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Gur Dial

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ON MEASURABLE SOLUTIONS OF A FUNCTIONAL EQUATION  
AND THEIR APPLICATION TO INFORMATION THEORY

GUR DIAL

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1. INTRODUCTION

Let  $\Gamma_n = \{P = (p_1, \dots, p_n); p_i \geq 0, i = 1, \dots, n; \sum p_i = 1\}$  for  $n \geq 1$  be a set of all  $n$ -complete probability distributions. Let  $R$  be the set of all real numbers and let  $I = [0, 1]$ .

Consider measurable functions  $f, g, h, k: I \rightarrow R$  satisfying the system of functional equations

$$(1.1) \quad \sum_i \sum_j h(x_i y_j) = \sum_j \sum_i f(x_i) g(y_j) + \sum_i k(x_i),$$

where  $X = (x_1, \dots, x_n) \in \Gamma_n, Y = (y_1, \dots, y_m) \in \Gamma_m$ .

The continuous solutions of (1.1) were given by Taneja [4].

The objective of this paper is to find the measurable solutions of (1.1). As an application, a joint characterization of Shannon's entropy and entropy of type  $\beta$  is given.

2. MEASURABLE SOLUTIONS OF (1.1)

In this section we will find the measurable solutions of system (1.1). This is done in the following theorem.

**Theorem 1.** *If  $f, g, h$  and  $k$  are measurable solutions of (1.1) for  $X \in \Gamma_n, Y \in \Gamma_m$  where  $n, m = 2, 3$ , then they are given by one of the following set of solutions for  $x \in [0, 1]$ . 1st set of solutions:*

$$(2.1) \quad h(x) = Bx + Ax \log x, \quad f(x) = Cx,$$

$$(2.2) \quad g(x) = Dx + A/Cx \log x, \quad k(x) = (B - CD)x + Ax \log x;$$

2nd set of solutions:

$$(2.3) \quad h(x) = Bx + A(x^\beta - x), \quad f(x) = Cx^\beta,$$

$$(2.4) \quad g(x) = Dx + (A/C)(x^\beta - x), \quad k(x) = (B - A)x + (A - CD)x^\beta,$$

where  $A, B, C$  and  $D$  are arbitrary constants and  $\beta > 0$  ( $1 \neq \beta > 0$ ) is a parameter.

In addition to the above two sets of solutions, we also get the trivial solution:

$$\begin{aligned} h(x) &= Bx, \quad f(x) = \text{arbitrary}, \quad g(x) = Dx \quad \text{and} \\ k(x) &= Bx - Df(x). \end{aligned}$$

**Proof.** Substituting  $Y = (y, u, l - y - u) \in \Gamma_3$  and  $Y = (y + u, l - y - u) \in \Gamma_2$  in (1.1), we get respectively

$$\begin{aligned} (2.5) \quad & \sum_i (h(x_i y) + h(x_i u) + h(x_i(1 - y - u))) = \\ & = \sum_i f(x_i) (g(y) + g(u) + g(1 - y - u)) + \sum_i k(x_i) \\ (2.6) \quad & \sum_i (h(x_i(y + u) + h(x_i(1 - y - u)))) = \sum_i f(x_i) (g(y + u) + \\ & + g(1 - y - u) + \sum_i k(x_i). \end{aligned}$$

Subtracting (2.6) from (2.5), we obtain

$$(2.7) \quad \sum_i (h(x_i y) + h(x_i u) - h(x_i(y + u))) = \sum_i f(x_i) (g(y) + g(u) - g(y + u)).$$

Let us define for  $X \in \Gamma_n$ ,  $n = 2, 3$ ,  $t \in I = [0, 1]$ :

$$(2.8) \quad A_X(t) = \sum_i h(x_i t) - \sum_i f(x_i) g(t).$$

By virtue of (2.8) it is easy to see that  $A_X(\cdot)$  is additive on  $I$ , i.e.

$$(2.9) \quad A_X(y + u) = A_X(y) + A_X(u).$$

It now follows from a result of Daroczy and Losonczi [3] that

$$(2.10) \quad A_X(t) = t A_X(1), \quad t \in I$$

is a measurable solution.

In order to obtain the expression for  $A_x(1)$ , we will find the expression for the function

$$(2.11) \quad \sum_i h(x_i) - \sum_i f(x_i) g(1).$$

Substituting  $Y = (1, 0)$  and  $Y = (1, 0, 0)$  in (1.1) we get respectively

$$(2.12) \quad \sum_i h(x_i) + n h(0) = \sum_i f(x_i) (g(1) + g(0)) + \sum_i k(x_i),$$

$$(2.13) \quad \sum_i h(x_i) + 2n h(0) = \sum_i f(x_i) (g(1) + 2g(0)) + \sum_i k(x_i).$$

Subtracting (2.12) from (2.13) we obtain

$$(2.14) \quad n h(0) = \sum_i f(x_i) g(0).$$

Using (2.14), we transform (2.12) into

$$(2.15) \quad \sum_i h(x_i) = \sum_i f(x_i) g(1) + \sum_i k(x_i)$$

From (2.15) and (2.10) we get

$$(2.16) \quad \sum_i h(x_i t) - \sum_i f(x_i) g(t) = t \sum_i k(x_i)$$

for all  $X \in \Gamma_n$ ,  $n = 2, 3$  and  $t \in I$ .

Let us substitute  $X = (x, v, 1 - x - v) \in \Gamma_3$  and  $X = (x + v, 1 - x - v) \in \Gamma_2$  in (2.16). We obtain respectively

$$(2.17) \quad h(xt) + h(vt) + h((1 - x - v)t) - (f(x) + f(v) + f(1 - x - v))g(t) = \\ = t(k(x) + k(v) - k(1 - x - v)),$$

$$(2.18) \quad h((x + v)t) + h((1 - x - v)t) - (f(x + v) + f(1 - x - v))g(t) = \\ = t(k(x + v) - k(1 - x - v)).$$

From (2.18) and (2.17), we get

$$(2.19) \quad h(xt) + h(vt) - h((x + v)t) = (f(x) + f(v) - f(x + v))g(t) + \\ + t(k(x) + k(v) - k(x + v)).$$

For  $t \in I$ , let us define

$$(2.20) \quad B_t(w) = h(wt) - f(w)g(t) - t h(w).$$

Then using (2.20), we can write (2.19) in the form

$$(2.21) \quad B_t(x + v) = B_t(x) + B_t(v), \quad \text{for } x, v, x + v \in [0, 1].$$

Again using the result of Daroczy and Losonczi [3], we have

$$(2.22) \quad B_t(x) = x B_t(1).$$

By substituting  $X = (1, 0)$  and  $X = (1, 0, 0)$  in (2.16) we get the relation

$$(2.23) \quad h(t) = f(1)g(t) + t k(1).$$

Using (2.23), (2.22) becomes

$$(2.24) \quad h(xt) = f(x)g(t) + t k(x), \quad \text{for all } x, t \in I.$$

Dividing (2.24) by  $xt$  ( $x \neq 0, t \neq 0$ ), we get

$$\frac{h(xt)}{xt} = \frac{f(x)}{x} \frac{g(t)}{t} + \frac{k(x)}{x}.$$

Let  $h_1(x) = h(x)/x$ ,  $f_1(x) = f(x)/x$ ,  $g_1(t) = g(t)/t$  and  $k_1(x) = k(x)/x$ .

Then we have

$$(2.25) \quad h_1(xt) = f_1(x) g_1(t) + k_1(x).$$

Putting first  $x = 1$  and then  $t = 1$  in (2.25) we get

$$(2.26) \quad \begin{aligned} h_1(t) &= f_1(1) g_1(t) + k_1(1), \\ h_1(1) &= f_1(1) g_1(1) + k_1(1). \end{aligned}$$

If  $f_1(1) = 0$  then (2.26) implies

$$h_1(t) = h_1(1) \quad \text{or} \quad h(t) = t h_1(1) = At \quad \text{where} \quad A = h_1(1) = h(1).$$

In this case  $h$  is a homogeneous linear function. Now suppose that  $f_1(1) \neq 0$ . Then from (2.25) and (2.26) we obtain

$$h_1(xt) = \frac{f_1(x)}{f_1(1)} h_1(t) + k_1(x) - \frac{f_1(x)}{f_1(1)} k_1(1).$$

Define  $f_2(x) = f_1(x)/f_1(1)$ ,  $k_2(x) = k_1(x) - f_2(x) k_1(1)$ . Then we have from the above equation that

$$(2.28) \quad h_1(xt) = f_2(x) h_1(t) + k_2(x).$$

Since  $f$ ,  $g$ ,  $h$ ,  $k$  are measurable functions, hence  $h_1$ ,  $f_2$ ,  $h_1$  and  $k_2$  are also measurable.

The general measurable solution of (2.28) with  $h_1$ ,  $f_2$ ,  $k_2$  measurable is given by (see Aczel [1])

$$(2.29) \quad h_1(x) = h_0(x) + \alpha; \quad f_2(x) = 1; \quad k_2(x) = h_0(x)$$

and

$$(2.30) \quad h_1(x) = \gamma e^{h_0(x)} + \alpha, \quad f_2(x) = e^{h_0(x)}, \quad k_2(x) = \alpha(1 - e^{h_0(x)})$$

with an additional trivial solution

$$(2.31) \quad h_1(x) = \alpha, \quad f_2(x) \text{ arbitrary}, \quad k_2(x) = \alpha(1 - f_2(x))$$

where  $\gamma \neq 0$  and  $\alpha$  are arbitrary constants and  $h_0$  is an arbitrary measurable solution of the equation

$$(2.32) \quad h_0(xt) = h_0(x) + h_0(t).$$

However the most general measurable solution of (2.32) is

$$(2.33) \quad h_0(x) = A \log x$$

where  $A$  is an arbitrary constant.

Thus the solutions (2.29), (2.30) and (2.31) together with (2.33), (2.28) and (2.25) give the required set of solutions.

### 3. APPLICATIONS TO INFORMATION THEORY

Shannon's measure of information is defined as

$$(3.1) \quad H(P) = - \sum_i p_i \log p_i, \quad P \in \Gamma_n.$$

A well known generalization of (3.1) is covered by the entropy of type  $\beta$  and is given as (see [2])

$$(3.2) \quad H^\beta(P) = (2^{1-\beta} - 1)^{-1} (\sum_i p_i^\beta - 1), \quad \beta \neq 1, \beta > 0, \quad P \in \Gamma_n.$$

In terms of measurable solution of (1.1), we can define  $H(P)$  or  $H^\beta(P)$  as

$$(3.3) \quad H(P) = \sum_i h(p_i)$$

under suitable boundary and normalization conditions.

In the following theorem a joint characterization of (3.1) and (3.2) is given.

**Theorem 2.** *The entropies of distribution  $P$  under the conditions  $h(1) = h(0)$  and  $h(1/2) = 1/2$  corresponding to the measurable solutions are (3.1) and (3.2), respectively.*

*Proof.* Putting  $x = 0$  in (2.1) and (2.3) we have  $h(0) = h(1) = 0$ . Using  $h(1/2) = 1/2$ , the constant  $A$  becomes  $-1$  and  $(2^{1-\beta} - 1)^{-1}$ , respectively. The result follows from (3.3).

#### References

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#### Souhrn

### MĚŘITELNÁ ŘEŠENÍ JISTÉ FUNKCIONÁLNÍ ROVNICE A JEJICH APLIKACE V TEORII INFORMACE

GUR DIAL

V článku jsou nalezena měřitelná řešení jisté funkcionální rovnice se čtyřmi neznámými funkcemi. Jako jejich aplikace je dána společná charakterizace Shannony entropie a entropie  $\beta$ .

*Author's address:* Prof. Gur Dial, Department of Mathematics, Federal University of Santa Catarina, 88000 Florianopolis, SC, Brasil.