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ISONEMALITY AND MONONEMALITY OF WOVEN FABRICS

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In the paper [2] combinational problems concerning woven fabrics are studied. The following conjecture is expressed: Every periodic mononemal fabric which is warp-isonemal is also weft-isonemal. We shall prove this conjecture for fabrics with \( n \times n \) square fundamental blocks for \( n \) odd.

We shall consider diagrams of woven fabrics as they are used in [2] or in the Czech book [1]. Such a diagram is formed by a plane lattice in which some squares are white and the others are black. A white square denotes a place where a weft strand passes over a warp strand, and a black square denotes a place where a warp strand passes over a weft strand. A fabric is called periodic, if it can be obtained from a fundamental \( n \times m \) block of squares by translations in horizontal and vertical directions through multiples of \( n \) and \( m \) units.

Consider a fundamental block of a given fabric \( F \). An example (the fabric No. 164 from [1]) is in Fig. 1. Let the warp strands be numbered by the numbers 1, \ldots, \( n \) from the left end to the right end and let the weft strands be numbered by the numbers 1, \ldots, \( m \) from the upper end to the lower end. For \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \) put \( a_{ij} = 1 \) if the intersection of the \( i \)-th warp strand with the \( j \)-th weft strand is
a black square, and $a_{ij} = 0$ if it is a white square. Now construct a bipartite graph $G(\mathcal{F})$ on the vertex sets $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_m\}$ in which the vertices $u_i, v_j$ are adjacent if and only if $a_{ij} = 1$. This graph will be called the graph of the fabric $\mathcal{F}$. The graph of the fabric from Fig. 1 is in Fig. 2.

![Fig. 2.](image)

A fabric $\mathcal{F}$ is called warp-isonemal (or weft-isonemal), if for every two warp strands (or weft strands, respectively) there exists a mapping which maps one onto the other and is either a symmetry of the whole fabric (taken as infinite in all directions), or such a symmetry superposed with the interchange of the colours black and white.

A fabric $\mathcal{F}$ is called mononemal, if any two strands of $\mathcal{F}$ (each of them may be either a warp strand or a weft one) have the property that the two-way infinite sequences of white and black squares formed by these strands either are equal, or become equal after interchanging the colours black and white. Evidently if a fabric is mononemal, then it has a fundamental block which is a square. We shall always consider an $n \times n$ square and suppose that $n$ is the least possible.

In the sequel we consider mononemal fabrics. The group of isometric mappings of the plane onto itself which map warp strands of a fabric $\mathcal{F}$ onto warp strands and weft strands onto weft strands will be denoted by $T_{0}(\mathcal{F})$. To each of these mappings, a certain mapping of the vertex set of $G(\mathcal{F})$ onto itself corresponds. The group $T_{0}(\mathcal{F})$ is generated by the elements $\varphi_0, \psi_0, \alpha_0, \beta_0$ described in the sequel.

The mapping $\psi_0$ is a translation in the horizontal direction which maps every warp strand onto its neighbour from the right and leaves all weft strands fixed. To the mapping $\varphi_0$, a mapping $\varphi$ of the vertex set of $G(\mathcal{F})$ onto itself corresponds; this mapping is defined by $\varphi(u_i) = u_{i+1}, \varphi(v_i) = v_i$ for $i = 1, \ldots, n$.

The mapping $\psi_0$ is a translation in the vertical direction which maps every weft strand onto its neighbour from below and leaves all warp strands fixed. To the mapping $\psi_0$, a mapping $\psi$ of the vertex set of $G(\mathcal{F})$ onto itself corresponds; this mapping is defined by $\psi(u_i) = u_1, \psi(v_i) = v_{i+1}$ for $i = 1, \ldots, n$. (The subscripts are always taken modulo $n$.)

The mapping $\alpha_0$ is an axial symmetry with respect to the vertical axis going through the centre of a fundamental block. The corresponding mapping $\alpha$ of the vertex set of $G(\mathcal{F})$ onto itself is defined by $\alpha(u_i) = u_{n+1-i}, \alpha(v_i) = v_i$ for $i = 1, \ldots, n$.

The mapping $\beta_0$ is an axial symmetry with respect to the horizontal axis going
through the centre of a fundamental block. The corresponding mapping \( \beta \) of the vertex set of \( G(\mathcal{F}) \) onto itself is defined by \( \beta(u_i) = u_i, \beta(v_i) = v_{n+1-i} \) for \( i = 1, \ldots, n \).

By \( T(\mathcal{F}) \) denote the group generated by the elements \( \varphi, \psi, \alpha, \beta \).

Now let \( \varphi_1, \alpha_1 \) be the restrictions of \( \varphi, \alpha \), respectively, onto \( U \) and let \( \varphi_2, \beta_2 \) be the restrictions of \( \psi, \beta \), respectively, onto \( V \). Let \( T_1(\mathcal{F}) \) (or \( T_2(\mathcal{F}) \)) be the group formed by the restrictions of elements of \( T(\mathcal{F}) \) onto \( U \) (or \( V \), respectively). As every mapping \( \eta \in T(\mathcal{F}) \) maps \( U \) onto \( U \) and \( V \) onto \( V \), there exist mappings \( \eta_1 \in T_1(\mathcal{F}) \) and \( \eta_2 \in T_2(\mathcal{F}) \) such that \( \eta(x) = \eta_1(x) \) for \( x \in U \) and \( \eta(x) = \eta_2(x) \) for \( x \in V \); we may write \( \eta = [\eta_1, \eta_2] \).

Evidently, \( T_1(\mathcal{F}) \) is generated by \( \varphi_1, \alpha_1 \) and \( T_2(\mathcal{F}) \) is generated by \( \beta_2, \psi_2 \). Let \( A(\mathcal{F}) \) be the automorphism group of \( G(\mathcal{F}) \) and let \( B(\mathcal{F}) \) be the group consisting of all automorphisms of \( G(\mathcal{F}) \) and all isomorphisms of \( G(\mathcal{F}) \) onto its bipartite complement. (The bipartite complement of \( G(\mathcal{F}) \) is the bipartite graph on the vertex sets \( U, V \) such that a vertex of \( U \) is adjacent to a vertex of \( V \) in it if and only if these vertices are not adjacent in \( G(\mathcal{F}) \).) Let \( A_0(\mathcal{F}) = A(\mathcal{F}) \cap T(\mathcal{F}), B_0(\mathcal{F}) = B(\mathcal{F}) \cap T(\mathcal{F}) \). The mappings from \( B_0(\mathcal{F}) \) are exactly those mappings of the vertex set of \( G(\mathcal{F}) \) onto itself which correspond to the symmetries of \( \mathcal{F} \) and to those symmetries superposed with the interchange of the colours black and white. If \( G(\mathcal{F}) \) is not isomorphic to its bipartite complement, then evidently \( B(\mathcal{F}) = A(\mathcal{F}) \) and \( B_0(\mathcal{F}) = A_0(\mathcal{F}) \).

Now let \( B_1(\mathcal{F}) \) (or \( B_2(\mathcal{F}) \)) be the set of all mappings \( \eta_1 \in T_1(\mathcal{F}) \) (or \( \eta_2 \in T_2(\mathcal{F}) \)) to which there exists a mapping \( \eta_2 \in T_2(\mathcal{F}) \) (or \( \eta_1 \in T_1(\mathcal{F}) \), respectively) such that \( \eta = [\eta_1, \eta_2] \in B_0(\mathcal{F}) \). Analogously \( A_1(\mathcal{F}), A_2(\mathcal{F}) \) may be defined.

We shall prove some theorems and a lemma. Here \( \mathcal{F} \) is always a fabric with an \( n \times n \) square fundamental block and \( n \) is supposed to be the least possible.

**Theorem 1.** Let \( \mathcal{F} \) be a warp-isonemal fabric with an \( n \times n \) square fundamental block for \( n \) odd. Then \( \varphi_1 \in B_1(\mathcal{F}) \).

**Proof.** There are two mappings from \( T_1(\mathcal{F}) \) which map \( u_1 \) onto \( u_2 \); they are \( \varphi_1 \) and \( \varphi_1^2 \alpha_1 \). As \( \mathcal{F} \) is warp-isonemal, at least one of them must be in \( B_1(\mathcal{F}) \). If \( \varphi_1 \in B_1(\mathcal{F}) \), the assertion is true; thus suppose that \( \varphi_1^2 \alpha_1 \in B_1(\mathcal{F}) \). Similarly there are two mappings from \( T_1(\mathcal{F}) \) which map \( u_1 \) onto \( u_3 \); they are \( \varphi_1^2 \) and \( \varphi_1^3 \alpha_1 \). If \( \varphi_1^2 \in B_1(\mathcal{F}) \), then \( \varphi_1 = (\varphi_1^2)^{(n+1)/2} \in B_1(\mathcal{F}) \). If \( \varphi_1^3 \alpha_1 \in B_1(\mathcal{F}) \), then \( \varphi_1 = (\varphi_1^3 \alpha_1)^{-1} \in B_3(\mathcal{F}) \).

**Theorem 2.** Let \( \mathcal{F} \) be a fabric with an \( n \times n \) square fundamental block, where \( n \) is odd. Then no mapping which is a superposition of an isometric mapping of the plane onto itself and the interchange of colour black and white maps \( \mathcal{F} \) onto itself.

**Proof.** All fundamental blocks of \( \mathcal{F} \) are obtained from one of them by cyclic permutations of warp strands and cyclic permutations of weft strands; therefore all
of them have the same number of black squares and the same number of white squares. As \( n \) is odd, the number of squares of any fundamental block is odd and such a block cannot contain the same number of black and white squares. Hence the interchange of colours black and white transforms the fabric \( F \) into a fabric non-isomorphic to \( F \).

**Lemma.** Let \( F \) be a warp-isonemal and mononemal fabric with an \( n \times n \) square fundamental block for \( n \) odd. Let \( \eta_2 \) be the mapping from \( B_2(F) \) such that \( \eta = [\varphi_1, \eta_2] \in A_0(F) = B_0(F) \). Then the degree of \( \eta_2 \) in \( T_2(F) \) is equal to \( n \).

**Remark.** The equality \( B_0(F) = A_0(F) \) follows from Theorem 2.

**Proof.** Evidently, the degree of \( \eta_2 \) is either 2 or a divisor of \( n \). Let it be \( k \neq n \). If \( \eta = [\varphi_1, \eta_2] \in B_2(F) \), then \( \eta^k = [\varphi_1^k, \eta_2^k] = [\varphi_1^k, \varepsilon_2] \in B_0(F) \), where \( \varepsilon_2 \) is the identity mapping of \( V \). The mapping \( \eta^k \) is an automorphism of \( G(F) \), therefore the neighbourhoods of \( u_i \) and \( u_{i+m} \) are equal for each \( i \), where \( m \) is the greatest common divisor of \( n \) and \( k \). (No mapping from \( B_0(F) \) maps \( G(F) \) onto its bipartite complement, therefore each of them maps it onto itself; this follows from Theorem 2.) Hence \( n \) is not the least possible period of the two-way infinite sequence of black and white squares on a strand; hence \( m \) is such a period and there exists an \( m \times m \) square fundamental block of \( F \), which is a contradiction with the assumption that the fundamental block of \( F \) is an \( n \times n \) square.

**Theorem 3.** Let \( F \) be a fabric with an \( n \times n \) square fundamental block, where \( n \) is odd. Let \( F \) be mononemal and warp-isonemal. Then \( F \) is weft-isonemal.

**Proof.** According to Theorem 1 we have \( \varphi_1 \in B_1(F) \). According to Lemma there exists \( \eta_2 \in B_2(F) \) such that \( \eta = [\varphi_1, \eta_2] \in B_0(F) = A_0(F) \), and the degree of \( \eta_2 \) is \( n \). As the degree of \( \psi_2^l \) is 2 for each \( k \), we have \( \eta_2 = \psi_2^l \), where \( l \) is relatively prime to \( n \). Among the powers of \( \psi_2^l \) there are all powers of \( \psi_2 \), hence each \( v_i \) can be mapped onto each \( v_j \) by a mapping from \( B_2(F) \) and \( F \) is weft-isonemal.

**References**

Souhrn
ISONEMALITA A MONONEMALITA TKANIN

BOHDAN ZELINKA

V článku se zkoumají diagramy tkanin složené z bílých a černých čtverčků jakožto geometrické útvary a popisují se jejich symetrie. Užívá se pojmů isonemality a mononemality, které zavedli B. Grünbaum a G. C. Shephard. Dokazuje se, že periodická mononemální útkově isonemální tkanina, jejíž střída je čtverec o straně liché délky, je rovněž osnovně isonemální.

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