, m + \delta>\), where \(m, \delta (\delta > 0)\) are fixed real numbers. This two-sided control is often performed by a graphical method, which can be found in [5], pp. 54–57 (cf. also [3]). The aim of this paper is to apply the maximum likelihood principle for the two-sided control. The second part of the paper contains an exact formula for the asymptotic distribution of the test statistic and the third part contains its critical values.

2. ASYMPTOTIC DISTRIBUTION OF THE MAXIMUM LIKELIHOOD STATISTIC

Let us denote

\[ \Theta = \left\{ \left( \begin{array}{c} \mu \\ \sigma \end{array} \right) : \mu \in R, \sigma > 0 \right\}, \]

where \(R\) is the real line, and for \(\theta = \left( \begin{array}{c} \mu \\ \sigma \end{array} \right) \in \Theta\) put

\[ F_{\mu, \sigma}(x) = \int_{-\infty}^{x} f_{\theta}(z) \, dz, \quad f_{\theta}(z) = \frac{1}{\sqrt{(2\pi)\sigma}} \exp \left( -\frac{(z - \mu)^2}{2\sigma^2} \right). \]

Let \(A \in (0, 1), \phi = F_{0,1}\) and \(\phi(c_A) = 1 - A/2\). If we denote

\[ H_A = \left\{ \left( \begin{array}{c} \mu \\ \sigma \end{array} \right) \in \Theta; \mu + c_A \sigma \leq m + \delta, \mu - c_A \sigma \geq m - \delta \right\}, \]

then for \((\mu, \sigma) \in H_A\) (where \(x'\) means the transpose of the vector \(x\)) we obtain

\[ P_{\mu, \sigma}[x \in (m - \delta, m + \delta)] \geq F_{\mu, \sigma}(\mu + c_A \sigma) - F_{\mu, \sigma}(\mu - c_A \sigma) = 2\phi(c_A) - 1 = 1 - A, \]

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and the population has the mentioned property. Now we describe the maximum likelihood statistic for testing the hypothesis \( H_A \) against the alternative \( \Theta - H_A \).

Let us put for \( \varphi \in \Theta \)

\[
L(x^{(n)}, \varphi) = \sup_{\Theta \ni \varphi} \prod_{k=1}^{n} f_\varphi(x_k), \quad L(x^{(n)}) = \sup_{\Theta \ni \Theta} \prod_{k=1}^{n} f_\theta(x_k).
\]

where \( x^{(n)} = (x_1, \ldots, x_n) \) consists of \( n \) independent realizations of the random variable \( X \), and define a mapping \( T_n : R^n \to H_A \) as follows. Let us denote

\[
\bar{x} = \frac{1}{n} \sum_{k=1}^{n} x_k, \quad s^2 = \frac{1}{n} \sum_{k=1}^{n} (x_k - \bar{x})^2
\]

and put

\[
T_n(x^{(n)}) = (M_n(x^{(n)}), D_n(x^{(n)})),
\]

where the mappings \( M_n, D_n \) are defined by the following formulas. If \( \bar{x} \in (m - \delta, m + \delta) \) we put

\[
M_n(x^{(n)}) = \bar{x}, \quad D_n(x^{(n)}) = \begin{cases} 
  s & (\bar{x}, s) \in H_A \\
  -(\bar{x} - m + \delta) c_A^{-1} & (\bar{x}, s) \notin H_A, \bar{x} \in (m - \delta, m) \\
  (m + \delta - \bar{x}) c_A^{-1} & (\bar{x}, s) \notin H_A, \bar{x} \in (m, m + \delta).
\end{cases}
\]

Further, if \( \bar{x} \notin (m - \delta, m + \delta) \), we put

\[
M_n(x^{(n)}) = m + \delta - c_A D_n(x^{(n)}),
\]

\[
D_n(x^{(n)}) = \min \{ \delta/c_A, c_A(\bar{x} - m - \delta)/2 + [s^2 + (\bar{x} - m - \delta)^2 (1 + c_A^2)]^{1/2} \}
\]

if \( \bar{x} \geq m + \delta \), and

\[
M_n(x^{(n)}) = m - \delta + c_A D_n(x^{(n)}),
\]

\[
D_n(x^{(n)}) = \min \{ \delta/c_A, c_A(m - \delta - \bar{x})/2 + [s^2 + (\bar{x} - m + \delta)^2 (1 + c_A^2)]^{1/2} \}
\]

if \( \bar{x} \leq m - \delta \).

**Theorem 1.** (i) If we denote \( f^{(n)}_{\Theta}(x^{(n)}) = \prod_{k=1}^{n} f_{\Theta}(x_k) \), then

\[
L(x^{(n)}) = L(x^{(n)}),
\]

where \( T_n = T_n(x^{(n)}) \).

(ii) If \( \cdot > 0 \), then for every \( \theta \in H_A \)

\[
\lim_{n \to \infty} P_{\theta}^{(n)} \left[ -2 \ln \frac{L(x^{(n)}, H_A)}{L(x^{(n)})} \geq t \right] \leq 1 - F_A(t),
\]

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where \( F_A(t) = 0 \) if \( t < 0 \). If \( t \geq 0 \), then

\[
F_A(t) = \left[ 2^{-1} - \left( \frac{1}{\pi} \right) \arctan \left( 2^{1/2}/c_\delta \right) \right] + 2^{-1} F_1(t) + \\
+ \left[ 2^{-1} - \left( \frac{1}{\pi} \right) \arctan \left( c_\delta/2^{1/2} \right) \right] F_2(t),
\]

where \( F_j \) is the chi-square distribution function on \( j \) degrees of freedom and the function \( \arctan \) takes its values in the interval \((-\pi/2, \pi/2)\). If \( \theta' = (m, c_\delta) \), then (8) holds with the equality sign.

**Proof.** First we prove the first part of the assertion. Since \( n s^2 \) is chi-square distributed, we may assume that \( s > 0 \).

If \((\bar{x}, s) \in H_4\), then (7) holds (cf. [4], p. 504). Let \( \bar{x} \in (m - \delta, m + \delta) \) and \((\bar{x}, s) \notin H_4\). If we put

\[
\hat{\lambda}_{\mu, \sigma}(x^{(n)}) = \ln f_{\mu, \sigma}^{(n)}(x^{(n)}),
\]

then

\[
\frac{\partial^2 \hat{\lambda}_{\mu, \sigma}}{\partial \mu^2} = n\sigma^{-2}(\bar{x} - \mu), \quad \frac{\partial^2 \hat{\lambda}_{\mu, \sigma}}{\partial \sigma^2} = -n\sigma^{-1} + \sum_{k=1}^n (x_k - \mu)^2\sigma^{-3},
\]

which means that

\[
\lambda_{\mu, \sigma} \leq \hat{\lambda}_{\bar{x}, \sigma}
\]

for every \( \mu \in \mathbb{R}, \sigma > 0 \). Further, if we denote \( g(x) = \hat{\lambda}_{\bar{x}, \sigma} \), we see that the function \( g \) is increasing on \((0, 1)\). Hence if \( \bar{x} \in (m - \delta, m + \delta) \), the relations

\[
s > c_\delta^{-1}(\bar{x} - (m - \delta)) \geq \sigma \quad \text{if} \quad (\bar{x}, \sigma) \in H_4,
\]

\[
(\bar{x}, c_\delta^{-1}(\bar{x} - (m - \delta))) \in H_4
\]

together with (11) and (4) imply (7). The case \( \bar{x} \in (m, m + \delta) \) can be treated similarly.

Now we assume that \( \bar{x} \geq m + \delta \). Making use of (10) we obtain that

\[
\ln L(x^{(n)}, H_4) = \sup \{ \lambda_{m + \delta, c_\delta, \sigma}(x^{(n)}); \sigma \in (0, c_\delta^{-1}) \}.
\]

Denoting \( \hat{\lambda}_\sigma = \hat{\lambda}_{m + \delta, c_\delta, \sigma}(x^{(n)}) \) we see that

\[
\frac{d\hat{\lambda}_\sigma}{d\sigma} = \frac{n}{\sigma} \left[ -1 + \sigma^{-2}(s^2 + (\bar{x} - (m + \delta))^2) + \sigma^{-1} c_\delta(\bar{x} - m + \delta) \right]
\]

and the equation \( d\hat{\lambda}_\sigma / d\sigma = 0 \) has a unique positive solution

\[
\sigma_1 = c_\delta(\bar{x} - (m + \delta))/2 + \bar{\nu}_{\bar{x}, \sigma},
\]

where

\[
\bar{\nu}_{\bar{x}, \sigma} = \left[ s^2 + (\bar{x} - (m + \delta))^2 (1 + c_\delta^2/4) \right]^{1/2}.
\]

Since the function \( \hat{\lambda}_\sigma \) is increasing on \((0, \sigma_1)\) and reaches its maximum in the right
end-point of this interval, taking into account both (12) and (5) we see that (7) holds. The case \( x \leq m - \delta \) can be treated similarly.

We begin the second part of the proof with the definition of the approximability (cf. also [1], [6]). A set \( \varphi \subset R^m \) is said to be approximable at a point \( \theta \in \partial \) by a cone \( C \subset R^m \), if

\[
\sup \{ \varrho(x, C + \theta); \ x \in \varphi, \| x - \theta \| \leq a_n \} = o(a_n),
\]
\[
\sup \{ \varrho(y + \theta, \varphi); \ y \in C, \| y \| \leq a_n \} = o(a_n)
\]

for every sequence \( \{a_n\} \) of positive numbers which tend to zero. By a cone we understand any closed convex set \( C \subset R^m \) satisfying the relation \( y \in C, \ a > 0 \Rightarrow a y \in C \), and

\[
\varrho(z, D) = \inf \{ \| z - d \|; \ d \in D \}
\]

is the usual distance of a point \( z \) from a set \( D \). To prove the second part of the theorem we shall need a version of the Chernoff theorem. Before stating it we introduce regularity conditions of the Rao-Cramer type (cf. also [1], [4] and [7]). We assume that a class of probabilities \( \mathcal{P} = \{ P_\theta; \ \theta \in \Theta \} \), where \( \Theta \subset R^m \) is an open set, is defined on \( (X, \mathcal{F}) \) by densities \( f_\theta(x) = dP_\theta(x)/d\mu \) which for every \( \theta \in \Theta \) satisfy

(C1) \( f_\theta(x) \) is positive on \( X \times \Theta \) and has all partial derivatives of the third order in \( \theta \).

(C2) There are a \( P_\theta \)-integrable non-negative function \( G \) and a neighbourhood \( U \subset \Theta \) of the point \( \theta \) such that

\[
\sup_{i,j,k} \sup_{\theta \in U} \left| \frac{\partial^3}{\partial \theta_i \partial \theta_j \partial \theta_k} \ln f_\theta(x) \right| \leq G(x)
\]

for every \( x \in X \).

(C3) The coordinates of the vector \( (\partial \ln f_\theta(x)/\partial \theta_i)_{i=1,\ldots,m} \) belong to \( L_2(P_\theta) \) and its covariance matrix \( J(\theta) \) is strictly positive definite.

(C4) The identities

\[
\int \frac{\partial}{\partial \theta_i} f_\theta(x) \ d\mu(x) = 0, \quad \int \frac{\partial^2}{\partial \theta_i \partial \theta_j} f_\theta(x) \ d\mu(x) = 0
\]

hold for \( i, j = 1, \ldots, m \).

If we denote by \( x^{(n)} = (x_1, \ldots, x_n) \in X^n \) independent realizations of the random variable \( X \), then under the preceding regularity conditions the following assertion holds.

**Theorem 2.** Let \( \omega, \tau \) be subsets of \( \Theta \) such that

(i) \( \theta_0 \in \omega \cap \tau \).
(ii) If \( \varphi \in \{ \omega, \tau \} \), then there is a sequence \( \{ \hat{\varphi}_n \}_{n=1}^{\infty} \) of measurable mappings \( \hat{\varphi}_n : X^n \to \varphi \) such that \( \hat{\varphi}_n \to \varphi \) in the probability \( P_{\theta_0} \) and (cf. (2))

\[
\lim_{n \to \infty} P_{\theta_0} \left[ L(x^{(n)}, \varphi) = L(x^{(n)}, \hat{\varphi}_n) \right] = 1.
\]

If \( \omega, \tau \) are approximable at \( \theta_0 \) by cones \( C_\omega, C_\tau \), then

\[
L(x^{(n)}, H_A)/L(x^{(n)}) \to 1 \quad \text{in the probability} \quad P_{\theta_0}
\]

and (8) holds.

Let \( \theta_0 \) be a boundary point of \( H_A \). If \( \theta_0' = (\mu_0, \sigma_0) \), where \( \mu_0 \in (m - \delta, m) \), then the set \( H_A \) can be approximated at \( \theta_0 \) by the cone

\[
K = \left\{ (y_1, y_2) ; \quad y_1 - c_d y_2 \geq 0 \right\}
\]

and according to Theorem 2,

\[
\mathcal{L} \left[ -2 \ln \frac{L(x^{(n)}, H_A)}{L(x^{(n)})} \right] \to \mathcal{L} \left[ \inf_{\theta \in J^{1/2} K} \| \theta - z \|^2 \left| N(0, I) \right| \right].
\]

where \( J = J(\theta_0) \) and \( I \) is the unit matrix. Since \( J^{1/2} K = \{ z \in \mathbb{R}^2 ; c'z \leq 0 \} \) where \( c \) is a non-zero vector, making use of the mapping \( x \to -x \) and the notation \( D = = J^{1/2} K, g_1(z) = e^2(z, D) \) (cf. (13)), we obtain for every \( t > 0 \)

\[
P[g_1(z) \leq t \left| N(0, I) \right|] = 1/2 + (1/2) P[g_1(z) \leq t, z \notin D \left| N(0, I) \right|].
\]

If we denote

\[
\pi_D(z) = z - c'z \|c\|^{-2} c,
\]


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then \( \pi_D(z) \) is the projection of \( z \) into \( D \) and making use of the relation

\[
\mathcal{L}\left[\|\pi_D(z) - z\|^2 \mid N(0, I)\right] = \mathcal{L}\left[x^2 \mid N(0, (\|c\|^{-1} c)^t I \|c\|^{-1} c)\right]
\]

we see that the right hand side in (16) is of the form

\[
1/2 + (1/2) F_1(t).
\]

But \( \arctan \gamma + \arctan \gamma^{-1} = \pi/2 \) implies

\[
F_\gamma(t) - (1/2 + (1/2) F_1(t)) \leq 0,
\]

which means that (8) holds. Since the case \( \mu_0 \in (m, m + \delta) \) can be treated similarly, we assume that \( \theta_0 = (m, \delta/c_d) \). It is easy to see that \( H_A \) can be approximated at \( \theta_0 \) by the cone

\[
K = \{ y \in \mathbb{R}^2; y_1 \in \langle c_d y_2, - c_d y_2 \rangle \},
\]

and Theorem 2 implies that

\[
\mathcal{L}\left[-2 \ln L(x^{(n)}, H_A) \mid L(x^{(n)}) \mid P_0\right] \to \mathcal{L}\left[g^2(z, J^{1/2}K) \mid N(0, I)\right],
\]

where

\[
J^{1/2}K = \{ x \in \mathbb{R}^2; x_2 \leq \gamma x_1, x_2 \leq -\gamma x_1 \}, \quad \gamma = \sqrt{(2)/c_d}.
\]

Hence to complete the proof of Theorem 1, we have to prove

**Lemma 1.** If \( D = \{ y \in \mathbb{R}^2; y_2 \leq \gamma y_1, y_2 \leq -\gamma y_1 \} \) with \( \gamma > 0 \), then for every \( t \in \mathbb{R} \)

\[
P[g^2(z, D) \leq t \mid N(0, I)] = F(t),
\]

where the function \( F \) is defined by (9) with \( \sqrt{(2)/c_d} \) replaced by \( \gamma \).

**Proof.** Since \( N(0, I) \) is a symmetric distribution, we have

\[
P[g^2(z, D) \leq t \mid N(0, I)] = 2P[z_1 < 0, \|z - \pi_D(z)\|^2 \leq t \mid N(0, I)],
\]

where \( \pi_D(z) \) is the projection of \( z \) on the convex set \( D \).

Let \( z_1 < 0, z_2 \in (\gamma z_1, -\gamma^{-1} z_1) \) (cf. Fig. 1). Then \( \pi_D(z) \) is the projection on the cone

\[
C = \{ x \in \mathbb{R}^2; c'x = 0 \},
\]

where \( c' = (\gamma, -1) \). Making use of the transformation

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -\gamma^{-1} z_1 - z_2 \\ \gamma z_1 - z_2 \end{pmatrix}
\]

we obtain

\[
P[z_1 < 0, \gamma z_1 < z_2 < -\gamma^{-1} z_1, \|z - \pi_D(z)\|^2 \leq t \mid N(0, I)] = \]

\[
= (1/2) P[y_2 < 0, y_2^2 \leq t(1 + \gamma^2) \mid N(0, \gamma^2 + 1)] = (1/4) F_1(t).
\]
Let $z_1 < 0$, $z_2 < yz_1$. Then $\pi_D(z) = z$ and the substitution
\begin{equation}
\tag{21}
z_1 = r \cos \psi, \quad z_2 = r \sin \psi
\end{equation}
yields
\begin{equation}
\tag{22}
P[z_1 < 0, z_2 < yz_1, \left\| z - \pi_D(z) \right\|^2 \leq t | N(0, 1)] = 1/4 - (1/2\pi) \arctan y.
\end{equation}

Finally, let $z_1 < 0$, $z_2 > -\gamma^{-1}z_1$. Then $\pi_D(z) = 0$ and the substitution (21) yields
\begin{equation}
\tag{23}
P[z_1 < 0, z_2 > -\gamma^{-1}z_1, \left\| z - \pi_D(z) \right\|^2 \leq t | N(0, 1)] =
\end{equation}
\[= (2\pi)^{-1} \nu_L[\psi \in (\pi/2, 3\pi/2); \tan \psi < -\gamma^{-1}] \int_0^t \exp(-r^2/2) r \, dr =
\]
\[= 2^{-1} \left[ 1/2 - \frac{1}{\pi} \arctan \gamma^{-1} \right] F_2(t),
\]
where $\nu_L$ is the Lebesgue measure on the line. Combining relations (19)–(23) we see that (18) holds.

3. REMARKS AND TABLES.

If we denote for $\mathcal{A} \in (0, 1)$
\[l_n^{(A)}(x_1, \ldots, x_n) = n \left( 2 \ln \frac{D_n}{s} - 1 \right) + \sum_{k=1}^{n} \frac{(x_k - M_n)^2}{D_n^2},
\]
where the quantities $D_n = D_n(x_1, \ldots, x_n)$, $M_n = M_n(x_1, \ldots, x_n)$ are defined by the formulas (4)–(6), then the inequality (8) implies
\[
\sup_{\theta \in \mathcal{H}_\mathcal{A}} \lim_{n \to \infty} P^{(n)}[l_n^{(A)} \geq t] = 1 - F_3(t),
\]
whenever $t > 0$. Obviously, if we find a suitable constant $t(A, \alpha)$, then the tests

$$
\Psi_n(x_1, \ldots, x_n) = \begin{cases} 
\text{reject } H_A & \text{if } t_n^{(A)}(x_1, \ldots, x_n) \geq t(A, \alpha) \\
\text{accept } H_A & \text{if } t_n^{(A)}(x_1, \ldots, x_n) < t(A, \alpha)
\end{cases}
$$

will have the asymptotic size $\alpha$. The values of $t(A, \alpha)$ for various $A, \alpha$ are given in the following Table 1.

<table>
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<th>$A$</th>
<th>$\alpha$</th>
<th>$t(A, \alpha)$</th>
</tr>
</thead>
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<td>0.1</td>
<td>0.05</td>
<td>4.11833</td>
</tr>
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<td></td>
<td>0.02</td>
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</tr>
<tr>
<td></td>
<td>0.01</td>
<td>7.16359</td>
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</tr>
<tr>
<td></td>
<td>0.01</td>
<td>6.77779</td>
</tr>
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</table>

We remark that for every $t > 0$

$$
\inf_{\theta \in \Theta - H_A} \lim_{n \to \infty} P^{(\alpha)}[t_n^{(A)} \geq t] = 1,
$$

which means that the test $\Psi_\mu$ not only have the asymptotic size $\alpha$ but are consistent as well.

**Acknowledgement.** I would like to thank Dr. Ivica Mišiková for computing the values of $t(A, \alpha)$ in Table 1.

**References**

Súhrn

O DVOJSTRANNEJ KONTROLE KVALITY

FRANTIŠEK RUBLÍK

Nech náhodná premenná $X$ má normálne rozdelenie $N(\mu, \sigma^2)$. V článku sú odvodené explicitné formuly pre odhad maximálnej viero hodnosti pre parametre $\mu, \sigma$ za predpokladu platnosti hypotézy $\mu + c\sigma \leq m + \delta$, $\mu - c\sigma \geq m - \delta$, kde $c, m, \delta$ sú hocijaké pevné zvolené čísla. Táto hypotéza je testovaná pomocou pomerené viero hodnosti, uvádzame jeho asymptotické rozdelenie a niektoré jeho kvantily.

Author’s address: RNDr. František Rublík, CSc., Ústav merania a meracej techniky CEFV SAV, Dúbravská cesta, 842 19 Bratislava.