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SOLUTION OF SIGNORINI'S CONTACT PROBLEM IN THE DEFORMATION THEORY OF PLASTICITY BY SECANT MODULES METHOD

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A problem of unilateral contact between an elasto-plastic body and a rigid friction less foundation will be solved within the range of the so called deformation theory of plasticity [1], [5]. Thus the famous Signorini's problem in linear elasticity [6] is generalized to non-linear stress-strain relations. The weak solution is defined on the basis of a variational inequality, which in turn is equivalent to the minimum of the potential energy. Then the so-called secant modules (Kačanov) iterative method is introduced, each step of which corresponds to a classical Signorini's problem in elastostatics. Thus a finite element analysis of the latter is available [7].

On an abstract level, we prove the convergence of the secant modules method to the exact solution. Special effort is devoted to some cases when rigid admissible displacements exist.

1. INTRODUCTION

Let us consider a bounded domain $\Omega \subset R^3$ with a Lipschitz boundary $\partial\Omega$ and assume that

$$\partial\Omega = \Gamma_u \cup \Gamma_\tau \cup \Gamma_K \cup \Gamma_M ,$$

where Γ_u, Γ_τ, Γ_K are open subsets of $\partial\Omega$, $\Gamma_K \neq \emptyset$ and the surface measure of Γ_M vanishes.

Let the elasto-plastic body, occupying the domain Ω, be governed by the following Hencky-Mises stress-strain relations

$$\tau_{ij} = (k - \tfrac{2}{3}\,\mu(\gamma))\,\delta_{ij}e_{ll} + 2\,\mu(\gamma)\,e_{ij}\,, \tag{1.1}$$

where k is a (constant) bulk modulus,

$$\gamma(\boldsymbol{u}, \boldsymbol{v}) = -\tfrac{2}{3}\,\vartheta(\boldsymbol{u})\,\vartheta(\boldsymbol{v}) + 2\,e_{ij}(\boldsymbol{u})\,e_{ij}(\boldsymbol{v})\,,$$

$$\gamma(\boldsymbol{u}, \boldsymbol{u}) \overset{\text{df}}{=} \gamma(\boldsymbol{u}) = \gamma\,, \quad \vartheta(\boldsymbol{u}) = \operatorname{div} \boldsymbol{u}\,,$$

$$e_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and a repeated index implies summation over the range 1, 2, 3. Assumptions on the function μ will be presented in Section 3.

Finally, let the functions $\boldsymbol{u}^0 \in [W^{1,2}(\Omega)]^3$, $\boldsymbol{f} \in [L^2(\Omega)]^3$ and $\boldsymbol{g} \in [L^2(\Gamma_\tau)]^3$ be given.

We are seeking a solution of the non-linear system

$$(1.2) \qquad -\frac{\partial}{\partial x_i}\left[\left(k - \tfrac{2}{3}\,\mu(\gamma)\right) \vartheta(\boldsymbol{u})\right] - 2\,\frac{\partial}{\partial x_j}\left[\mu(\gamma)\, e_{ij}(\boldsymbol{u})\right] = f_i\,, \quad i = 1, 2, 3\,,$$

in Ω, such that

$$(1.3) \qquad \boldsymbol{u} = \boldsymbol{u}^0 \quad \text{on} \quad \Gamma_u\,,$$

$$\tau_{ij}\nu_j = g_i \quad \text{on} \quad \Gamma_\tau\,,$$

where $\boldsymbol{\nu}$ denotes the unit outward normal to $\partial\Omega$. We denote $u_\nu = u_i\nu_i$, $T_\nu = \tau_{ij}\nu_i\nu_j$, $(T_t)_i = T_i - T_\nu\nu_i$, where $T_i = \tau_{ij}\nu_j$, and assume that

$$(1.5) \qquad u_\nu \leqq 0\,, \quad T_\nu \leqq 0\,, \quad u_\nu T_\nu = 0 \text{ on } \Gamma_K\,.$$

The solution of the problem (1.2) till (1.5) leads to minimizing the following functional of potential energy (cf. [1])

$$(1.6) \quad \mathscr{L}(\boldsymbol{u}) = \tfrac{1}{2}k \int_\Omega \vartheta^2(\boldsymbol{u})\,\mathrm{d}x + \tfrac{1}{2}\int_\Omega \left(\int_0^{\gamma(\boldsymbol{u})} \mu(t)\,\mathrm{d}t\right)\mathrm{d}x - \int_\Omega f_i u_i\,\mathrm{d}x - \int_{\Gamma_\tau} g_i u_i\,\mathrm{d}s$$

over the convex set

$$(1.7) \qquad K = \{\boldsymbol{u} \in [W^{1,2}(\Omega)]^3 \mid \boldsymbol{u} = \boldsymbol{u}^0 \text{ on } \Gamma_u, u_\nu \leqq 0 \text{ on } \Gamma_K\}\,.$$

The latter problem is equivalent to the solution of the variational inequality:

$$(1.8) \quad \boldsymbol{u} \in K\,, \int_\Omega \left[\left(k - \tfrac{2}{3}\mu(\gamma)\right) \vartheta(\boldsymbol{u})\, \vartheta(\boldsymbol{v} - \boldsymbol{u}) + 2\,\mu(\gamma)\, e_{ij}(\boldsymbol{u})\, e_{ij}(\boldsymbol{v} - \boldsymbol{u}) - \right.$$

$$\left. - f_i(v_i - u_i)\right]\mathrm{d}x - \int_{\Gamma_\tau} g_i(v_i - u_i)\,\mathrm{d}s \geqq 0 \quad \forall \boldsymbol{v} \in K\,.$$

Method of secant modules (or Kačanov method, see [1] – chapter 8 and 11·5) consists in solving a sequence of the following variational inequalities:

$$(1.9) \qquad \boldsymbol{u}_{n+1} \in K\,, \int_\Omega \left[\left(k - \tfrac{2}{3}\mu(\gamma(\boldsymbol{u}_n))\right) \vartheta(\boldsymbol{u}_{n+1})\, \vartheta(\boldsymbol{v} - \boldsymbol{u}_{n+1}) + \right.$$

$$\left. + 2\,\mu(\gamma(\boldsymbol{u}_n))\, e_{ij}(\boldsymbol{u}_{n+1})\, e_{ij}(\boldsymbol{v} - \boldsymbol{u}_{n+1}) - f_i(v_i - (u_{n+1})_i)\right]\mathrm{d}x -$$

$$-\int_{\Gamma_\tau} g_i(v_i - (u_{n+1})_i)\, \mathrm{d}s \geqq 0\,, \quad n = 1, 2, \ldots.$$

Under certain assumptions on the function μ we shall prove convergence of the method. We use an abstract approach, parallel to that of [1]. The problem is transferred to the solution of a sequence of variational inequalities with variable coefficients, in general.

2. ABSTRACT FORMULATION

Let a functional Φ be given on a Hilbert space H. Assume that Φ has the second Gateaux differential $D^2\, \Phi(u, h, k)$ and the mapping $u \mapsto D^2\, \Phi(u, h, k)$ is continuous on every line segment.

Assume further that

$$(2.1) \qquad D^2\, \Phi(u, h, h) \geqq m\|h\|^2\,, \quad m = \text{cost.} > 0\,.$$

Let a bilinear form $B(u; x, y)$ be given, symmetric in x, y and such that

$$(2.2) \qquad B(u; x, x) \geqq c_1\|x\|^2\,, \quad c_1 = \text{const.} > 0\,,$$

$$(2.3) \qquad |B(u; x, y)| \leqq c_2\|x\|\,\|y\|\,,$$

$$(2.4) \qquad B(u; u, v) = D\,\Phi(u, v)\,,$$

$$(2.5) \qquad \tfrac{1}{2}\,B(x; y, y) - \tfrac{1}{2}\,B(x; x, x) - \Phi(y) + \Phi(x) \geqq 0 \quad \forall x, y \in H\,.$$

Moreover, let K be a closed convex subset of H.

Theorem 2.1. *Let the assumptions* (2.1) *till* (2.5) *be satisfied and let an element $\varphi \in H$ be given.*

Then the problem: find $u \in K$ such that

$$(2.7) \qquad D\,\Phi(u, v - u) \geqq (\varphi, v - u) \quad \forall v \in K$$

has a unique solution.

Let $u_n \in K$, $n = 1, 2, \ldots$, be such that

$$(2.8) \qquad B(u_n; u_{n+1}, v - u_{n+1}) \geqq (\varphi, v - u_{n+1}) \quad \forall v \in K\,.$$

Then

$$\lim_{n\to\infty} u_n = u$$

holds in the space H.

Proof. The existence and uniqueness of a solution of the problem (2.7) is easy to verify.

Let us introduce the notation

$$(2.9)\qquad \pi_n(v) = \Phi(u_n) - (\varphi, v) + \tfrac{1}{2} B(u_n; v, v) - \tfrac{1}{2} B(u_n; u_n, u_n) .$$

By virtue of (2.5) we may write

$$(2.9')\qquad \pi_n(u_{n+1}) = \Phi(u_n) - (\varphi, u_{n+1}) + \tfrac{1}{2} B(u_n; u_{n+1}, u_{n+1}) - \\ - \tfrac{1}{2} B(u_n; u_n, u_n) - \Phi(u_{n+1}) + \Phi(u_{n+1}) \geqq \\ \geqq \Phi(u_{n+1}) - (\varphi, u_{n+1}) \overset{\text{df}}{=} \psi(u_{n+1}) .$$

We have defined

$$(2.10)\qquad \psi(v) = \Phi(v) - (\varphi, v) .$$

Using (2.8) we obtain

$$(2.11)\qquad \tfrac{1}{2} B(u_n; u_{n+1}, u_{n+1}) - (\varphi, u_{n+1}) \leqq \tfrac{1}{2} B(u_n; v, v) - (\varphi, v) \quad \forall v \in K ,$$

consequently,

$$(2.12)\qquad \pi_n(u_n) = \psi(u_n) \geqq \pi_n(u_{n+1}) .$$

From (2.9') and (2.12) it follows that

$$(2.13)\qquad \psi(u_n) \geqq \psi(u_{n+1}) .$$

Assumption (2.1) yields the coerciveness of ψ:

$$(2.14)\qquad \psi(v) \geqq c_3 \|v\|^2 - c_4 \quad \forall v \in H .$$

Therefore, using (2.13) and (2.14) we obtain

$$\lim_{n \to \infty} \psi(u_n) = c < - \infty .$$

We have

$$(2.15)\qquad c_1 \|u_{n+1} - u_n\|^2 \leqq B(u_n; u_{n+1} - u_n, u_{n+1} - u_n) = \\ = B(u_n; u_n, u_n) + B(u_n; u_{n+1}, u_{n+1} - 2u_n) ;$$

on the other hand, we may write

$$(2.16)\qquad 2\,\psi(u_n) - 2\,\pi_n(u_{n+1}) \geqq B(u_n; u_n, u_n) - B(u_n; u_{n+1}, u_{n+1}) + \\ + 2\,B(u_n; u_{n+1} - u_n) ,$$

using (2.8). Therefore (2.16) and (2.15) yield that

$$2\,\psi(u_n) - 2\,\pi_n(u_{n+1}) \geqq B(u_n; u_n, u_n) + B(u_n; u_{n+1}, u_{n+1} - 2u_n) \geqq c_1 \|u_{n+1} - u_n\|^2 .$$

Using (2.12), (2.13), (2.9′) and the convergence of $\psi(u_n)$, we obtain

$$(2.16')\qquad \lim_{n \to \infty} \|u_{n+1} - u_n\| = 0 .$$

Moreover, we have

$$(2.17) \qquad \tfrac{1}{2}m\|u_n - u\|^2 \leqq D\,\Phi(u_n, u_n - u) - D\,\Phi(u, u_n - u) =$$
$$= B(u_n; u_n, u_n - u) - D\,\Phi(u, u_n - u) \leqq B(u_n; u_n, u_n - u) + (\varphi, u - u_n)\,,$$

by virtue of (2.7).

We also may write

$$(2.18) \qquad B(u_n; u_n, u_n - u) + (\varphi, u - u_n) = B(u_n; u_n - u_{n+1}, u_n - u) +$$
$$+ B(u_n; u_{n+1}, u_n - u) + (\varphi, u - u_n) = B(u_n; u_n - u_{n+1}, u_n - u) +$$
$$+ B(u_n; u_{n+1}, u_n - u_{n+1}) + (\varphi, u_{n+1} - u_n) +$$
$$+ B(u_n; u_{n+1}, u_{n+1} - u) + (\varphi, u - u_{n+1}) \leqq$$
$$\leqq B(u_n; u_n - u_{n+1}, u_n - u) + B(u_n; u_{n+1}, u_n - u_{n+1}) + (\varphi, u_{n+1} - u_n)$$

according to (2.8).

Using (2.18), (2.17), the boundedness of u_n, (2.3) and (2.16'), we obtain $u_n \to u$.
Q.E.D.

Moreover, let us consider the s e m i - c o e r c i v e c a s e, corresponding to the original problem and $\Gamma_u = \emptyset$.

Let $P \subsetneq H$ be a subspace of H such that $\dim P < \infty$. Let $H = P \oplus Q$ be the orthogonal decomposition and assume that

$$\Phi(v)\,, \quad D\,\Phi(v, h)\,, \quad D^2\,\Phi(v, h, k) \quad \text{and} \quad B(u; x, y)$$

are independent of an addition of $p \in P$ in all variables: for example, $\Phi(v + p) = \Phi(v)\ \forall p \in P$, etc.

Assume that the only element $p \in P \cap K$ such that also $-p \in P \cap K$ is $p = 0$. Let $\varphi \in H$ be such that

$$(2.19) \qquad (\varphi, p) < 0 \quad \forall p \in P \cap K \dotminus \{0\}\,.$$

Assume that K is a closed convex cone with the vertex at the origin.

Lemma 2.1. *Let the conditions* (2.1), (2.2), (2.3) *be fulfilled for elements* h, x, $y \in Q$ *and let* (2.4), (2.19) *hold.*

Then the functionals $\psi(v)$ *and*

$$\omega(x) = \tfrac{1}{2}\,B(v; x, x) - (\varphi, x)$$

are coercive, weakly lower semi-continuous in K. *Consequently, solutions of the variational inequalities* (2.7), (2.8) *exist.*

If e.g. $\bar{u}$ *and* u *are two solutions of* (2.7), *then* $\bar{u} = u + p$, *where* $p \in P$, $u + p \in K$, $(\varphi, p) = 0$. *Each such* $\bar{u}$ *represents a solution of* (2.7). *A parallel assertion holds for solutions of* (2.8).

Proof. To prove the coerciveness of ψ, it is sufficient to show that positive constants c_5, c_6 exist such that

$$\psi(v) \geqq c_5\|v\| - c_6 \quad \forall v \in K . \tag{2.21}$$

The latter inequality is equivalent to the following

$$\liminf_{\substack{\|v\| \to \infty \\ v \in K}} \frac{\psi(v)}{\|v\|} \geqq c_5 > 0 . \tag{2.22}$$

Assume that (2.22) is false. Then there exist $v_n \in K$ such that for $n \to \infty$

$$\|v_n\| \to \infty , \quad \lim \frac{\psi(v_n)}{\|v_n\|} = c_7 \leqq 0 .$$

From (2.1) (for $h \in Q$) and (2.4) we obtain

$$\Phi(v) \geqq c_8\|\Pi_Q v\|^2 - c_9 , \tag{2.23}$$

where Π_Q stands for the projector of H onto Q and $c_8 > 0$. Consequently, we have

$$\psi(v) \geqq c_8\|\Pi_Q v\|^2 - c_9 - (\varphi, v) . \tag{2.24}$$

Setting $v_n' = v_n/\|v_n\|$, we may write

$$\frac{\psi(v_n)}{\|v_n\|} \geqq c_8\|v_n\| \, \|\Pi_Q v_n'\|^2 - \frac{c_9}{\|v_n\|} - (\varphi, v_n') .$$

Therefore it must hold that

$$\|\Pi_Q v_n'\| \to 0 .$$

We can assume that $v_n' \to v'$ in H and therefore $v' \in P \cap K$, $\|v'\| = 1$. We thus obtain

$$\lim \frac{\psi(v_n)}{\|v_n\|} \geqq -(\varphi, v') > 0$$

in accordance with (2.19), which is a contradiction. Consequently, (2.21) is valid.

The rest of the existence proof is easy, since ψ is convex in virtue of the assumptions on Φ.

Let u and $\bar{u}$ be two solutions of (2.7). Then

$$D\,\psi(u, \bar{u} - u) \geqq 0 , \quad D\,\psi(\bar{u}, u - \bar{u}) \geqq 0 ,$$

$$m\|\Pi_Q\bar{u} - \Pi_Q u\|^2 \leqq D\,\Phi(\bar{u}, \bar{u} - u) - D\,\Phi(u, \bar{u} - u) \leqq 0 .$$

Consequently, $\bar{u} - u \in P$. Let us denote $\bar{u} - u = p$. We have

$$D\,\Phi(u, p) - (\varphi, p) \geqq 0 , \quad -D\,\Phi(u, p) + (\varphi, p) \geqq 0 ;$$

since $D\,\Phi(u, p) = 0$ by assumption, we are led to the conclusion that $(\varphi, p) = 0$.

It is easy to verify that $u + p$ is a solution, provided u is a solution and $p \in P$ fulfils $u + p \in K$, $(\varphi, p) = 0$.

The analysis of the problem (2.8) could be carried out in a parallel way. Q.E.D.

Theorem 2.2. *Let the assumptions of Lemma 2.1 be fulfilled. Moreover, let (2.5) hold and for any $h, k \in H$,*

$$w_n \rightharpoonup w \Rightarrow B(w_n; h, k) \to B(w; h, k) . \tag{2.25}$$

Let K be a closed convex cone with the vertex at the origin, let u_n and u be as in Theorem 2.1.

Then

$$\Pi_Q u_n \to \Pi_Q u$$

and if $\lim_{k\to\infty} u_{n_k} \rightharpoonup v$, then v is a solution of (2.7); we have $\|u_n\| \leqq c < \infty$.

Proof. As previously, we deduce that positive constants c_{10}, c_{11} exist such that

$$\tfrac{1}{2} B(v; w, w) - (\varphi, w) \geqq c_{10}\|w\| - c_{11} , \quad \forall w \in K \tag{2.26}$$

holds uniformly with respect to v and

$$\psi(w) \geqq c_{10}\|w\| - c_{11} . \tag{2.27}$$

Hence Lemma 2.1 implies the existence of a sequence $\{u_n\}$. There exists a constant c_{12} such that $\|u_n\| \leqq c_{12}$ $\forall n$. Indeed, this is a consequence of (2.8), if we insert $v = 0$ and use (2.2), (2.26).

Now the proof follows the same lines as the proof of Theorem 2.1 till (2.15), where we obtain

$$c_1\|\Pi_Q u_{n+1} - \Pi_Q u_n\|^2 \leqq B(u_n; u_n, u_n) + B(u_n; u_{n+1}, u_{n+1} - 2u_n) . \tag{2.28}$$

Consequently,

$$\|\Pi_Q u_{n+1} - \Pi_Q u_n\| \to 0 . \tag{2.28'}$$

Next, we may write

$$\begin{aligned} \tfrac{1}{2}m\|\Pi_Q u_n - \Pi_Q u\|^2 &\leqq D\,\Phi(u_n, u_n - u) - D\,\Phi(u, u_n - u) = \\ &= B(u_n; u_n, u_n - u) - D\,\Phi(u, u_n - u) = D\,\Phi(u, u_{n+1} - u_n) + \\ &\quad + B(u_n; u_{n+1}, u_{n+1} - u) + B(u_n; u_{n+1}, u_n - u_{n+1}) + \\ &\quad + B(u_n; u_n - u_{n+1}, u_n - u) - D\,\Phi(u, u_{n+1} - u) \leqq \\ &\leqq D\,\Phi(u, u_{n+1} - u_n) + B(u_n; u_{n+1}, u_{n+1} - u) + \\ &\quad + B(u_n; u_{n+1}, u_n - u_{n+1}) + B(u_n; u_n - u_{n+1}, u_n - u) + \end{aligned} \tag{2.29}$$

$$+ (\varphi, u - u_{n+1}) \leqq D\,\Phi(u, u_{n+1} - u_n) + B(u_n; u_{n+1}, u_n - u_{n+1}) +$$
$$+ B(u_n; u_n - u_{n+1}, u_n - u) \to 0\,,$$

where we have used the boundedness of $\|u_n\|$ and the convergence (2.28').

Suppose now that a subsequence $u_{n_k} \to v$. Then we have for all $w \in K$

$$B(u_{n_k-1}; u_{n_k}, w - u_{n_k}) \geqq (\varphi, w - u_{n_k})\,,$$

consequently,

$$B(u_{n_k-1}; v, w - v) \geqq (\varphi, w - v) + \varepsilon_{n_k}(w)\,,$$

where

$$\varepsilon_{n_k}(w) \to 0\,.$$

By the assumption (2.25) and using (2.28') we obtain

$$B(u_{n_k-1}; v, w - v) = B(u_{n_k} + u_{n_k-1} - u_{n_k}; v, w - v) =$$
$$= B(u_{n_k} + \Pi_Q u_{n_k-1} - \Pi_Q u_{n_k}; v, w - v) \to B(v; v, w - v) =$$
$$= D\,\Phi(v, w - v)\,.$$

Q.E.D.

3. APPLICATION TO AN ELASTO-PLASTIC BODY

We assume that the function μ is continuously differentiable in $[0, \infty)$ and satisfies the following conditions

$$0 < \mu_0 \leqq \mu(\gamma) \leqq \tfrac{3}{2}k\,, \tag{3.1}$$

$$0 < \alpha \leqq \mu(\gamma) + 2\gamma \frac{\mathrm{d}\mu}{\mathrm{d}\gamma}(\gamma) \leqq \beta < \infty\,. \tag{3.2}$$

Then the inequalities (2.1) till (2.4) and (2.25) in the sense of Theorems 2.1 and 2.2 are fulfilled. For the details see [1] – chapter 8 and 11.5. Obviously, we put

$$B(\mathbf{v}; \mathbf{w}, \mathbf{u}) = \int_\Omega [(k - \tfrac{2}{3}\,\mu(\gamma(\mathbf{v})))\,\vartheta(\mathbf{w})\,\vartheta(\mathbf{u}) + 2\,\mu(\gamma(\mathbf{v}))\,e_{ij}(\mathbf{w})\,e_{ij}(\mathbf{u})]\,\mathrm{d}x\,,$$

$$\Phi = \mathscr{L} \quad (\text{see } (1.6))\,,$$

$$(\varphi, \mathbf{v}) = \int_\Omega f_i v_i\,\mathrm{d}x + \int_{\Gamma_\tau} g_i v_i\,\mathrm{d}s\,,$$

$$P = \{\boldsymbol{p} \in [W^{1,2}(\Omega)]^3 \mid e_{ij}(\boldsymbol{p}) = 0 \text{ a.e.}\} = \{\boldsymbol{p} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}\}\,,$$

where $\boldsymbol{a}, \boldsymbol{b} \in R^3$ are arbitrary constant vectors.

Let us recall a result from [1] – 11·5 (see also the references in [1]).

Theorem 3.1. *If* $\mathrm{d}\mu/\mathrm{d}\gamma \leqq 0$, *then the condition* (5.2) *is fulfilled.*

Proof. Setting

$$M(\gamma) = \int_0^\gamma \mu(t)\,\mathrm{d}t\,,$$

the condition (2.5) takes the following form

$$\int_\Omega [\mu(\gamma(\boldsymbol{u}))\,(\gamma(\boldsymbol{v}) - \gamma(\boldsymbol{u})) - (M(\gamma(\boldsymbol{v})) - M(\gamma(\boldsymbol{u})))]\,\mathrm{d}x \geqq 0\,. \tag{3.3}$$

Consequently, (3.3) is satisfied if the function $M(\gamma)$ is concave. Q.E.D.

Remark 4.1. The above conclusion can be verified also in two-dimensional problems of elastoplastic bodies.

4. SOME FURTHER SEMICOERCIVE CASES

In the present section we consider two-dimensional problems and the cases when $\Gamma_u = \emptyset$ but on a part Γ_0 of the boundary of the domain $\Omega \subset R^2$ the conditions of the so called bilateral contact, i.e.

$$u_\nu = 0\,, \quad T_t = 0 \quad \text{on} \quad \Gamma_0\,, \tag{4.0}$$

are prescribed. The latter conditions hold for example on the axis of symmetry.

Then the space of virtual rigid displacements has the dimension one and we can formulate uniquely solvable original and approximate contact problems. Besides, we shall prove that the solutions of the approximate problems (2.8) converge to the solution of the original problem (2.7).

First of all we study the cases when the whole problem can be solved in a subspace $Q \subset H$. We start again with an abstract analysis.

4.1. Solution of the Signorini problem in a subspace

Let $P \hookrightarrow H$ be a subspace of a Hilbert space H, $H = P \oplus Q$ the orthogonal decomposition of H, $\dim P < \infty$.

Assume that $\Phi(v)$, $D\,\Phi(v, h)$, $D^2\,\Phi(v, h, k)$ and $B(u; x, y)$ are independent of an addition of $p \in P$ in all variables.

Let an element $\varphi \in Q$ and a convex cone K with its vertex at the origin be given such that

$$P \subset K\,. \tag{4.1}$$

Lemma 4.1. *Let the conditions* (2.1), (2.2) (2.3) *hold for* $h, x, y \in Q$ *and let* (2.4) *be satisfied.*

Then the functionals

$$\psi(v) \quad and \quad \omega(x) = \tfrac{1}{2} B(v; x, x) - (\varphi, x)$$

are coercive and weakly lower semicontinuous on Q.

There exists a unique solution $\hat{u} \in K \cap Q$ *and* $\hat{u}_{n+1} \in K \cap Q$ *of the inequality*

$$D\,\Phi(\hat{u}, v - \hat{u}) \geqq (\varphi, v - \hat{u}) \quad \forall v \in K \cap Q \tag{2.7'}$$

and

$$B(\hat{u}_n; \hat{u}_{n+1}, v - \hat{u}_{n+1}) \geqq (\varphi, v - \hat{u}_{n+1}) \quad \forall v \in K \cap Q\,, \tag{2.8'}$$

respectively. Any solution of (2.7) *and* (2.8) *can be written in the form*

$$u = \hat{u} + p \quad and \quad u_{n+1} = \hat{u}_{n+1} + p\,,$$

where $\hat{u}$ *and* $\hat{u}_{n+1}$ *are the solutions of* (2.7') *and* (2.8'), *respectively, and* $p \in P$.

If $\hat{u}$ *and* $\hat{u}_{n+1}$ *are solutions of* (2.7') *and* (2.8'), *respectively, then* $u = \hat{u} + p$ *and* $u_{n+1} = \hat{u}_{n+1} + p$, *where* p *is any element of* P, *represent solutions of* (2.7) *and* (2.8), *respectively.*

Proof. From (2.1) it follows that $D^2 \psi = D^2 \Phi$ is positive definite on Q and therefore ψ is coercive on Q. ψ is also strictly convex and differentiable, $K \cap Q$ convex and closed. Hence a unique solution of (2.7') exists.

Similar conclusions are valid for the functional $\omega(x)$, as follows from (2.2).

Since $\varphi \in Q$, we have

$$\psi(v) = \psi(v + p) \quad \forall p \in P\,. \tag{4.2}$$

The assumption (4.1) implies

$$K \cap Q = \Pi_Q(K)\,, \tag{4.3}$$

where Π_Q denotes the projector of H onto Q. In fact, let $v \in K$. Then

$$\Pi_Q v = v - \Pi_P v = v + (-\Pi_P v) \in K\,,$$

consequently $\Pi_Q(K) \subset K \cap Q$. The converse inclusion is obvious:

$$K \cap Q = \Pi_Q(K \cap Q) \subset \Pi_Q(K)\,.$$

Let u be a solution of (2.7). Using (4.2), we may write

$$\psi(\Pi_Q v) = \psi(\Pi_Q v + \Pi_P v) = \psi(v) \quad \forall v \in H\,.$$

Since $\Pi_Q u \in \Pi_Q(K) = K \cap Q$ and we have

$$\psi(\Pi_Q u) = \psi(u) \leqq \psi(v) = \psi(\Pi_Q v) \quad \forall v \in K\,,$$

$\Pi_Q u = \hat{u}$ is a solution of (2.7'), $u = \hat{u} + p$, $p \in P$.

In a parallel way we may prove that $\Pi_Q u_{n+1} = \hat{u}_{n+1}$ is a solution of (2.8′), hence $u_{n+1} = \hat{u}_{n+1} + p$, $p \in P$.

Let $\hat{u}$ be a solution of (2.7'). Then for $u = \hat{u} + p$, $p \in P$ we have

$$\psi(u) = \psi(\hat{u}) \leqq \psi(z) \quad \forall z \in K \cap Q . \tag{4.4}$$

Let $v \in K$. Then $\Pi_Q v \in \Pi_Q(K) = K \cap Q$ and

$$\psi(\Pi_Q v) = \psi(v) . \tag{4.5}$$

Combining (4.4) and (4.5) we obtain

$$\psi(u) \leqq \psi(v) \quad \forall v \in K .$$

By the assumption (4.1) $p \in P \subset K$, consequently $u = \hat{u} + p \in K$ and u is a solution of (2.7).

The same argument is applicable to the functional ω.

Theorem 4.1. *Let the assumptions of Lemma* 4.1 *be fulfilled. Moreover, let* (2.5) *hold and let for all* $h, k \in H$ *the condition* (2.25) *be satisfied.*

Denote by $\hat{u}$ *and* $\hat{u}_{n+1}$ *the solutions of* (2.7′) *and* (2.8'), *respectively. Then*

$$\lim_{n \to \infty} \hat{u}_n = \hat{u} .$$

Proof. By the assumption (2.2) we have

$$\tfrac{1}{2} B(v; w, w) - (\varphi, w) \geqq \tfrac{1}{2} c_1 \|w\|^2 - c_2 \|w\| \quad \forall v \in Q ,$$

with c_1, c_2 independent of v. Furthermore, we may write (by virtue of (2.1))

$$\psi(v) \geqq c_3 \|w\|^2 - c_4 \|w\| \quad \forall w \in Q .$$

Lemma 4.1 implies existence of a sequence $\hat{u}_n \in Q \cap K$ and $c_0 = \text{const}$ such that

$$\|u_n\| \leqq c_0 \quad \forall n .$$

The proof then proceeds like that of Theorem 2.1 with the only change — the space H is replaced everywhere by the subspace Q.

Application. Let $\Omega \subset R^2$ be a bounded domain with a Lipschitz boundary $\partial\Omega$ and let

$$\partial\Omega = \Gamma_0 \cup \Gamma_\tau \cup \Gamma_K \cup \Gamma_M ,$$

where Γ_0 and Γ_K have a positive length, whereas Γ_M has zero length. Let the conditions (4.0) hold on Γ_0. We define

$$K = \{\boldsymbol{u} \in [W^{1,2}(\Omega)]^2 \mid u_\nu = 0 \text{ on } \Gamma_0, u_\nu \leqq 0 \text{ on } \Gamma_K\} ,$$

$$H = V = \{v \in [W^{1,2}(\Omega)]^2 \mid v_\nu = 0 \text{ on } \Gamma_0\},$$
$$\mathscr{R} = \{\mathbf{v} \in [W^{1,2}(\Omega)]^2 \mid v_1 = a_1 - bx_2, v_2 = a_2 + bx_1\},$$

where a_1, a_2, b are arbitrary real constants;

$$P = \{\mathbf{p} \in \mathscr{R} \cap K \mid -\mathbf{p} \in \mathscr{R} \cap K\} = \{\mathbf{p} \in \mathscr{R} \mid p_\nu = 0 \text{ on } \Gamma_0 \cup \Gamma_K\}.$$

The same bilinear form B and the functional ψ will be chosen as in Section 3. Only the coefficient $(-2/3)$ has to be replaced by (-1) and $3k/2$ in the formula (3.1) by k.

Obviously, the condition $P \subset K$ is fulfilled. Assume that Γ_0 and Γ_K consist of straight segments parallel with the x_1-axis. Then

$$P = \{\mathbf{p} = (p_1, p_2) \mid p_1 = a_1 = \text{const}, p_2 = 0\};$$

$\varphi \in Q$ if and only if

$$V_1 \equiv \int_\Omega f_1 \, dx + \int_{\Gamma_\tau} g_1 \, ds = 0$$

holds for the resultant of the external forces.

The space V will be decomposed by means of some suitable inner product. We may choose for instance

$$(\mathbf{u}, \mathbf{v})_v = \int_\Omega e_{ij}(\mathbf{u}) \, e_{ij}(\mathbf{v}) \, dx + p(\mathbf{u}) \, p(\mathbf{v}),$$

$$p(\mathbf{v}) = \int_{\Gamma_1} v_1 \, ds, \; \Gamma_1 \subset \bar{\Omega}, \; \Gamma_1 \text{ has a positive length.}$$

Then

$$Q = V \ominus P = \{\mathbf{v} \in V \mid p(\mathbf{v}) = 0\}.$$

4.2. Solution of more general problems with unilateral contact

Lemma 4.2. *Let the assumptions of Lemma* 2.1 *be satisfied. Moreover, let us assume that*

(4.5) $$(\varphi, p) \neq 0 \quad \forall p \in P \dot{-} \{0\}.$$

Then there exist unique solutions of the problems (2.7) *and* (2.8).

Proof. Like at the beginning of the proof of Lemma 2.1 we derive for any two solutions $\bar{u}$ and u of the inequality (2.7) that

$$\bar{u} - u = p \in P, \quad (\varphi, p) = 0.$$

By means of (4.5) we conclude that $p = 0$ and there exists at most one solution. The argument for the inequality (2.8) is quite analogous.

The proof of coerciveness of ψ and ω on K follows the same lines as that of Lemma 2.1. Both functionals are convex and differentiable, hence they are weakly lower semicontinuous. Consequently, the solutions exist.

Theorem 4.2. *Let the assumptions of Lemma 4.2. be fulfilled. Moreover, let (2.5) and (2.25) hold. Denote by u and u_{n+1} the solutions of the problem (2.7) and (2.8), respectively.*

Then

$$\lim_{n \to \infty} u_n = u .$$

Proof. Following the proof of Theorem 2.2, we arrive at the conclusion (cf. (2.29)) that

$$\|\Pi_Q u_n - \Pi_Q u\| \to 0 , \quad n \to \infty . \tag{4.6}$$

Besides, we derive the boundedness of norms $\|u_n\|$. Hence a subsequence $\{u_m\}$ exists such that

$$u_m \rightharpoonup u^* \quad (\text{weakly in } H) , \quad m \to \infty. \tag{4.7}$$

Since K is weakly closed, $u^* \in K$. It follows from (4.7) that

$$\Pi_Q u_m \rightharpoonup \Pi_Q u^* . \tag{4.8}$$

On the other hand, from (4.6) we obtain that

$$\Pi_Q u_m \to \Pi_Q u ,$$

consequently, $\Pi_Q u^* = \Pi_Q u$ and the convergence (4.8) is even strong. Moreover, by virtue of (4.7), we have

$$\Pi_P u_m \to \Pi_P u^* \tag{4.9}$$

(the subspace P being finite-dimensional).

Combining (4.8) and (4.9) we obtain the convergence

$$\|u_m - u^*\| \to 0 .$$

In the end of the proof of Theorem 2.2, however, we have shown that the limit element u^* solves the inequality (2.7). The uniqueness of the solution implies $u^* = u$ and the whole sequence $\{u_n\}$ converges to u strongly.

Application. The assumption (4.5) can be satisfied if and only if $\dim (\mathcal{R} \cap V) = 1$. Indeed, let $H = V$, K, $\mathcal{R}$, B, ψ be defined as above, $P = \mathcal{R} \cap V$. If $\Gamma_0 = \emptyset$, then

$\mathscr{R} = \mathscr{R} \cap V$, dim $\mathscr{R} = 3$ and

$$(\varphi, \mathbf{p}) = a_1 V_1 + a_2 V_2 + bM \,, \quad \mathbf{p} \in \mathscr{R} \,,$$

where V_i are the components of the external forces resultant and M is the moment resultant. The condition (4.5) does not hold, since $(\varphi, \mathbf{p}) = 0$ for each vector $(a_1, a_2, b) \in R^3$ orthogonal to the vector (V_1, V_2, M). The case $\dim(\mathscr{R} \cap V) = 2$ is not possible.

Let Γ_0 consists of straight segments parallel with the x_1-axis. Then obviously

$$\mathscr{R} \cap V = \{\mathbf{p} \in \mathscr{R} \mid p_1 = a_1 = \text{const.}, p_2 = 0\} \,, \quad \dim(\mathscr{R} \cap V) = 1 \,.$$

The condition (4.5) is fulfilled if and only if the component V_1 of the force resultant does not vanish.

Next, let Γ_K be such that (see Fig. 1)

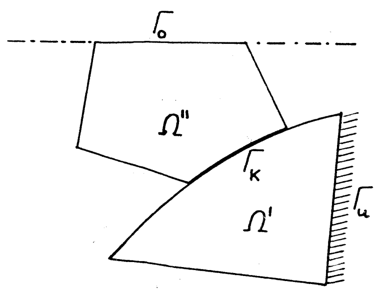

Fig. 1.

$$P \cap K = \mathscr{R} \cap K = \{\mathbf{p} \in \mathscr{R} \mid p_1 = a_1 \leqq 0, p_2 = 0\} \,.$$

Then the condition (2.19) is satisfied exactly if V_1 is positive.

Lemma 4.3. *Let* $P \neq \{0\}$, let *P be a subspace of H as in Section* 2, $\dim P < \infty$. *Let* (4.5) *and*

$$P \cap K = \{0\} \tag{4.10}$$

hold. Assume that (2.1), (2.2), (2.3) *hold for* $h, x, y \in Q$ *and* (2.4) *is valid.*

Then there exist unique solutions of (2.7) *and* (2.8).

Proof. The assumption (2.1) yields

$$\Phi(v) \geqq c_3 - c_4 \|v\| + c_5 \|\Pi_Q v\|^2 \quad \forall v \in H \,. \tag{4.11}$$

On the other hand, we have

$$\|\Pi_Q v\| \geqq c_6 \|v\| \quad \forall v \in K \,. \tag{4.12}$$

In fact, $\|\Pi_Q v\|$ is a seminorm in H such that the assumptions of Theorem 2.2 in [4] are satisfied.

By combining (4.11) and (4.12) the coerciveness of ψ on K follows. Since ψ is weakly lower semicontinuous, we obtain the existence of a solution of (2.7). The uniqueness is a consequence of (4.5) as in Lemma 4.2.

The argument for ω and (2.8) is analogous.

Theorem 4.3. *Let the assumptions of Lemma* 4.3 *and* (2.5), (2.25) *be satisfied. Denote by* u *and* u_{n+1} *the solutions of* (2.7) *and* (2.8), *respectively.*

Then

$$\lim_{n \to \infty} u_n = u .$$

Proof is the same as that of Theorem 4.2.

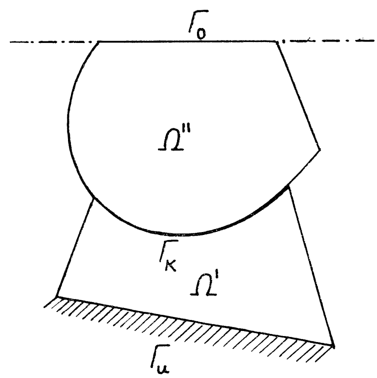

Fig. 2.

Applications. Define $H = V, K, \mathscr{R}, B, \psi$ and $P = \mathscr{R} \cap V$ as above. Let Γ_0 consist of straight segments parallel with the x_1-axis. Then the condition (4.10) is satisfied, if Γ_K has a proper form (see Fig. 2), i.e. if the component ν_1 of the normal changes the sign. The condition (4.5) is again equivalent to $V_1 \neq 0$.

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Souhrn

ŘEŠENÍ SIGNORINIHO KONTAKTNÍHO PROBLÉMU V DEFORMAČNÍ TEORII PLASTICITY METODOU SEČNÝCH MODULŮ

JINDŘICH NEČAS, IVAN HLAVÁČEK

Řeší se úloha jednostranného kontaktu mezi pružně plastickým tělesem a dokonale hladkou tuhou podporou v mezích tzv. deformační teorie plasticity. Řešení je formulováno pomocí variační nerovnice, ekvivalentní s principem minima potenciální energie. Metodou sečných modulů (Kačanova) je sestrojen iterační algoritmus, jehož každý krok odpovídá klasické Signoriniho úloze v teorii pružnosti. Dokazuje se konvergence této metody k přesnému řešení a studují se také některé úlohy, kdy existují přípustná pole posunutí tuhého tělesa.

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