Ta Van Dinh

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SOME FAST FINITE-DIFFERENCE SOLVERS FOR TWO-DIMENSIONAL EVOLUTIONARY EQUATIONS ON SPECIAL DOMAINS

TA VAN DINH

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Our aim is to prove the existence of asymptotic error expansions to the Peaceman-Rachford finite-difference scheme for the first boundary value problem of the two-dimensional evolutionary equation on special domains. These expansions lead, by Richardson extrapolation, to fast finite-difference solvers for the problems mentioned [1].

1. UNIFORM AND NEARLY UNIFORM DOMAINS

Let D be a bounded domain in the (x, y)-plane with a boundary G. For some real numbers x_0, y_0 let us consider a uniform grid over the (x, y)-plane:

$$(1) \qquad (x_i, y_j)\,, \quad x_i = x_0 + ih\,, \quad h = \text{const} > 0\,,$$
$$y_j = y_0 + jk\,, \quad k = \text{const} > 0\,,$$
$$0 < \text{const} < h/k < \text{const}\,.$$

The domain D will be called uniform if there exist two values x_0, y_0 and two sequences of positive numbers $\{h\}$ and $\{k\}$ tending simultaneously to zero such that the grid lines $x = x_i$ and $y = y_j$ cut the boundary G only at points of the form (x_m, y_n). Then the points (1) cover D with a uniform grid which consists of the set D_h of interior grid points which belong to the interior of D and the set G_h of boundary grid points lying just on G. The domain D will be called nearly uniform if there exist four real numbers a, b, c, d, a sequence of positive numbers $\{h\}$ tending to zero and two strictly increasing and smooth functions $x(t)$ $(a \leqq t \leqq c)$, $y(t)$ $(b \leqq t \leqq d)$ such that D lies in the rectangle $x(a) \leqq x \leqq x(c)$, $y(b) \leqq y \leqq y(d)$ and the lines $x = x_i = x(a + ih)$ and $y = y_j = y(b + jh)$, i, j integers, cut the boundary G only at the points of the form (x_m, y_n). So we can cover D with a grid (x_i, y_j), $x_i =$

$= x(a + ih)$, $y_j = y(b + jh)$, $i, j = 0, 1, 2, 3, \ldots$, which consists of the set D_h of interior grid points which belong to the interior of D and the set G_h of boundary grid points lying just on G. This grid is not uniform but depends uniformly on one parameter h.

2. THE DIFFERENTIAL PROBLEM

On a uniform domain D consider the differential problem

$$(2) \qquad \frac{\partial u}{\partial t} - Lu = f(x, y, t), \quad (x, y) \in D, \quad t \in (0, 1],$$

$$u(x, y, t) = g(x, y, t), \quad (x, y) \in G, \quad t \in (0, 1],$$

$$u(x, y, 0) = g^0(x, y), \quad (x, y) \in D,$$

where

$$Lu = \frac{\partial}{\partial x}\left(p(x, y, t)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(q(x, y, t)\frac{\partial u}{\partial y}\right) - c(x, y, t)\, u,$$

p, q, c, f, g, g^0 being given smooth enough functions with $p \geqq p_0 = \text{const} > 0$, $q \geqq q_0 = \text{const} > 0$, $c \geqq 0$.

3. THE DISCRETE PROBLEM

We cover D with a uniform grid as described above. Furthermore we consider the points

$$t_m = ml, \quad l = 1/M, \quad M \text{ a positive inger}, \quad m = 0, \ldots, M.$$

For each discrete function v defined on $D_h \times \{t_m\}$ we use the notation

$$v_{ij}^m = v(x_i, y_j, t_m), \quad v^m = v(\cdot, \cdot, t_m),$$

and the discrete operators

$$v_{\bar{t}ij}^m = \frac{v_{ij}^m - v_{ij}^{m-1}}{l},$$

$$L_{1h}^m v_{ij}^m = [p(x_i + 0{\cdot}5h, y_j, t_m)(v_{i+1j}^m - v_{ij}^m) - $$
$$- p(x_i - 0{\cdot}5h, y_j, t_m)(v_{ij}^m - v_{i-1j}^m)]/h^2 - \tfrac{1}{2}c(x_i, y_j, t_m)\, v_{ij}^m;$$

$$L_{2h}^m v_{ij}^m = [q(x_i, y_j + 0{\cdot}5k, t_m)(v_{ij+1}^m - v_{ij}^m) - $$
$$- q(x_i, y_j - 0{\cdot}5k, t_m)(v_{ij}^m - v_{ij-1}^m)]/k^2 - \tfrac{1}{2}c(x_i, y_j, t_m)\, v_{ij}^m.$$

The discrete problem is of the Peaceman-Rachford type:

$$(3) \qquad v_{\bar{t}ij}^{2n+1} - L_{1h}^{2n+1} v_{ij}^{2n+1} - L_{2h}^{2n} v_{ij}^{2n} = f(x_i, y_j, t_{2n+1}),$$

$$v_{\bar{t}ij}^{2n+2} - L_{1h}^{2n+1} v_{ij}^{2n+1} - L_{2h}^{2n+2} v_{ij}^{2n+2} = f(x_i, y_j, t_{2n+1}), \quad (i, j) \in D_h,$$
$$v_{ij}^m = g(x_i, y_j, t_m), \quad (i, j) \in G_h,$$
$$v_{ij}^0 = g^0(x_i, y_j), \quad (i, j) \in D_h;$$

here and in what follows we write (i, j) instead of (x_i, y_j).

4. MAIN RESULT

Theorem 1. *Assume that*

1) *the problem* (2) *has a unique solution*

$$u(x, y, t) \in C^7(\bar{D} \times [0, 1]),$$

2) *each problem*

$$\frac{\partial w_i}{\partial t} - Lw_i = F_i(x, y, t), \quad (x, y) \in D, \quad t \in (0, 1],$$

$$w_i(x, y, t) = 0, \quad (x, y) \in G, \quad t \in (0, 1],$$

$$w_i(x, y, 0) = 0, \quad (x, y) \in D,$$

$$F_1 = \frac{1}{24}\left[\frac{\partial}{\partial x}\left(p\frac{\partial^3 u}{\partial x^3}\right) + \frac{\partial^2}{\partial x^2} L_1 u\right], \quad L_1 u = \frac{\partial}{\partial x}\left(p\frac{\partial u}{\partial x}\right),$$

$$F_2 = \frac{1}{24}\left[\frac{\partial}{\partial y}\left(q\frac{\partial^3 u}{\partial y^3}\right) + \frac{\partial^2}{\partial y^2} L_2 u\right], \quad L_2 u = \frac{\partial}{\partial y}\left(q\frac{\partial u}{\partial y}\right),$$

$$F_3 = -u'''/6 + (L_1 + L_2)\, u''/2 - L_1 L_2 u' - (cu)''/2,$$

$$u' = \frac{\partial u}{\partial t},$$

has a unique solution and $w_i \in C^5(\bar{D} \times [0, 1])$,

3) $0 < \text{const} < 1/h < \text{const}$, $0 < \text{const} < 1/k < \text{const}$.

Then the discrete problem has a unique solution v_{ij}^m *and we have*

$$v_{ij}^{2n} - u(x_i, y_j, t_{2n}) = h^2 w_1(x_i, y_j, t_{2n}) + k^2 w_2(x_i, y_j, t_{2n}) + l^2 w_3(x_i, y_j, t_{2n}) + O(h^4 + k^4 + l^4).$$

Proof. We put

$$z = v - u - h^2 w_1 - k^2 w_2 - l^2 w_3.$$

Then we have

$$z_{\bar{t}ij}^{2n+1} - L_{1h}^{2n+1} z_{ij}^{2n+1} - L_{2h}^{2n} z_{ij}^{2n} = \varphi_{ij}^{2n+1}, \quad (i, j) \in D_h, \tag{4}$$
$$z_{\bar{t}ij}^{2n+2} - L_{1h}^{2n+1} z_{ij}^{2n+1} - L_{2h}^{2n+2} z_{ij}^{2n+2} = \psi_{ij}^{2n+1}, \quad (i, j) \in D_h,$$

$$z_{ij}^m = 0\,, \quad (i, j) \in G_h\,,$$
$$z_{ij}^0 = 0\,, \quad (i, j) \in D_h\,,$$

where, by Taylor's formula, we have

$$(5) \qquad \Phi_{ij}^{2n+1} \equiv \varphi_{ij}^{2n+1} + \psi_{ij}^{2n+1} - l^2 L_{1h}^{2n+1} \frac{\psi_{ij}^{2n+1} - \varphi_{ij}^{2n+1}}{l} = O(h^4 + k^4 + l^4)\,.$$

We consider the finite dimensional space H of discrete functions defined on $D_h \cup G_h$ and equal to zero on G_h. We introduce a scalar product and a norm in H:

$$(v, w) = \sum_{(i,j) \in D_h} v_{ij} w_{ij} hk\,, \quad \|v\| = ((v, v))^{1/2}\,.$$

We define two linear operators A_s^m ($s = 1, 2$) by putting

$$(A_s^m v^m)_{ij} = -L_{sh}^m v_{ij}^m\,, \quad v^m \in H\,.$$

It is clear that $z^m \in H$ and the discrete problem (4) can be rewritten as

$$(z^{2n+1} - z^{2n})/l + A_1^{2n+1} z^{2n+1} + A_2^{2n} z^{2n} = \varphi^{2n+1}\,,$$
$$(z^{2n+2} - z^{2n+1})/l + A_1^{2n+1} z^{2n+1} + A_2^{2n+2} z^{2n+2} = \psi^{2n+1}\,,$$
$$z^0 = 0\,.$$

Hence we have

$$(6) \qquad (E + lA_1^{2n+1})\, z^{2n+1} = (E - lA_2^{2n})\, z^{2n} + l\varphi^{2n+1}\,,$$
$$(E + lA_2^{2n+2})\, z^{2n+2} = (E - lA_1^{2n+1})\, z^{2n+1} + l\psi^{2n+1}\,,$$

E being the unit operator in H. It is clear that A_1^m and A_2^m are positive definite so that $(E + lA_s^m)^{-1}$ and $T_s^m = (E - lA_s^m)(E + lA_s^m)^{-1}$ always exist (for $s = 1, 2$) and

$$(7) \qquad \|(E + lA_s^m)^{-1}\| \leqq 1\,, \quad \|T_s^m\| \leqq 1\,.$$

Now we deduce from (6)

$$(E + lA_2^{2n+2})\, z^{2n+2} = (E - lA_1^{2n+1})(E + lA_1^{2n+1})^{-1} (E - lA_2^{2n})\, z^{2n} +$$
$$+ l\psi^{2n+1} + (E - lA_1^{2m+1})(E + lA_1^{2n+1})^{-1}\, l\varphi^{2n+1}\,.$$

Then we can write

$$(E + lA_2^{2n+2})\, z^{2n+2} = T_1^{2n+1} T_2^{2n} (E + lA_2^{2n})\, z^{2n} +$$
$$+ l(E + lA_1^{2n+1})^{-1} \left[(E + lA_1^{2n+1})\, \psi^{2n+1} + (E - lA_1^{2n+1})\, \varphi^{2n+1}\right].$$

Thus we have

$$(8) \qquad (E + lA_2^{2n+2})\, z^{2n+2} = T_1^{2n+1} T_2^{2n} (E + lA_2^{2n})\, z^{2n} + (E + lA_1^{2n+1})^{-1}\, l\Phi^{2n+1}\,,$$

where Φ^m have been determined by (5). We define a new norm

$$|||z^m|||_{(m)} = \|(E + lA_2^m)\, z^m\|\,.$$

Hence, from (8) and (7) we have

$$|||z^{2n+2}|||_{(2n+2)} \leqq |||z^{2n}|||_{(2n)} + l\|\Phi^{2n+1}\| .$$

We deduce

$$|||z^{2n}|||_{(2n)} \leqq |||z^0|||_{(0)} + \sum_{r=0}^{n-1} l\|\Phi^{2r+1}\| .$$

Then the theorem follows from $z^0 = 0$ and the estimate (5).

5. NOTE ABOUT THE NEARLY UNIFORM DOMAINS

The previous result is still available when D is nearly uniform domain. Considering the grid depending uniformly on one parameter h described in Section 1, we put

$$h_i = x_i - x_{i-1} , \quad k_j = y_j - y_{j-1} ,$$

$$L^m_{1h}v^m_{ij} = (2/(h_i + h_{i+1}))\,[p(x_i + 0{\cdot}5h_{i+1}, y_j, t_m)\,(v^m_{i+1\,j} - v^m_{ij})/h_{i+1} - \\ - p(x_i - 0{\cdot}5h_i, y_j, t_m)\,(v^m_{ij} - v^m_{i-1\,j})/h_i] - \tfrac{1}{2}c(x_i, y_j, t_m)\,v^m_{ij} ,$$

$$L^m_{2h}v^m_{ij} = (2/(k_j + k_{j+1}))\,[q(x_i, y_j + 0{\cdot}5k_{i+1}, t_m)\,(v^m_{ij+1} - v^m_{ij})/k_{j+1} - \\ - q(x_i, y_j - 0{\cdot}5k_j, t_m)\,(v^m_{ij} - v^m_{ij-1})/k_j] - \tfrac{1}{2}c(x_i, y_j, t_m)\,v^m_{ij} .$$

Then the discrete problem takes the form (3) with the operators L^m_{sh} just introduced. The asymptotic error expansion takes the form

$$v^{2n}_{ij} - u(x_i, y_j, t_{2n}) = h^2 w_1(x_i, y_j, t_{2n}) + l^2 w_2(x_i, y_j, t_{2n}) + O(h^4 + l^4) ,$$

if in addition to the assumptions Theorem 1 we assume that

$$x(t) \in C^4([a, c]) , \quad y(t) \in C^4([b, d]) .$$

6. A NUMERICAL EXAMPLE

Let D be the square $0 < x < 1$, $0 < y < 1$. We consider the problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = (1 + 2\pi^2 t) \sin \pi x \,.\, \sin \pi y , \quad (x, y) \in D ,$$

$$u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0 ,$$

$$u(x, y, 0) = 0 , \quad (x, y) \in D .$$

Its solution is $u(x, y, t) = t \,.\, \sin \pi x \,.\, \sin \pi y$. In the square D we consider the uniform grid

$$D_h = \{(x_i, y_j)\} , \quad x_i = ih , \quad y_j = jh , \quad h = 1/N ,$$

N being an integer > 0, $i, j = 0, \ldots, N$, and the points

$$t_m = ml\,, \quad l = 1/M\,, \quad M \text{ being an integer } >0\,, \quad m = 0, \ldots, M\,.$$

The discrete problem is that described in Section 3. We assume that $l/h = \text{const} > 0$ and denote the approximate value of $u(x_P, y_P, 1)$ calculated on this grid at a grid point $P \in D_h$ and at the moment $t = 1$ by $v(P; h)$. Theorem 1 with $l = \text{const} . h$ yields

$$v(P; h; h/2) \equiv \tfrac{4}{3}v(P; h/2) - \tfrac{1}{3}v(P; h) = u(x_P, y_P, 1) + O(h^4)$$

at each point P common to two grids with grid spacings h and $h/2$. The numerical results in the case $l/h = \frac{1}{2}$ at the point $P(\frac{1}{2}, \frac{1}{2})$ for $t = 1$ are presented in Table 1.

Table 1

h/l	$1/h$	$v(P; h)$	$v(P; h; h/2)$
2	2	1·1419679	0·9939090
2	4	1·0309237	0·9996481
2	8	1·0074670	0·9999785
2	16	1·0018506	

Reference

[1] *O. B. Widlund:* Some recent applications of asymptotic error expansions to finite difference schemes. Proc. Roy. Soc. London, A 323, N. 1553, 1971, 167–177.

Souhrn

RYCHLÉ ŘEŠENÍ DVOUDIMENZIONÁLNÍ EVOLUČNÍ ROVNICE NA SPECIÁLNÍ OBLASTI METODOU KONEČNÝCH DIFERENCÍ

TA VAN DINH

Autor dokazuje existenci mnohoparametrického asymptotického rozvoje pro chybu obvyklého pětibodového diferenčního schématu pro první okrajovou úlohu pro dvoudimenzionální evoluční rovnici na jistých speciálních (tzv. uniformních) oblastech. Tento rozvoj dává s použitím Richardsonovy extrapolace jednoduchý způsob zrychlení konvergence dané metody. Postup je ilustrován na numerickém příkladě.

Author's address: Ta Van Dinh, Bo Mon Toan-Tinh, Khoa Toan-Ly, Toan-Ly, Truong Dai Hoc Bach Khoa, Ha-Noi, Vietnam.