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LOCALLY AND UNIFORMLY BEST ESTIMATORS IN REPLICATED REGRESSION MODEL

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1. INTRODUCTION

Consider a linear regression model $(Y, X\beta, \Sigma)$ with an unknown k-dimensional parameter β and covariance matrix Σ. The aim is to estimate a function $\gamma = \operatorname{tr}(D\beta\beta') + \operatorname{tr}(C\Sigma)$, where D and C are symmetric $k \times k$ and $n \times n$ known matrices, respectively. Let us suppose that Y is normally distributed, $Y \sim N_n(X\beta, \Sigma)$, and that there are m independent replications of an experiment, i.e.

$$Y_i = X\beta + \varepsilon_i , \quad i = 1, \ldots, m , \quad E(\varepsilon_i) = 0 , \quad E(\varepsilon_i \varepsilon_j') = \delta_{ij}\Sigma ,$$

$$\delta_{ij} = \begin{cases} 0, & i \neq j , \\ 1, & i = j \end{cases}$$

which, written as $Y = (Y_1', \ldots, Y_m')'$, follow a model

$$Y = (\mathbf{1} \otimes X)\,\beta + \varepsilon , \quad E(\varepsilon) = 0 , \quad E(\varepsilon\varepsilon') = I \otimes \Sigma ,$$

where $\mathbf{1} = (1, \ldots, 1)'$.

This model offers the well known estimators

$$\bar{Y} = \frac{1}{m} \sum_{i=1}^{m} Y_i , \quad \hat{\Sigma} = \frac{1}{m-1} \sum_{i=1}^{m} (Y_i - \bar{Y})(Y_i - \bar{Y})'$$

for $X\beta$ and Σ.

The paper gives method for locally and uniformly best estimators of γ based on $\bar{Y}$ and $\hat{\Sigma}$.

2. SOLUTION

Let $\mathscr{S}$ be the space of $mn \times mn$ symmetric matrices. The class of estimators for $\gamma = \operatorname{tr}(D\beta\beta') + \operatorname{tr}(C\Sigma)$ will be $\mathscr{A} = \{Y'AY : A \in \mathscr{S}\}$. Let $\mathscr{L} = \{M_m \otimes S_1 +$

$+ P_m \otimes S_2 : S_1, S_2$ symmetric $n \times n$ matrices$\}$, where P_m is the projection matrix onto the space generated by the vector $\mathbf{1} = \{1, \ldots, 1)'$ and M_m is the projection matrix onto its orthogonal complement. In view of Lemma 1 in Kleffe and Volaufová [2] the class of estimators

$$\bar{\mathscr{L}} = \{Y'(M_m \otimes S_1 + P_m \otimes S_2)\, Y : S_1, S_2 \text{ symmetric } n \times n \text{ matrices}\}$$

constitutes a complete class of estimators for γ in the following sense. The estimator $Y'AY$, $A \in \mathscr{S}$, has the same mean value and variance greater than or equal to those of the estimator $Y'A_1Y$, where A_1 is the projection of the matrix A on to the closed subspace $\mathscr{L}$ of the space $\mathscr{S}$.

Lemma 1. *The estimator $Y'(M_m \otimes S_1 + P_m \otimes S_2)\, Y$ is unbiased for*

$$\gamma = \operatorname{tr}(D\beta\beta') + \operatorname{tr}(C\Sigma) \quad \textit{iff} \quad mX'S_2X = D \quad \textit{and} \quad (m-1)\, S_1 + S_2 = C\,.$$

The proof immediately follows from the expression for the mean value of the estimator.

Remark 1. The matrix equation $mX'S_2X = D$ is consistent iff there exists a symmetric matrix U such that $D = X'UX$.

Theorem 1. a) *The locally minimum variance unbiased estimator* (LMVUE) *for* $\gamma_1 = \operatorname{tr}(D\beta\beta')$ at Σ_0 *and uniformly best with respect to β is*

$$\hat{\gamma}_1 = \frac{1}{m} \operatorname{tr}\{(X')^-_{m(\Sigma_0)}\, D[(X')^-_{m(\Sigma_0)}]'\, \hat{\Sigma}\} + \bar{Y}'(X')^-_{m(\Sigma_0)}\, D[(X')^-_{m(\Sigma_0)}]'\, \bar{Y}\,,$$

where

$$\hat{\Sigma} = \frac{1}{m-1} \sum_{j=1}^{m} (Y_j - \bar{Y})(Y_j - \bar{Y})'\,, \quad \bar{Y} = \frac{1}{m} \sum_{j=1}^{m} Y_j \quad \text{and} \quad (X')^-_{m(\Sigma_0)}$$

is the minimum Σ_0-seminorm g-inverse of the matrix X' (see [4]).

b) LMVUE *for* $\gamma_2 = \operatorname{tr}(C\Sigma)$ *at* β_0, Σ_0 *is*

$$\hat{\gamma}_2 = \operatorname{tr}(C\hat{\Sigma}) - \frac{1}{m} \operatorname{tr}[(C - P'_{T_0}CP_{T_0})\, \hat{\Sigma}] + (\bar{Y} - X\beta_0)'\,(C - P'_{T_0}CP_{T_0})\,(\bar{Y} - X\beta_0)\,,$$

where

$$T_0 = \Sigma_0 + XX' \quad \textit{and} \quad P_{T_0} = X(X'T_0^-X)^-\, X'T_0^-\,.$$

Proof. a) Let us consider the class

$$\mathscr{B} = \{M_m \otimes T_1 + P_m \otimes T_2 : (m-1)\, T_1 + T_2 = 0\,, \quad X'T_2X = 0\,, \ T_1, T_2$$

symmetric matrices$\}$.

The class $\bar{\mathscr{B}}$ of estimators of the form $Y'BY$, $B \in \mathscr{B}$, is the class of all unbiased estimators in $\bar{\mathscr{L}}$ of the function $\gamma(\beta, \Sigma) \equiv 0$. According to the fundamental lemma

(Rao [3], p. 257) it is sufficient to verify that the covariance of $\hat{\gamma}_1$ and $Y'BY$, $B \in \mathscr{B}$, at Σ_0, is equal to zero.

b) Let

$$\overline{\mathscr{M}} = \{(Y - (1 \otimes X)\beta_0)' B(Y - (1 \otimes X)\beta_0) : B \in \mathscr{B}\}.$$

$\overline{\mathscr{M}}$ constitutes the class of unbiased estimators of the function $\gamma(\beta, \Sigma) \equiv 0$.

Similarly as in a) the evaluation of the covariance of $\hat{\gamma}_2$ and an arbitrary estimator from $\overline{\mathscr{M}}$ at β_0, Σ_0 proves b).

Remark 2. According to the fundamental lemma the LMVUE for $\gamma = \operatorname{tr}(D\beta\beta') + \operatorname{tr}(C\Sigma)$ is the sum of the LMVUE for the term $\operatorname{tr}(D\beta\beta')$ and the LMVUE for the term $\operatorname{tr}(C\Sigma)$.

Remark 3. The estimator $\operatorname{tr}(C\hat{\Sigma})$ is LMVUE for $\gamma_2 = \operatorname{tr} C\Sigma$ at $\sum_0$ and uniformly best with respect to β iff $C = P'_{T_0}CP_{T_0}$, which is equivalent to the existence of a symmetric matrix Q such that $C = T_0^- XQX'T_0$.

Remark. 4. The LMVUE for Σ at β_0, Σ_0 is given by

$$\hat{\hat{\Sigma}} = \hat{\Sigma} - \frac{1}{m}(\hat{\Sigma} - P_{T_0}\hat{\Sigma}P'_{T_0}) + (\bar{Y} - X\beta_0)(\bar{Y} - X\beta_0)' -$$

$$- P_{T_0}(\bar{Y} - X\beta_0)(\bar{Y} - X\beta_0)' P'_{T_0}.$$

To avoid the dependence of the estimator $\hat{\gamma}_2$ from Theorem 1 on the unknown parameter β_0 the class of unbiased invariant estimators is considered.

Lemma 2. *The estimator* $Y'(M_m \otimes S_1 + P_m \otimes S_2) Y$, S_1, S_2 symmetric matrices is unbiased and invariant for $\gamma = \operatorname{tr} C\Sigma$ iff $(m - 1) S_1 + S_2 = C$, $S_2X = 0$.

Proof is obvious.

Theorem 2. *The locally minimum variance invariant unbiased estimator* (LMVIUE) *for* $\gamma = \operatorname{tr} C\Sigma$ *at* Σ_0 *is*

$$\hat{\gamma} = \operatorname{tr}((C - \frac{1}{m} M'_{T_0}CM_{T_0})\hat{\Sigma}) + \bar{Y}'M'_{T_0}CM_{T_0}\bar{Y}, \quad \textit{where} \quad M_{T_0} = I - P_{T_0}.$$

For the proof check the covariance of $\hat{\gamma}$ and the quadratic invariant unbiased estimator of zero $Y'BY$, $B \in \mathscr{B}_1$,

$$\mathscr{B}_I = \{M_m \otimes T_1 + P_m \otimes T_2 : (m - 1) T_1 + T_2 = 0, T_2X = 0\}.$$

Remark 5. It can be shown that the LMVIUE from Theorem 2 for $\gamma = \operatorname{tr} C\Sigma$ coincides with the MINQUE at Σ_0.

Remark 6. The LMVIUE for Σ at Σ_0 is

$$\hat{\Sigma}_I^* = \hat{\Sigma} - \frac{1}{m} M_{T_0} \hat{\Sigma} M'_{T_0} + M_{T_0} \overline{YY}' M'_{T_0} .$$

Theorem 3. *A necessary and sufficient condition for* tr $C\hat{\Sigma}$ *to be* LMVIUE *at* Σ_0 *for* $\gamma = \operatorname{tr} C\Sigma$ *is*

$$M\Sigma_0 C\Sigma_0 M = 0 , \quad \textit{where} \quad M = I - XX^+ .$$

Proof immediately follows from the expression for the covariance of $\operatorname{tr}(C\hat{\Sigma})$ with $Y'BY$, $B \in \mathscr{B}_I$, from the proof of Theorem 2.

Remark 7. A sufficient condition for $\operatorname{tr}(C\hat{\Sigma})$ to be LMVIUE at Σ_0 for $\gamma = \operatorname{tr}(C\Sigma)$ is $M'_{T_0}CM_{T_0} = 0$ (cf. Theorem 2). The condition $M'_{T_0}CM_{T_0} = 0$ implies $M\Sigma_0 C\Sigma_0 M = 0$ as follows. The relation $M'_{T_0}CM_{T_0} = 0$ implies the existence of some symmetric matrices R_1 and R_2 such that $C = P'_{T_0}R_1 + R_1 P_{T_0} + P'_{T_0}R_2 P_{T_0}$. Because of $P_{T_0}T_0 M$ matrices R_1 and R_2 such that $C = P'_{T_0}R_1 + R_1 P_{T_0} + P'_{T_0}R_2 P_{T_0}$. Because $P_{T_0}T_0 M = X(X'T_0^- X)^- X'T_0^- T_0 M = 0$ we have $0 = MT_0 CT_0 M = M\Sigma_0 C\Sigma_0 M$.

Theorem 4. *The uniformly minimum variance invariant unbiased estimator* (UMVIUE) *for* $\gamma = \operatorname{tr}(C\Sigma)$ *exists iff*

$$M(\Sigma C\Sigma - \Sigma M'_{T_0}CM_{T_0})\, M = 0$$

for all Σ.

Proof. The LMVIUE for $\gamma = \operatorname{tr}(C\Sigma)$ at Σ_0 is (cf. Theorem 2)

$$\hat{\gamma} = Y'\left(M_m \otimes \frac{1}{m-1}\left(C - \frac{1}{m} M'_{T_0}CM_{T_0}\right) + P_m \otimes \frac{1}{m} M'_{T_0}CM_{T_0}\right)Y .$$

Let $\zeta = Y'(M_m \otimes T_1 + P_m \otimes T_2)\, Y = Y'BY$ be an invariant unbiased zero estimator, i.e. $B \in \mathscr{B}_I$, then

$$\operatorname{cov}_\Sigma(\hat{\gamma}, \zeta) = 2 \operatorname{tr}\left[\left(C - \frac{1}{m}(M'_{T_0}CM_{T_0})\right)\Sigma T_1 \Sigma\right] + \frac{2}{m} \operatorname{tr}(M'_{T_0}CM_{T_0}\Sigma T_0 \Sigma) .$$

The estimator is UMVIUE iff $\operatorname{cov}_\Sigma(\hat{\gamma}, \zeta) = 0$ for all Σ p.s.d. and for all $B \in \mathscr{B}_I$. Because of $T_1 X = 0$, which implies $T_1 = MUM$ for a suitable symmetric matrix U, this means $\operatorname{cov}_\Sigma(\hat{\gamma}, \zeta) = \operatorname{tr}[(M\Sigma C\Sigma M - M\Sigma M'_{T_0}CM_{T_0}\Sigma M)\, U] = 0$ for all symmetric matrices U and for all Σ which is equivalent to $M(\Sigma C\Sigma - \Sigma M'_{T_0}CM_{T_0}\Sigma)\, M = 0$ for all Σ.

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References

[1] *Jürgen Kleffe:* C. R. Rao's MINQUE for replicated and multivariate observations. Lecture Notes in Statistics 2. Mathematical Statistics and Probability Theory. Proceedings Sixth International Conference. Wisla (Poland) 1978. Springer N. York, Heidelberg, Berlin 1979, 188—200.

[2] *Jürgen Kleffe* and *Júlia Volaufová:* Optimality of the sample variance-covariance matrix in repeated measurement designs. (Submitted to Sankhyā).

[3] *C. R. Rao:* Linear Statistical Inference and Its Applications. J. Wiley, N. York 1965.

[4] *C. R. Rao, S. K. Mitra:* Generalized Inverse of Matrices and Its Applications. J. Wiley, N. York 1971.

[5] *R. Thrum, J. Kleffe:* Inequalities for moments of quadratic forms with applications to a.s. convergence. Math. Operationsforsch. Statistics Ser. Statistics (in print).

Súhrn

LOKÁLNE A ROVNOMERNE NAJLEPŠIE ODHADY V OPAKOVANOM REGRESNOM MODELI

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V regresnom modeli $(Y, X\beta, \Sigma)$ s neznámym parametrom β a s neznámou kovariančnou maticou Σ sa má určiť odhad funkcie $\gamma = \operatorname{tr}(D\beta\beta') + \operatorname{tr}(C\Sigma)$, kde D a C sú známe matice. K dispozícii sú stochasticky nezávislé opakované realizácie $Y_1, \ldots$ $\ldots, Y_m$ náhodného vektora Y. Nevychýlenými odhadmi vektora $X\beta$ a matice Σ sú

$$\bar{Y} = \frac{1}{m} \sum_{i=1}^{m} Y_i \quad \text{a} \quad \hat{\Sigma} = \frac{1}{m-1} \sum_{i=1}^{m} (Y_i - \bar{Y})(Y_i - \bar{Y})' .$$

V práci sú uvedené lokálne a rovnomerne najlepšie nevychýlené odhady funkcie γ, ktoré sú založené na odhade $\bar{Y}$ a $\hat{\Sigma}$.

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