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APPROXIMATION AND NUMERICAL SOLUTION OF CONTACT PROBLEMS WITH FRICTION

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INTRODUCTION

In applications we often meet with problems, in which one deformable body comes in contact with another. Contact problems are non-classical in the sense that they cannot be directly formulated as usual boundary value problems. The reason for this is the fact that neither the contact surface nor the distribution of contact forces are known a — priori. It is well known that the mathematical formulation leads to variational inequalities. The aim of the present paper is to study the approximation and the computation of contact problems between an elastic body and a rigid foundation, taking into account the influence of friction on the contact surface. We analyse the simplest model involving friction, namely that with “a given friction”. The importance of this model consists in the fact that it serves as an auxiliary problem, by means of which we can approximate contact problems with friction obeying the classical Coulomb law. The main attention here is paid to the modification of Uzawa’s algorithm, using specific features of our problem.

1. SETTING OF THE PROBLEM

Let us assume a plane elastic body subjected to a body force $F$ and to surface tractions on a portion of its boundary, unilateraly supported by a rigid foundation. Our aim is to determine the behaviour of the structure, taking into account the influence of friction between the body and the foundation. We start with the classical formulation of the problem.

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with Lipschitz boundary $\partial \Omega$, which is decomposed as follows:

$$\partial \Omega = \Gamma_u \cup \Gamma_\tau \cup \Gamma_0 \cup \Gamma_K,$$

where $\Gamma_u, \Gamma_\tau, \Gamma_0, \Gamma_K$ are mutually disjoint parts of $\partial \Omega$, $\Gamma_u$ and $\Gamma_K$ are non-empty. By a classical solution of the problem in question we mean a displacement field
\( u = (u_1, u_2) \) satisfying the boundary conditions

\[
(1.1) \quad u = 0 \quad \text{on } \Gamma_u,
\]
\[
(1.2) \quad u_n = 0, \quad T_i(u) = 0 \quad \text{on } \Gamma_0,
\]
\[
(1.3) \quad \tau_{ij} n_j = p_i \quad \text{on } \Gamma_e, \quad 1)
\]
\[
(1.4) \quad u_n \leq 0, \quad T_n(u) \leq 0, \quad u_n T_n(u) = 0 \quad \text{on } \Gamma_K,
\]
\[
(1.5) \quad |T_i(u)| \leq g \quad \text{on } \Gamma_K;
\]

if \( |T_i(u)(x)| < g(x) \), then \( u_i(x) = 0 \)

if \( |T_i(u)(x)| = g(x) \), then \( \exists \lambda \geq 0 \) \( u_i(x) = -\lambda T_i(u)(x) \)

and the system of equilibrium equations

\[
(1.6) \quad \frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0 \quad \text{on } \Omega, \quad i = 1, 2.
\]

Here \( u_n, u_t \) denote the normal and tangential components of the displacement field \( u \), i.e. \( u_n = u \cdot n, u_t = u \cdot t; n = (n_1, n_2), t = (t_1, t_2) = (-n_2, n_1) \) are the outward unit normal and tangential vectors to \( \partial \Omega \). Similarly \( T_n(u) \) and \( T_t(u) \) are the normal and tangential components of the stress vector \( T = (T_1, T_2) = (\tau_{1j} n_j, \tau_{2j} n_j) \).

The stress tensor \( \tau = (\tau_{ij})_{i,j=1} \) and the strain tensor \( \varepsilon = (\varepsilon_{ij})_{i,j=1} \),

\[
\varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),
\]

are related by means of the generalized Hooke’s law

\[
\tau_{ij} = \tau_{ij}(u) = c_{ijkl} e_{kl}(u).
\]

The elastic coefficients \( c_{ijkl} \) are supposed to be bounded and measurable in \( \Omega \), i.e. \( c_{ijkl} \in L^\infty(\Omega) \), satisfying the symmetry conditions

\[
c_{ijkl} = c_{jikl} = c_{klji} \quad \text{a.e. in } \Omega
\]

and the condition of ellipticity

\[
\exists c_0 = \text{const} > 0 \quad \text{such that}
\]
\[
\left\{ \begin{array}{l}
c_{ijkl} e_{ij} e_{kl} \geq c_0 e_{ij} e_{ij} \\
e_{ij} = e_{ji}, e_{ij} \in \mathbb{R}^1
\end{array} \right.
\]

In order to give the variational form of our problem, we introduce the following notations.

By \( H^k(\Omega) (k \geq 0 \text{ integer}, H^0(\Omega) = L^2(\Omega)) \) we denote the Sobolev space of functions, the derivatives of which up to order \( k \) are square integrable in \( \Omega \), equipped with the classical norm denoted by \( \| \cdot \|_{k, \Omega} \).

\( 1) \) Throughout the paper, summation convention is used.

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Let \( V \) be the space of virtual displacements:
\[
V = \{ v \in (H^1(\Omega))^2 \mid v = 0 \text{ on } \Gamma_u, v_n = 0 \text{ on } \Gamma_o \},
\]
and
\[
K = \{ v \in V \mid v_n \leq 0 \text{ on } \Gamma_K \}
\]
the set of admissible displacements.

Finally, let us set
\[
a(u, v) = \int_\Omega \tau_{ij}(u) e_{ij}(v) \, dx,
\]
\[
L(v) = \int_\Omega F_i v_i \, dx + \int_{r_t} P_i v_i \, ds,
\]
\[
\mathcal{J}_0(v) = \frac{1}{2} a(v, v) - L(v),
\]
\[
j(v) = \int_{r_K} g v_i \, ds,
\]
\[
\mathcal{J}(v) = \mathcal{J}_0(v) + j(v),
\]
where \( F = (F_1, F_2) \in (L^2(\Omega))^2, P = (P_1, P_2) \in (L^2(\Gamma))^2, g \in L^\infty(\Gamma_K), g \geq 0 \) a.e. on \( \Gamma_K \).

**Definition 1.1.** By a variational solution of the contact problem with a given friction we mean a function \( u \in K \) such that
\[
(\mathcal{P}) \quad \mathcal{J}(u) \leq \mathcal{J}(v) \quad \forall v \in K.
\]

Using a classical result of calculus of variations (see [1], [2]), one obtains the following result:

**Theorem 1.1.** Under the above mentioned hypotheses there exists a unique solution \( u \) of (\( \mathcal{P} \)).

The problem (\( \mathcal{P} \)) can be equivalently characterized by the relation
\[
(1.7) \quad u \in K : a(u, v - u) + j(v) - j(u) \geq L(v - u) \quad \forall v \in K.
\]

Applying Green's formula to (1.7) one can prove formal equivalence between (1.1) to (1.6) and (\( \mathcal{P} \)).

2. THE APPROXIMATION OF (\( \mathcal{P} \))

Let all the assumptions and notations of Section 1 be satisfied. Moreover, let us suppose that \( \Omega \) is a polygonal domain. Let \( \{\tau_h\}, h \to 0^+ \) be a regular family of triangulations of \( \overline{\Omega} \), satisfying the usual requirements on the mutual position of triangles \( T_i \in \tau_h \).
With every $\tau_h$ we associate the space
\begin{equation}
V_h = \{ v \mid v \in (C(\Omega))^2, \frac{v}{T} \in (P_1)^2, v = 0 \text{ on } \Gamma_u, v_n = 0 \text{ on } \Gamma_0, \forall T \in \tau_h \},
\end{equation}
where $P_1$ is the set of all linear polynomials.

Denote by $A_j, j = 1, 2, \ldots, k$, the vertices of $\Omega$. By $n^- = (n_1^-, n_2^-)$, $n^+ = (n_1^+, n_2^+)$ we denote the outward unit normal vector to $A_{j-1}A_j$, $A_jA_{j+1}$, respectively (see Fig. 1).

Let $a_1, a_2, \ldots, a_m$ by the nodes of $\tau_h$, lying on $\Gamma_K$ and define
\begin{equation}
K_h = \{ v \mid v \in V_h, v_n(a_i) \leq 0 \text{ for } a_i \neq A_j, j = 1, 2, \ldots, k;
(v_1n_1^- + v_2n_2^-)(a_i) \leq 0, (v_1n_1^+ + v_2n_2^+)(a_i) \leq 0, \text{ if } a_i = A_j
\end{equation}
for some $j = 1, 2, \ldots, k; i = 1, 2, \ldots, m$).

**Definition 2.1.** An element $u_h \in K_h$ is said to be the approximation of the contact problem with a friction, if
\begin{equation}
(\mathcal{P}_h)
J(u_h) \leq J(v) \forall v \in K_h.
\end{equation}

It is readily seen that for every $h \in (0, 1)$, there exists a unique solution $u_h$ of $(\mathcal{P}_h)$.

The question arises, what is the relation between $u$ and $u_h$.

**Theorem 2.1.** Let us suppose that $\bar{\Gamma}_K \cap \bar{\Gamma}_u = \emptyset$, $\bar{\Gamma}_K \cap \bar{\Gamma}_0 = \emptyset$ and let there exist a finite number of boundary points $\Gamma_t \cup \Gamma_K$, $\Gamma_0 \cup \Gamma_t \cup \bar{\Gamma}_0$. Then
\begin{equation}
u_h \to u \text{ in } (H^1(\Omega))^2, h \to 0^+.
\end{equation}

**Proof.** If the assumptions concerning $\Gamma_u, \Gamma_t, \Gamma_K, \Gamma_0$ are satisfied, then the system of $\{K_h\}, h \in (0, 1)$ is dense in $K$ in the following sense:
\begin{equation}

\forall v \in K, \exists v_h \in K_h \text{ such that } v_h \to v \text{ (see [3]).}
\end{equation}

As $K_h \subset K, \forall h \in (0, 1), (2.4)$ yields (2.3).

If some additional smoothness assumptions concerning $u$ are satisfied, then the following rate of convergence result can be established (see [4]):

**Theorem 2.2.** Let us suppose that $u \in (H^2(\Omega))^2 \cap K, T(u) \in (L^\infty(\Gamma_K))^2, u \in (W^{1,\infty}(\Gamma_K))^2$, let the number of points from $\Gamma_K$, where $u_n, u_t$ change from $u_n < 0$ to $u_n = 0$ and
from $u_t \neq 0$ to $u_t = 0$, respectively, be finite. Then
$$\|u - u_h\|_{1,\Omega} = \mathcal{O}(h), \quad h \to 0^+.$$ 
Under weaker assumptions, lower order of convergence can be derived, namely:

if $u \in (H^2(\Omega))^2 \cap K$ and $T_n(u) \in L^2(\Gamma_K)$, then
$$\|u - u_h\|_{1,\Omega} = \mathcal{O}(h^{3/4}), \quad h \to 0^+.$$ 

3. SADDLE-POINT FORMULATION OF CONTACT PROBLEMS WITH A GIVEN FRICTION

Formulation ($\mathcal{P}$) is very simple from the theoretical point of view, but it is not suitable for practical computations. The reason for this is the fact that the direct application of finite elements to ($\mathcal{P}$) leads to a non-differentiable optimization problem. To avoid this difficulty, we proceed as follows (see [5]):

Let us define

(3.1) $A = \{\mu \mid \mu \in L^2(\Gamma_K), |\mu| \leq 1 \text{ a.e. on supp } g, \mu = 0 \text{ on } \Gamma_K \setminus \text{supp } g\}$.

Clearly

$$j(v) = \sup_{\mu \in A} \int_{\Gamma_K} \mu g v_i \, ds.$$ 

Now the problem ($\mathcal{P}$) can be equivalently formulated:

$$\begin{cases}
\text{find } u \in K \text{ such that } \\
\mathcal{F}(u) = \min_{v \in K} \sup_{\mu \in A} \left\{ \frac{1}{2} a(v, v) - L(v) + \int_{\Gamma_K} \mu g v_i \, ds \right\}.
\end{cases}$$

Let us set

$$\mathcal{L}(v, \mu) = \frac{1}{2} a(v, v) - L(v) + \int_{\Gamma_K} \mu g v_i \, ds.$$ 

Now, instead of ($\mathcal{P}$), we shall study the problem of finding a saddle-point $(w, \lambda)$ of $\mathcal{L}$ on $K \times A$, i.e.:

(P) $\mathcal{L}(w, \mu) \leq \mathcal{L}(w, \lambda) \leq \mathcal{L}(v, \lambda) \quad \forall v \in K, \quad \forall \mu \in A$.

The relation between ($\mathcal{P}$) and (P) is given by

Theorem 3.1. There exists a unique saddle-point of $\mathcal{L}$ on $K \times A$. Moreover,

$$w = u, \quad \lambda g = -T_n(u),$$

where $u \in K$ is the solution of ($\mathcal{P}$).

Proof. As the bilinear form $a$ is $V$-elliptic due to Korn's inequality and $A$
is a bounded, convex subset of $L^2(\Gamma_k)$, the existence of a saddle-point $(w, \lambda)$ is a direct consequence of [6], Prop. 6.2A.

If $(w, \lambda)$ is such a saddle-point, then (see [6], Prop. 6.1.6)

$$\frac{\partial \mathcal{L}}{\partial v} (w, \lambda) (v - w) \geq 0 \quad \forall v \in K,$$

$$\frac{\partial \mathcal{L}}{\partial \mu} (w, \lambda) (\mu - \lambda) \leq 0 \quad \forall \mu \in \Lambda.$$ 

Applying Green’s formula to the first inequality we have:

(3.2) \[ g\lambda = -T_\lambda (w) \quad \text{on} \quad \text{supp} \ g . \]

As $T_\lambda (w) = 0$ on $\Gamma_k \setminus \text{supp} \ g$, the previous identity is true on the whole $\Gamma_k$.

Moreover, the pair $(w, \lambda) \in K \times \Lambda$, being a saddle-point of $\mathcal{L}$ on $K \times \Lambda$, satisfies

$$\mathcal{J}(w) = \sup_{\mu \in \Lambda} \mathcal{L}(w, \mu) = \mathcal{L}(w, \lambda) = \inf_{v \in K} \sup_{\mu \in \Lambda} \mathcal{L}(v, \mu) = \inf_{v \in K} \mathcal{J}(v) = \mathcal{J}(u).$$

As the solution $u$ of $(\mathcal{P})$ is unique, necessarily $w = u$. The uniqueness of $\lambda$ on $\text{supp} \ g$ follows from (3.2).

Hence, the contact problem with a given friction can be approximated, starting from the saddle-point formulation $(\mathcal{P})$. This approach has some advantages. First of all, the minimization problem for a non-differentiable functional is replaced by the saddle-point formulation for a functional, which is smooth in all components. Secondly, the Lagrange multiplier $\lambda$ has a “nice” physical meaning and the saddle-point formulation makes it possible to approximate it independently of the displacement field $u$. Finally, the approximation of our problem, which is based on the saddle-point formulation, offers a very effective algorithm for its numerical realization (see below).

Remark 3.1. In the approach, described above, only the dualization of the non-differentiable term is used. In order to release the geometrical constraint $v \in K$, another set of Lagrange multipliers has to be introduced. This formulation, involving two fields of Lagrange multipliers, together with its approximation by finite elements, is analysed in [7]. The approach, presented in the present paper, is suitable for the approximation of the so-called semicoercive cases, when the bilinear form is a $V$/$P$-elliptic only ($P$ denotes the set of rigid body motions).

4. APPROXIMATION OF $(\mathcal{P})$

The general idea of the approximation of the saddle-point formulation is the following:

Let $\{\mathcal{X}_h\}, \mathcal{X}_h \subset V$, $\{A_h\}, A_h \subset L^2(\Gamma_k)$ be the families of sets approximating $K$
and \( A \), respectively. Instead of (P), we shall consider the following problem:

\[
(\Psi_h) \quad \text{find} \quad (w_h, \lambda_h) \in \mathcal{H}_h \times A_h \quad \text{such that}
\]

\[
\mathcal{L}(w_h, \mu_h) \leq \mathcal{L}(w_h, \lambda_h) \leq \mathcal{L}(v_h, \lambda_h) \quad \forall v_h \in \mathcal{H}_h, \quad \forall \mu_h \in A_h.
\]

One can formulate conditions concerning \( \mathcal{H}_h, A_h \) under which \((w_h, \lambda_h)\) tends to \((w, \lambda)\) in the corresponding norms, where \((w, \lambda)\) is the solution of (P) (see [8]). Moreover, the first component \( w_h \) minimizes the functional

\[
\mathcal{J}_0(v_h) + j_h(v_h) \quad \text{over} \quad \mathcal{H}_h,
\]

where

\[
j_h(v_h) = \sup_{\mu_h \in A_h} \int_{\Gamma_K} g \mu_h v_h \, ds.
\]

In what follows we use the following choice of \( \mathcal{H}_h, A_h \):

\[
(4.1) \quad \mathcal{H}_h = K_h, \quad A_h = A,
\]

with \( K_h \) and \( A \) given by (2.2) and (3.1), respectively. In this case

\[
j_h(v_h) = j(v_h) \quad \forall v_h \in V_h, \quad h \in (0, 1)
\]

and \( w_h \in K_h \) coincides with the solution \( u_h \) of \((\Psi_h)\).

Remark 4.1. In (4.1) the family \( \{A_h\} \) does not depend on \( h \). Nevertheless, some other constructions of \( A_h \) are possible. One can use for example “finite” dimensional approximations” of \( A \), when \( A_h \) is a closed, convex and bounded subset of some finite-dimensional space \( L_h \) (see [8]).

Let us assume a regular family \( \{\tau_h\}, h \to 0_+ \), of triangulations of \( \bar{Q} \). We shall study what happens if \( h \to 0_+ \). According to Theorem 2.1 we have

\[
w_h \to u_h \quad \rightarrow 0_+,
\]

provided all the assumptions of Theorem 2.1 are satisfied. Let us study the behaviour of the second component \( \lambda_h \). (It should be noted that the second component \( \lambda_h \) depends on \( h \), even if \( A_h = A \) are independent of \( h \).)

**Theorem 4.1.** Let all the assumptions of Theorem 2.1 be satisfied. Then

\[
\lambda_h \to \lambda, \quad h \to 0_+, \quad \text{weakly in} \quad L^2(\Gamma_K),
\]

where \( \lambda \) is the second component of the saddle-point of \( \mathcal{L} \) on \( K \times A \).

Proof. From the definition of \( A \) it follows that the sequence \( \{\lambda_h\}, h \in (0, 1) \), is bounded in \( L^2(\Gamma_K) \). Hence there exists a subsequence \( \{\lambda_{h'}\} \subseteq \{\lambda_h\} \) and an element \( \chi \in A \) such that

\[
(4.2) \quad \lambda_{h'} \to \chi, \quad h' \to 0_+, \quad \text{in} \quad L^2(\Gamma_K).
\]
We shall show that $\chi = \lambda$. Following the definition of $(\mathcal{P}_h)$ we have

\begin{equation}
(4.3) \quad a(w_{h'}, v_{h'} - w_{h'}) + \int_{K} g\lambda_h(v_{h'} - w_{h'}) \, ds \geq L(v_{h'} - w_{h'}),
\end{equation}

\begin{equation}
\int_{K} g(\mu - \lambda_h) w_{h'} \, ds \leq 0 \quad \forall \mu \in A, \quad \forall v_{h'} \in K_{h'}.
\end{equation}

As noted before,

\begin{equation}
(4.4) \quad w_{h'} \rightarrow u, \quad h' \rightarrow 0_+, \quad \text{in } (H^1(\Omega))^2.
\end{equation}

Let $v \in K$ be an arbitrarily chosen element. Then there exists a sequence $\{v_{h'}\}, v_{h'} \in K_{h'}$, such that

\begin{equation}
(4.5) \quad v_{h'} \rightarrow v \quad \text{in } (H^1(\Omega))^2 \quad \text{see } (2.4).
\end{equation}

Passing to the limit for $h' \rightarrow 0_+$ in (4.3), taking into account (4.4), (4.5) as well as the compactness of the embedding of $H^1(\Omega)$ into $L^2(K)$, we obtain the following system of inequalities:

\begin{equation*}
\begin{split}
a(u, v - u) + \int_{K} g\lambda(v - u) \, ds & \geq L(v - u) \quad \forall v \in K, \\
\int_{K} g(\mu - \lambda) u \, ds & \leq 0 \quad \forall \mu \in A,
\end{split}
\end{equation*}

which means that $(u, \chi)$ is a saddle point of $\mathcal{L}$ on $K \times A$. Since the saddle point is unique (Theorem 3.1),

\begin{equation*}
\chi = \lambda = -g T(u).
\end{equation*}

As $\lambda$ is unique, the whole sequence $\{\lambda_h\}$ tends weakly to $\lambda$ in the $L^2(K)$-norm.

Remark 4.2. As the embedding of $L^2(K)$ into $H^{-1/2}(K)$ is compact, (4.2) implies the strong convergence of $\lambda_h$ to $\lambda$ in the $H^{-1/2}(K)$-norm.

For the numerical realization of $(\mathcal{P}_h)$ one can use the classical Uzawa's algorithm (see [6] or [1]).

Uzawa's algorithm proceeds as follows:

we choose an element $\lambda_0 \in A$;
if $\lambda_k \in A$ is known, we look for $u_h^{(k)} \in K_h$ satisfying

\begin{equation*}
\mathcal{L}(u_h^{(k)}, \lambda_k) = \min_{v \in K_h} \left\{ \mathcal{J}_0(v) + \int_{K} \lambda_k g v \, ds \right\},
\end{equation*}

after which we replace $\lambda_k$ by $\lambda_{k+1}$ as follows:

\begin{equation*}
\lambda_{k+1} = \Pi_A (\lambda_k + \varrho u_h^{(k)}),
\end{equation*}

where $\Pi_A$ denotes the projection of $L^2(K)$ on $A$. $\varrho$ is a positive parameter satisfying $\varrho_1 \leq \varrho \leq \varrho_2$, $\varrho_1, \varrho_2 > 0$. 

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It is readily seen that the projection $\Pi_A$ is given by the following relations:

$$
\Pi_A \mu(x) = \begin{cases} 
-1, & \text{if } \mu(x) < -1 \\
\mu(x), & \text{if } |\mu(x)| \leq 1 \\
1, & \text{if } \mu(x) > 1.
\end{cases} \quad x \in \Gamma_K.
$$

It is well known (see [6]) that there exist $\varrho_1, \varrho_2 > 0$ such that for any choice $\varrho \in (\varrho_1, \varrho_2)$, the components $u_h^{(k)}$ tend to $u_h$ (i.e. to the first component of the solution of $(P_h)$).

5. NUMERICAL REALIZATION OF CONTACT PROBLEMS WITH A GIVEN FRICTION

The present section deals with implementation of Uzawa’s algorithm on computers.

Each iterative step of Uzawa’s method consists of two substeps. The first is the problem of quadratic programming, the second is the computation of the projection $\Pi_A$. The latter substep is very simple. The Lagrange multipliers $\lambda$ will be defined by their values at a finite number of points from $\Gamma_K$. These points will coincide with the nodes of the quadrature formula used for the approximation of $\int_{\Gamma_K} g \lambda u_i \, ds$. Therefore we define the set

$$
A_E = \{ z \in R^n \mid |z_i| \leq 1 \}.
$$

$z_i$ will be interpreted as the values of $\lambda \in A$ at the nodes of the chosen quadrature formula. The projection $\Pi_A$ is now replaced by “pointwise” projection $\Pi_{AE}$ of $R^n$ on $A_E$, the $i$-th component of which is given by

$$
(\Pi_{AE} \mu)_i = \begin{cases} 
-1, & \text{if } \mu_i < -1 \\
\mu_i, & \text{if } |\mu_i| \leq 1 \\
1, & \text{if } \mu_i > 1.
\end{cases}
$$

As was already mentioned, the first part of each iterative step is the minimization problem for a quadratic function over a convex set given by linear constraints. Expressed in the matrix form, we look for $x^* \in K_E$ such that

$$
\mathcal{G}(x^*) \leq \mathcal{G}(x) \quad \forall x \in K_E ,
$$

where

$$
\mathcal{G}(x) = \frac{1}{2} (x, Cx)_{R^m} - (\mathcal{F}, x)_{R^m},
$$

$$
K_E = \{ x \in R^m \mid B'x \leq 0, B''x = 0 \}.
$$

Here $C$ denotes the stiffness matrix, $\mathcal{F} \in R^n$ is a vector obtained by the integration of the linear term of $\mathcal{L}(v_h, \lambda_i)$. $B'$, $B''$ are rectangular matrices of the ranges $n_1 \times m$ and $n_2 \times m$, respectively. If we suppose that $\Gamma_K$ is composed from one straight line segment we see that each row contains at least one and at most two non-zero elements and at most one non-zero element in each column.
Elements of $B', B''$ are formed by components of the outward unit normal vector with respect to $\Gamma_K$ and to $\Gamma_0$, respectively. The set of equality constraints can be empty, if the conditions on $\Gamma_0$ are respected during the construction of the matrix $C$.

The minimization problem (5.1) can be realized by means of the conjugate gradient method, which proceeds as follows (see [9]):

(i) choose $x_0 \in K_E$;
(ii) set $\mathcal{J}(x_0) = \{1, 2, \ldots, n_2\} \cup \{i \mid (b'_i, x_0)_{Rm} = 0\}$, where $b'_i$ is the $i$-th row of $B'$;
(iii) form a matrix $A_{\mathcal{J}(x_0)}$, the rows of which are the vectors $b'_i$ and $b''_i$, $i \in \mathcal{J}(x_0)$, and calculate

$$\mathcal{G}'(x_0) = Cx_0 - \mathcal{F},$$

$$P_{\mathcal{J}(x_0)} = A_{\mathcal{J}(x_0)}(A_{\mathcal{J}(x_0)}A_{\mathcal{J}(x_0)}^T)^{-1} A_{\mathcal{J}(x_0)}^T,$$

$$u(x_0) = -(A_{\mathcal{J}(x_0)}A_{\mathcal{J}(x_0)}^T)^{-1} A_{\mathcal{J}(x_0)} \mathcal{G}'(x_0),$$

where $A_{\mathcal{J}(x_0)}^T$ is the transpose of $A_{\mathcal{J}(x_0)}$.

(iv) We distinguish two cases:

(I) $(I - P_{\mathcal{J}(x_0)}) \mathcal{G}'(x_0) \neq 0$. Then set

$$p_1 = -(I - P_{\mathcal{J}(x_0)}) \mathcal{G}'(x_0)$$

and calculate

$$(*) \quad \alpha_{k+1} = -\frac{(\mathcal{G}'(x_k), p_{k+1})_{Rm}}{(p_{k+1}, C p_{k+1})_{Rm}},$$

$$\bar{\alpha}'_{k+1} = \min_{i = 1, \ldots, n_1} \frac{-(b'_i, x_k)_{Rm}}{(b'_i, p_{k+1})_{Rm}},$$

$$\bar{\alpha}''_{k+1} = \min_{i = 1, \ldots, n_2} \frac{-(b''_i, x_k)_{Rm}}{(b''_i, p_{k+1})_{Rm}},$$

where the minimum is taken for $i$ satisfying

$(b'_i, p_{k+1})_{Rm} > 0$ and $(b''_i, p_{k+1})_{Rm} > 0$.

Define

$$\bar{\alpha}_{k+1} = \min(\bar{\alpha}'_{k+1}, \bar{\alpha}''_{k+1}).$$

If $\alpha_{k+1} \leq \bar{\alpha}_{k+1}$ set

$$x_{k+1} = x_k + \alpha_{k+1} p_{k+1},$$

$$p_{k+2} = -(I - P_{\mathcal{J}(x_0)}) \mathcal{G}'(x_{k+1}) + \frac{((I - P_{\mathcal{J}(x_0)}) \mathcal{G}'(x_{k+1}))^2}{((I - P_{\mathcal{J}(x_0)}) \mathcal{G}'(x_{k}))^2} - p_{k+1},$$

$k = 0, 1, \ldots$.
If $x_{k+1}$ minimizes $\mathcal{G}(x)$ on the convex set given by

$$(b'_i, x) = 0, \quad i \in \mathcal{I}(x_0) - \{1, \ldots, n_2\},$$

and

$$(b'_i, x) = 0, \quad i = 1, 2, \ldots, n_2,$$

set $x_0 = x_{k+1}$ and go to (v).

Otherwise go to (*).

If $\alpha_{k+1} > \bar{\alpha}_{k+1}$ set

$$x_{k+1} = x_k + \bar{\alpha}_{k+1} p_{k+1}$$

and go to (ii).

(II) If $(I - P_{\mathcal{I}(x_0)}) \mathcal{G}'(x_0) = 0$ go to (v).

(v) If $(u(x_0))_i \geq 0 \quad \forall i \in \mathcal{I}(x_0) - \{1, 2, \ldots, n_2\},$
then $x_0$ is the desired minimum of $\mathcal{G}$ on the set $K_E$.

If there exists $i \in \mathcal{I}(x_0) \setminus \{1, 2, \ldots, n_2\}$ such that

$$(u(x_0))_i < 0,$$
set $\mathcal{I}(x_0) \setminus \{i\} \rightarrow \mathcal{I}(x_0)$ and go to (iii).

Clearly, this step is a limiting factor of the efficiency of Uzawa's algorithm. However, certain features of our problem enable us to modify it in such a way that the outcoming version represents an economic tool for the numerical realization.

The above mentioned features are the following ones:

– the stiffness matrix is the same during the whole iterative process:

– the number of constrained variables is small as compared with the number of all variables;

– only a few components on the right hand side change, namely the normal and tangential tractions along $\Gamma_K$, which are related to constrained variables.

Taking into account all these circumstances, the substructuring technique for the minimization of $\mathcal{G}$ over $K_E$ may be used (see [10]), where this method is applied to the frictionless contact problem.

Let us suppose that unknowns are arranged in such a way that all constrained variables are listed last. Due to these assumptions, we may split a vector $x \in R^m$ and write

$$x = (x_1, x_2) \in R^{m-k} \times R^k, \quad k = n_1 + n_2,$$

where $x_1, x_2$ correspond to the free and constrained variables, respectively. The closed convex set $K_E$ can be now written as

$$K_E = \{x \in R^m, \quad x = (x_1, x_2), \quad x_1 \in R^{m-k}, \quad x_2 \in R^k, \quad B'x_2 \leq 0, \quad B''x_2 = 0\}.$$

Similarly, we can split the vector $\mathcal{F}$ and decompose the matrix $C$;

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

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where \( C_{11}, C_{22} \) are square matrices of the range \( m - k \) and \( k \), respectively, and \( C_{12}, C_{21} \) are rectangular matrices \( (m - k) \times k \) and \( k \times (m - k) \), respectively. Since \( C \) is symmetric, we have

\[
C_{21} = C_{12}^T.
\]

Let \( x^* \in K_E \) be a solution of (5.1), i.e. \( x^* = (x_1^*, x_2^*) \in K_E \) fulfills

\[
(C x^*, y - x^*)_{R^m} \geq (\mathcal{F}, y - x^*)_{R^m} \quad \forall y \in K_E.
\]

Let \( y = (y_1, y_2) \in K_E \) be a vector of the form \( y_1 = x_1^* \pm z_1, z_1 \in \mathbb{R}^{m-k} \) arbitrary, \( y_2 = x_2^* \). Substituting \( y \) into (5.2) we get

\[
C_{11}x_1^* + C_{12}x_2^* = \mathcal{F}_1
\]

which implies

\[
x_1^* = -C_{11}^{-1}C_{12}x_2^* + C_{11}^{-1}\mathcal{F}_1.
\]

Now we choose \( y = (y_1, y_2) \) in such a way that \( y_1 = x_1^*, y_2 = z_2 \), where

\[
z_2 \in \tilde{K}_E = \{z \in \mathbb{R}^k \mid B'z \leq 0, B''z = 0\}.
\]

Clearly \( y \in K_E \).

Inserting it into (5.2) we arrive at

\[
(z_2 - x_2^*, C_{21}x_1^* + C_{22}x_2^*)_{R^k} \geq (z_2 - x_2^*, \mathcal{F}_2)_{R^k} \quad \forall z_2 \in \tilde{K}_E.
\]

Taking into account (5.4), we deduce that \( x_2^* \in \tilde{K}_E \) is the solution of the variational inequality

\[
(z_2 - x_2^*, \tilde{C}x_2^*)_{R^k} \geq (\mathcal{G}_2, z_2 - x_2^*)_{R^k} \quad \forall z_2 \in \tilde{K}_E
\]

with

\[
\tilde{C} = C_{22} - C_{21}C_{11}^{-1}C_{12} \quad \text{and} \quad \mathcal{G}_2 = \mathcal{F}_2 - C_{21}C_{11}^{-1}\mathcal{F}_1,
\]

or equivalently: \( x_2^* \) is a minimizer of

\[
\mathcal{G}(z) = \frac{1}{2}(z, \tilde{C}z)_{R^k} - (\mathcal{G}_2, z)_{R^k}, \quad z \in \mathbb{R}^k
\]

over the closed convex set \( \tilde{K}_E \). It is readily seen \( \tilde{C} \) is symmetric and it can be shown that it is obtained from \( C \) by the elimination of all unconstrained unknowns. Hence \( \tilde{C} \) is positive definite.

**NUMERICAL EXAMPLES**

In order to illustrate the approach discussed above, let us consider the Signorini problem for an elastic body given by Fig. 2, subjected to a horizontal surface traction \( P_1 = 1000 \) only. The decomposition of the boundary \( \partial \Omega \) into \( \Gamma_u, \Gamma_0, \Gamma_r \) and \( \Gamma_K \) as well as the triangulation \( \tau_h \) of \( \Omega \) are clear from the figure.
We discuss two cases:

(i) the frictionless problem, corresponding to \( g \equiv 0 \) on \( \Gamma_k \);

(ii) the contact problem with a given friction and \( g = 150 \) along \( \Gamma_k \).

The values of normal and tangential components of displacements \( u_n, u_t \), respectively, at the points lying on \( \Gamma_k \) are given in Tables 1, 2. The distribution of normal stresses for both cases is shown in Fig. 3. Finally, the computed values of tangential components of stresses are compared with the product \( g \lambda \) in the case (ii) (Fig. 4).
Fig. 3.

Tab. 3

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Tab. 2

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**Souhrn**

APROXIMACE A NUMERICKÉ ŘEŠENÍ KONTAKTNÍ ÚLOHY SE TŘENÍM

**JAROSLAV HASLINGER, MIROSLAV TVRDÝ**


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