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A NOTE ON THE OPTIMAL REPLACEMENT POLICY

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The present paper deals with a system with a single activated unit. We do not assume (as is usually done) that the unit is completely effective until it fails. We suppose that the unit can be in $k + 1$ states denoted by $0, 1, \ldots, k$ ($k \geqq 2$ and finite) at any time. A state i, $i \in \{0; 1; \ldots, k\}$ can be interpreted as the level of the wear of the unit. The states 0 and k correspond respectively to the full operative ability of the unit, and to a failure of the unit. Let us put $K = \{0; 1; \ldots; k\}$.

Let us suppose that inspections of the system are carried out at discrete time instants $t = 0, 1, 2, \ldots$ and that we have the possibility to replace the unit inspected at t by a new one, i.e. by a unit which is in state 0, at t, for every $t = 0, 1, 2, \ldots$.

We assume that the unit inspected at time 0, i.e. at the moment of the activation of the system, was in state 0 at time 0 and that the changes of states of a single unit form a time-homogeneous Markov chain with the state-space K and with the transition probabilities p_{ij} fulfilling

$$p_{ij} = 0 \quad \text{for all} \quad i \in K - \{k\}\,, \quad j \in K - \{i; i + 1; k\}\,, \tag{1}$$

$$p_{i,i+1} \neq 0 \quad \text{for all} \quad i \in K - \{k\}\,. \tag{2}$$

It means that if the unit is in state i at t and is used during $(t; t + 1)$, then its only possible states at $t + 1$ are i, $i + 1$, and k. If the unit fails during $(t; t + 1)$ between two successive inspections of the system, we must replace it at $t + 1$. On the other hand, if it does not fail during $(t; t + 1)$, then one of two possible actions (to replace or not to replace) can be taken at $t + 1$.

Let R be the costs for the replacement of a unit and let $m(i)$, for $i \in K - \{k\}$, be the expected income of the system achieved during the interval (say $(t; t + 1)$) between two successive inspections of the system under the condition that the unit used during this interval is in state i at t. We suppose that

$$\text{the sequence } \{m(i)\}_{i=0}^{k-1} \text{ is decreasing}\,. \tag{3}$$

In the infinite time horizon case the criterion of the quality of replacement strategies, which is widely used, is the value of the average income per unit time (AIUT). Its value equals the limit of the expected income of the system achieved till time n divided by n as n tends to infinity. It is well-known (see e.g. [1]) that the optimum within the class of the replacement strategies depending on the whole past of the system has the stationary property, i.e. the decisions according to the optimal replacement strategy are the same with all inspections. The assumption (1) implies that these decisions must have the form: to replace the unit inspected at t, $t = 0, 1, 2, \ldots,$ if and only if its state at t is equal to v or k, or equivalently if its state at t is $\geqq v$, for a properly chosen $v \in K$. We denote by $\mathscr{S}_v$, for all $v \in K$, the stationary replacement strategy corresponding to these decisions with the parameter v, and by C_v the AIUT under the condition that $\mathscr{S}_v$ is used. The paper [2] gives formulas for C_v for all $v \in K$ in terms of $m(i)$ and p_{ij} and introduces two algorithms for finding the value of

$$(4) \qquad k^* = \min \{w; C_w \geqq C_v \text{ for all } v \in K\}.$$

Theorem 2 of [2] implies, moreover, that

$$(5) \qquad C_{k^*} > C_v \quad \text{for all} \quad v \in K - \{k^*; k^* + 1\},$$

i.e. there are at most two stationary replacement strategies with the maximal AIUT.

On the other hand, if we know that we shall use the system only during a finite time interval, say $[0; n)$, then it is easy to see that the optimal replacement strategy, i.e. the strategy $\mathscr{S}^{(n)}$ with the greatest expected total income of the system (ETIS) achieved during $[0; n)$, need not be stationary. The optimal decision at t, for $t = 0, 1, 2, \ldots, n - 1$, may depend both on the state of the unit inspected at t and on t itself. The ETIS contains neither the rest value at time n of the unit used during $(n - 1; n)$ nor the costs for its prospective replacement at n. Thus we need not be interested in a good state of this unit at time n. It is evident e.g. that the optimal decision at time $n - 1$ is to replace the unit inspected at $n - 1$ if and only if its state at $n - 1 \geqq r$, where

$$(6) \qquad r = \min [\{i; i \in K - \{k\}, m(i) \leqq m(0) - R\} \cup \{k\}].$$

It is evident, however, that the optimal decision at every instant t, $t = 0, 1, \ldots, n - 1$, is to carry out the replacement at least if the inspected unit is in state $\geqq r$ at t because the cost for this replacement is payed completely during $(t; t + 1)$ and, moreover, when using a unit in a better state the ETIS with the optimal replacement strategy cannot be smaller. Concerning the infinite horizon case, we know according to Theorem 3 of [2] that $k^* \leqq r$. Thus we find that the optimal decision at time $n - 1$ described above is the smallest reasonable interference with the development of the system.

On the ground of these facts we can be led to the supposition that only the distance from the end of the period of the exploitation of the system is essential for the determination of the optimal decision. Namely, we may expect that the farther we are from the end of this period, the greater interferences with the development of the system (i.e. the more frequent replacements of units) are prescribed by the optimal replacement strategy. Let $t \in N \cup \{0\}$, $n \in N$ be such that $t < n$ and let $A_t^{(n)} \subset K$ be such that the decision at t corresponding to $\mathscr{S}^{(n)}$ is to replace the unit inspected at t if and only if its state at t belongs to the set $A_t^{(n)}$. Our supposition is that

(7) the sequence $\{A_t^{(n)}\}_{n=t+1}^{\infty}$ is non-decreasing for every $t \in N \cup \{0\}$.

If the interval of exploitation of the system is long enough, then the optimal decisions at its beginning are close to those corresponding to the infinite horizon case, i.e. to the decisions corresponding to the stationary replacement strategy $\mathscr{S}_{k^*}$ ($\mathscr{S}_{k^*}$ or $\mathscr{S}_{k^*+1}$ if $C_{k^*} = C_{k^*+1}$). Thus for every $n \in N$ and $t \in N \cup \{0\}$ such that $t < n$ we obtain the relation

$$A_t^{(n)} \subset \{k^*; k^* + 1; \ldots; k\}. \tag{8}$$

Let us mention that the suppositions (7) and (8) agree with the experience from the every-day life that the care given to a device becomes greater when the period of its exploitation becomes longer.

The following example shows that our conclusions (7) and (8) are not valid.

Example. Let

$$k = 3, \quad R = 1,$$

$$m(0) = 1, \quad m(1) = 0{\cdot}34, \quad m(2) = 0{\cdot}33,$$

$$p_{00} = 0{\cdot}260, \quad p_{01} = 0{\cdot}735, \quad p_{03} = 0{\cdot}005,$$

$$p_{11} = 0{\cdot}250, \quad p_{12} = 0{\cdot}740, \quad p_{13} = 0{\cdot}010,$$

$$p_{22} = 0{\cdot}015, \quad p_{23} = 0{\cdot}985,$$

and let the system be used during the interval $[0; 4)$, i.e. $n = 4$. The values of C_v, $v \in \{0; 1; 2; 3\}$ are

$$C_0 = 0,$$

$$C_1 = 0{\cdot}26,$$

$$C_2 = 0{\cdot}2995959596,$$

$$C_3 = 0{\cdot}3078370399,$$

so that $k^* = 3$ and using (8) we have

$$A_t^{(4)} = \{3\} \quad \text{for every} \quad t \in \{0; 1; 2; 3\}.$$

In this way (8) implies that the optimal replacement strategy $\mathscr{S}^{(4)}$ prescribes no replacement during $[0; 4)$. The ETIS achieved during $[0; 4)$ with the strategy $\mathscr{S}^{(4)}$ equals 2·0538437.

On the other hand, if we control the development of the system according to the stationary replacement strategy $\mathscr{S}_2$, then the ETIS achieved during $[0; 4)$ is equal to 2·0565632, so that it is greater than that with the strategy $\mathscr{S}^{(4)}$ which was supposed to be optimal with respect to the interval $[0; 4)$ of the exploitation of the system. It is obvious, however, that even the better replacement strategy $\mathscr{S}_2$ is not optimal with respect to the interval $[0; 4)$ because we know that the optimal decision at time 3 is "no replacement".

Let us now try to explain why the supposition (8) does not hold. It is easy to see that the optimal replacement strategy depends also on the initial state of the system at time 0. This dependence did not have to be emphasized above because of the assumption that the unit inspected at time 0 was in state 0 at time 0. It was, however, not taken into account properly when we deduced that the optimal decision with every inspection is fully determined by its time distance from the end of the period of the exploitation of the system, i.e. that

$$A_t^{(n)} = A_0^{(n-t)} \quad \text{for every} \quad n \in N\,, \quad t \in N \cup \{0\}\,, \quad t < n\,. \tag{9}$$

The state of the unit inspected at time 1 can be 0, 1 or k with positive probabilities p_{00}, p_{01} and p_{0k}, respectively. The corresponding distribution is denoted by D. There are generally three different replacement strategies $\mathscr{S}_0^{(n)} = \mathscr{S}^{(n)}$, $\mathscr{S}_1^{(n)}$, and $\mathscr{S}_k^{(n)}$ optimal, respectively, with respect to these three possibilities. It is obvious that the replacement strategy $\mathscr{S}_D^{(n)}$ optimal with respect to the distribution D of the initial state of the system may be different from $\mathscr{S}^{(n)}$. The strategy $\mathscr{S}^{(n+1)}$ can be expressed as the optimal decision at time 0 and the decisions corresponding to the strategy $\mathscr{S}_D^{(n)}$ applied one unit of time later. Thus it may happen that there exist $t \in N \cup \{0\}$ and $n \in N$ such that $t < n$ and

$$A_t^{(n)} \neq A_{t+1}^{(n+1)}\,. \tag{10}$$

This evidently contradicts the supposition (9).

It is worth-mentioning that the example given above was also used in [3] for showing that a better initial state of the system does not guarantee for every $n \in N$ a greater ETIS achieved during $[0; n)$ if the strategy $\mathscr{S}_{k^*}$ is used. The paper [4] generalizes the results of [2]. It considers the case that the inspected unit can be replaced by a unit which is in a better state, i.e. not necessarily in state 0 as supposed in the present paper.

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Souhrn

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POZNÁMKA O OPTIMÁLNÍ ZAMĚŇOVACÍ STRATEGII

V článku je uvažován systém s jedním prvkem, který může být v konečně mnoha stavech. Inspekce prvku jsou prováděny v diskrétních časových okamžicích, ve kterých je ho možné vyměnit za nový, tj. za prvek, který je v nejlepším z uvažovaných stavů. Uživatel systému se rozhoduje o podmínkách výměn prvků se snahou maximalizovat svůj zisk, který však nezahrnuje zůstatkovou hodnotu prvků. Prostřednictvím numerického příkladu se ukazuje, že aktivita uživatele systému nemusí obecně být tím větší, čím delší je doba jeho exploatace.

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