ON A NON-MARKOVIAN QUEUEING PROBLEM UNDER A CONTROL OPERATING POLICY AND START-UP TIMES

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1. INTRODUCTION

In most queueing processes, the system operates whenever a unit arrives and it becomes idle as soon as the system becomes empty. Lately, the concept of a modified operating rule called "control operating policy" and "start-up" times in the queueing theory have been introduced by various authors, see Heyman [3], Baker [2] etc.

The control operating policy and the start-up times are: (a) Turn the system on when n (an integer) number of customers are present in the queue and then turn the system off when it is empty. Once the system becomes empty, it remains idle till the queue length again reaches n or more. (b) When the queue lentth reaches n the system starts and the server blocks the counter for a random amount of time called "start-up" time while doing certain pre-service work. When the period of start-up time is over the server takes units for service one after another under FCFS discipline till the system becomes empty.

In the study of the economic behaviour of the system $M/G/1$, Heyman [3] assumed that the successive start-up times are identically zero. Baker [2] showed that although the addition of start-up times generally makes the non-Markovian queueing problem difficult to analyse, the simple queue with exponential start-up can be handled easily. The objective of our present note is to obtain the distribution of the number of customers in the queue (those who are actually waiting in the waiting line) for the model $M/G/1$ under the control operating policy and start-up times.

We consider here that customers arrive in accordance with a Poisson process with rate λ and form a waiting line. When the queue length reaches $n(>1)$, the server beings start-up. The duration of the start-up times is assumed to be distributed exponentially with the mean $1/\nu$. The customers are served in a system of one server according to a general service time distribution with the density $\mathrm{d}(t) = \eta(x) \{\exp - \int_0^x \eta(y)\, \mathrm{d}y\}$, where $\eta(y)\, \mathrm{d}y$ is the conditional probability that the service will be completed in the interval $(y, y + \mathrm{d}y)$ provided it has not been completed till the time y. When the system becomes empty it remains idle till the waiting line reaches n or more.

3. STATE EQUATIONS

Let us define;

$P_{r,0}(t)$ $\equiv$ Probability that at time t, the queue length is equal to r and the service channel is idle ($r \geqq 0$).

$P_{i,1}(t)$ $\equiv$ Probability that at time t, the queue length is equal to i and the service channel is busy ($i \geqq 0$).

$P_{i,1}(x, t)$ $\equiv$ Probability that at time t, the queue length is equal to i and a unit is being serviced with elapsed service time lying between x and $x + dx$ ($i \geqq 0$).

$\bar{F}(s)$ $\equiv$ Laplace transform (L.T.) of the function $F(t)$.

Clearly,

$$P_{i,1}(t) = \int_0^\infty P_{i,1}(x, t)\,dx.$$

The difference-differential equations govering the system are as follows:

$$(1) \qquad (d/dt)\,P_{0,0}(t) = -\lambda\,P_{0,0}(t) + \int_0^\infty P_{0,1}(x, t)\,\eta(x)\,dx\,,$$

$$(2) \qquad (d/dt)\,P_{r,0}(t) = -\lambda\,P_{r,0}(t) + \lambda\,P_{r-1,0}(t)\,; \quad 1 \leqq r \leqq n - 1\,,$$

$$(3) \qquad (d/dt)\,P_{r,0}(t) = -(\lambda + \nu)\,P_{r,0}(t) + \lambda\,P_{r-1,0}(t)\,; \quad r \geqq n\,,$$

$$(4) \qquad (\partial/\partial x)\,P_{0,1}(x, t) + (\partial/\partial t)\,P_{0,1}(x, t) = -[\lambda + \eta(x)]\,P_{0,1}(x, t)\,,$$

$$(5) \qquad (\partial/\partial x)\,P_{i,1}(x, t) + (\partial/\partial t)\,P_{i,1}(x, t) = -[\lambda + \eta(x)]\,P_{i,1}(x, t) + \lambda\,P_{i-1,1}(x, t);\ i \geqq 1\,,$$

$$(6) \qquad P_{i,1}(0, t) = \int_0^\infty P_{i+1,1}(x, t)\,\eta(x)\,dx\,; \quad 0 \leqq i \leqq n - 2\,,$$

$$(7) \qquad P_{i,1}(0, t) = \int_0^\infty P_{i+1,1}(x, t)\,\eta(x)\,dx + \nu\,P_{i+1,0}(t); \quad i \geqq n - 1\,.$$

3. SOLUTION OF THE STATE EQUATIONS

Let the initial condition of the system be $P_{0,0}(0) = 1$, i.e., the time is reckoned from the instant when the service channel is idle and there is no one waiting in the queue.

Define the following generating functions:

$$P(x, t; z) = \sum_{i=0}^{\infty} z^i P_{i,1}(x, t), \quad |z| \leqq 1,$$

$$P(t, z) \quad = \sum_{i=0}^{\infty} z^i P_{i,1}(t), \quad |z| \leqq 1.$$

Obviously,

$$P(t, z) = \int_0^{\infty} P(x, t; z) \, dx.$$

Multiplying (4) and (5) by the appropriate powers of z and then adding; we find

(8) $$(\partial/\partial x) P(x, t; z) + (\partial/\partial t) P(x, t; z) + [\lambda + \eta(x) - \lambda z] P(x, t; z) = 0.$$

Similarly, from (6) and (7), we get

(9) $$z P(0, t; z) = \int_0^{\infty} P(x, t; z) \eta(x) \, dx - \int_0^{\infty} P_{0,1}(x, t) \eta(x) \, dx + \\ + v \sum_{i=n}^{\infty} P_{i,0}(t) z^i.$$

The solution of the Lagrangian-type equation given by (8) is

(10) $$P(x, t; z) = K(t - x; z) \exp\{-N(x) - (\lambda - \lambda z) x\},$$

where $N(x) = \int_0^x \eta(y) \, dy$ and $K(.; z)$ is given by

$$K(-y; z) = 0 \qquad \text{for} \quad y \geqq 0,$$

$$K(t; z) \quad = P(0, t; z) \quad \text{for} \quad t > 0.$$

Performing the L.T. on (10), we get

(11) $$\bar{P}(x, s; z) = \bar{P}(0, s; z) \exp\{-N(x) - x(s + \lambda - \lambda z)\}.$$

Further, solving recursively the L.T. of (2) and (3) we find, respectively,

(12) $$\bar{P}_{r,0}(s) = [\lambda/(s + \lambda)]^r \bar{P}_{0,0}(s); \quad 1 \leqq r \leqq n - 1,$$

(13) $$\bar{P}_{r,0}(s) = \lambda^r (s + \lambda)^{1-n} (s + \lambda + v)^{n-r-1} \bar{P}_{0,0}(s); \quad r \geqq n.$$

Making use of (11), (13) and the L.T. of (1), (9) we conclude

(14) $$\bar{P}(0, s; z) = Q(s, z)/[z - \bar{d}(s + \lambda - \lambda z)],$$

where

$$Q(s, z) = 1 - [(s + \lambda) - v\lambda^n z^n/(s + \lambda + v - \lambda z)(s + \lambda)^{n-1}] \bar{P}_{0,0}(s).$$

From (11) and (14) we have

(15) $$\bar{P}(s, z) = \frac{1 - \bar{d}(s + \lambda - \lambda z)}{s + \lambda - \lambda z} \cdot \frac{Q(s, z)}{z - \bar{d}(s + \lambda - \lambda z)}.$$

The above expression gives the L.T. of the probability generating function (pgf) of the queue length when the service channel is busy.

Expanding the right-hand side of (15) in a power series of z and collecting the coefficient of z^i we can determine theoretically all $\bar{P}_{i,1}(s)$, $i \geqq 0$ in terms of $\bar{P}_{0,0}(s)$, whence $\bar{P}_{0,0}(s)$ can be derived by the normalizing condition of the state probabilities. Inversion of $\bar{P}_{i,1}(s)$ and $\bar{P}_{r,0}(s)$ (given by (12) and (13)) leads to the determination $P_{i,1}(t)$ and $P_{r,0}(t)$.

4. MARKOVIAN QUEUEING MODEL

As a special case, let us now discuss the queueing system where the service time distribution is exponential with the mean $(1/\mu)$. Accordingly, we get from (15)

(16)
$$\bar{P}(s, z) = [1 - \{(s + \lambda) - - \nu z^n \lambda^n / (s + \lambda + \nu - \lambda z)(s + \lambda)^{n-1}\} \bar{P}_{0,0}(s)] / [z\{(s + \lambda + \mu) - \lambda z\} - \mu] .$$

The denominator of (16) can be written as

$$f(z) = (-\lambda)(z - \alpha)(z - \beta) ,$$

where α and β are the two roots of the quadratic equation $f(z) = 0$.

Now expanding the right-hand side of (16) in powers of z and collecting the coefficients of z^i, we get

(17)
$$\bar{P}_{i,1}(s) = [\{((s + \lambda)/\lambda)(1/\alpha\beta) \sum_{k=0}^{i} 1/\alpha^{i-k}\beta^k\} \bar{P}_{0,0}(s) - - \{(1/\lambda)(1/\alpha\beta) \sum_{k=0}^{i} 1/\alpha^{i-k}\beta^k\}] , \quad 0 \leqq i \leqq n - 1 ,$$

(18)
$$\bar{P}_{i,1}(s) = [\{((s + \lambda)/\lambda)(1/\alpha\beta) \sum_{k=0}^{i} 1/\alpha^{i-k}\beta^k\} - - \{\nu\lambda^{n-1}/(s + \lambda)^{n-1} (s + \lambda + \nu)(1/\alpha\beta) \sum_{j=1}^{i+1-n} \sum_{k=0}^{j-1} (1/\alpha^{j-1-k}\beta^k)(\lambda/(s + \lambda + \nu))^{i+1-j-n}\}] \bar{P}_{0,0}(s) - - [(1/\lambda)(1/\alpha\beta) \sum_{k=0}^{i} 1/\alpha^{i-k}\beta^k] , \quad i \geqq n .$$

The value of $\bar{P}_{0,0}(s)$ can be obtained by using the normalizing condition, viz.,

$$\bar{P}_{0,0}(s) + \sum_{r=1}^{n-1} \bar{P}_{r,0}(s) + \sum_{r=n}^{\infty} \bar{P}_{r,0}(s) + \sum_{i=0}^{n-1} \bar{P}_{i,1}(s) + \sum_{i=n}^{\infty} \bar{P}_{i,1}(s) = (1/s) ,$$

which we omit here due to the lack of space.

Equilibrium Results. Let $P_{r,0}$ and $P_{i,1}$ be the steady state probabilities corresponding to $P_{r,0}(t)$ and $P_{i,1}(t)$, respectively. Applying the well known Tauberian theorem to (12), (13), (17), and (18), we get

$$P_{r,0} = P_{0,0}\,, \quad 1 \leqq r \leqq n - 1\,,$$

$$P_{r,0} = \theta^{r-n+1} P_{0,0}\,, \quad r \geqq n\,,$$

$$P_{i,1} = \varrho \frac{1 - \varrho^{i+1}}{1 - \varrho} P_{0,0}\,, \quad 0 \leqq i \leqq n - 1\,,$$

$$P_{i,1} = \left[\varrho \frac{1 - \varrho^{i+1}}{1 - \varrho} - \sum_{k=0}^{i-n} w\theta^{k+1} \left\{\frac{1 - \varrho^{i+1-n-k}}{1 - \varrho}\right\}\right] P_{0,0}\,, \quad i \geqq n\,,$$

where

$$\varrho = \lambda/\mu(<1)\,, \quad \theta = \lambda/(\lambda + v) \quad \text{and} \quad w = v/\mu = \varrho(1 - \theta)/\theta\,.$$

The normalizing condition of the steady state probabilities leads to the determination of $P_{0,0}$ as

$$P_{0,0} = v(1 - \varrho)\,,$$

where $v = [(1 - \theta)/\{\theta + n(1 - \theta)\}]$.

For convenience, let us write $P_i = P_{i,0} + P_{i,1}$. Thus, we have

$$P_i = v(1 - \varrho^{i+2})\,, \quad 0 \leqq i \leqq n - 1\,,$$

$$P_i = v[\varrho^{i-n+3}(1 - \varrho^{n-1}) + (1 - \varrho)\{\varrho^{i-n+3} + \varrho\theta^{i-n+1} - (1 + \varrho)\,\theta^{i-n+2}/(\varrho - \theta)\}]\,, \quad i \geqq n\,,$$

where $\theta \neq \varrho$.

The mean number of customers in the queue, say, $L(n)$ is given by

$$L(n) = \sum_{i=1}^{n-1} iP_i + \sum_{i=n}^{\infty} iP_i = v[n(n - 1)/2 - \varrho^3/(1 - \varrho)^2 + \{(\theta - \varrho + \varrho^2 - \varrho^2\theta)(n\theta - n\theta^2 + \theta^2)/(\theta - \varrho)(1 - \theta)^2\} - \{\varrho(1 - \theta)(n\varrho^2 - n\varrho^3 + \varrho^3)/(1 - \varrho)^2(\theta - \varrho)\}]\,.$$

5. INVENTORY QUEUEING ANALOGUE

Consider a single product inventory system for which replenishment items are supplied by a single production facility one by one. The system operates under a base-stock control policy in which each production run continues until on-hand stock is raised to the desired base-stock level, denoted by M. When the inventory is at the base-stock level, the production facility is idle. It remains idle until customer's demand reduces inventory to a pre-determined re-order level denoted by R. As soon as the inventory level first drops to R, a start-up interval occurs after which replenishment items are produced till the stock is raised to M. The behaviour of the Markovian model of the production inventory system may be viewed as an $M/M/1$ queue with exponential start-up, where start-up begins when the queue size reaches $M - R = n$.

In order to study the economic behaviour of the system it is necessary to derive the mean inventory level I, the mean backlog level B and the mean set up rate S, as these three factors commonly account for the operating cost in most inventory analysis. These factors are defined by

$$(19) \qquad I = \sum_{i=0}^{M} (M - i)(P_{i,0} + P_{i-1,1}),$$

$$B = \sum_{i=M}^{\infty} (i - M)(P_{i,0} + P_{i-1,1}),$$

$$S = v\lambda(1 - \varrho).$$

Moreover, it is well known that the performance of an inventory system is reflected by the cost of operation per unit time. Accordingly, a representative cost function that suits the present inventory model may be written as

$$(20) \qquad C(M, R) = C_0 + hI + pB + kS,$$

where C_0 is the preparatory cost which is fixed for the inventory level, h is the holding cost associated with I, p is the penalty cost associated with B and k is the fixed cost associated with each start-up.

The expressions represented in (19) may be substituted into (20) to obtain the desired cost function. As I and B depend on whether R is positive or negative, we present below two expressions for $C(M, R)$ according as $R \geqq 0$ and $R < 0$ for a Markovian production inventory system.

$R \geqq 0$

$$C(M, R) = C_0 + hv[M(M - R) - (M - R)(M - R - 1)/2 - M\varrho/(1 - \varrho) + R\{\varrho/(1 - \varrho) + (\lambda/v)\} + \varrho^2/(1 - \varrho) - \{(1 - \varrho)(\lambda + v)/(\varrho\lambda + \varrho v - \lambda)\} \times \{\varrho^3/(1 - \varrho)^2 - \lambda^3/v^2(\lambda + v)\}] + (h + p)v[(\varrho^2(\varrho^{R+1} - \varrho^M)/(1 - \varrho)^2 + \{(1 - \varrho)(\lambda + v)/(\varrho\lambda + \varrho v - \lambda)\}\{\varrho^{R+3}/(1 - \varrho)^2 - \lambda^{R+3}/v^2(\lambda + v)^{R+1}\}] + kv\lambda(1 - \varrho);$$

$R < 0$

$$C(M, R) = C_0 + hv[M(M + 1)/2 - M\varrho/(1 - \varrho) + \varrho^2(1 - \varrho^M)/(1 - \varrho)^2] + pv[R(R + 1)/2 - R\{\varrho/(1 - \varrho) + (\lambda/v)\} +$$

$$+ \varrho^2(\varrho - \varrho^M)/(1 - \varrho)^2 + \{(1 - \varrho)(\lambda + v)/(\varrho\lambda + \varrho v - \lambda)\} \times$$
$$\times \{\varrho^3/(1 - \varrho)^2 - \lambda^3/v^2(\lambda + v)\}] + kv\lambda(1 - \varrho).$$

The above results are in agreement with that of Baker [1] with $C_0 = 0$.

PARTICULAR CASES

By varying n and θ, a number of particular cases can be derived. We present below the results for some of them:

Case I. $\theta = 0$; this implies that there is no start-up in the system, rather the system operates as soon as the queue length reaches n. Note that $w\theta \to e$ as $\theta \to 0$. Therefore we have

$$P_{r,0} = (1 - \varrho)/n, \qquad 0 \leqq r \leqq n - 1,$$
$$P_{i,1} = (\varrho - \varrho^{i+2})/n, \qquad 0 \leqq i \leqq n - 1,$$
$$P_{i,1} = (\varrho^{i-n+2} - \varrho^{i+2})/n, \quad i \geqq n,$$
$$L(n) = (n - 1)/2 + \varrho^2/(1 - \varrho).$$

Case II. $\theta = 0, n = 1$, this implies the simple queue $M/M/1$. Here

$$P_{0,0} = (1 - \varrho),$$
$$P_{i,1} = (1 - \varrho)\varrho^{i+1}, \quad i \geqq 0,$$
$$L(n) = \varrho^2/(1 - \varrho).$$

Remarks. While the mathematical approach to this problem is useful to the theorists of OR, the numerical analysis of the problem will be of much help to the applied researchers. The computational aspects of the operational parameters of the single server and bulk service queue are the subject matter of another paper, which will appear elsewhere.

References

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Souhrn

O NEMARKOVOVSKÉM MODELU HROMADNÉ OBSLUHY SE ZPOŽDĚNÍM A ROZBĚHOVÝM ČASEM

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V článku se popisuje model systému hromadné obsluhy, v němž obsluhová linka začne reagovat na požadavky zákazníků teprve, když délka fronty dosáhne určité pevné hranice; každý pracovní interval navíc začíná náhodným rozběhovým časem. Autoři ukazují rovněž souvislost s jistým modelem řízení zásob.

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