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ON ONE TYPE OF SIGNORINI PROBLEM WITHOUT FRICTION IN LINEAR THERMOELASTICITY

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1. INTRODUCTION

The study of the mechanism of motion of lithospheric plates along asthenosphere and their collision in the sense of new global tectonics leads in the first approximation (see [1]) to the study of thermoelastic displacements, strains and stresses at the contact between lithospheric plates and blocks and between them and asthenosphere, particularly in the area of plate collision (Fig. 1).

We shall assume that the collision model can be investigated from the point of view of thermo-elasticity. The problem leads to a coupled problem consisting of dynamic equations and an expanded equation of heat conduction ([1]).

The fundamental dynamic equations can be written as

$$\tau_{ij,j} + f_i = \varrho u_{i,tt}\,, \quad i = 1, 2, 3 \quad \text{in} \quad G(t) \tag{1.1}$$

(we adopt the convention on summation over repeated indices and notation $f_{i,j} = \partial f_i/\partial x_j$, $f_{i,t} = \partial f_i/\partial t$) with the stress tensor τ_{ij} defined by Duhamel-Neumann's law (the generalized Hook's law in thermo-elasticity)

$$\tau_{ij} = c_{ijkl} e_{kl}(\boldsymbol{u}) + \beta_{ij}(T - T_0)\,. \tag{1.2}$$

The second term $\beta_{ij}(T - T_0)$ represents thermal expansibility.

The expanded equation of heat conduction can be written in the form

$$\varrho \beta_{ij} T_0 e_{ij,t} + \varrho c_e T_{,t} = W + (\varkappa_{ij} T_{,j})_{,i} \quad \text{in} \quad G(t)\,. \tag{1.3}$$

The first term on the left hand side represents the deformation energy dissipated in the form of heat in the lithospheric plate. The equations (1.1)–(1.2) and (1.3) are coupled in the terms $(\beta_{ij}(T - T_0))_{,j}$ and $\varrho \beta_{ij} T_0 e_{ij,t}$.

In the present paper we shall adopt the following simplifications of the equations (1.1)–(1.3):

For reasons of numerical treatment we shall study only the 2-dimensional problem. We neglect a) the term $\varrho u_{i,tt}$, because we assume that the motion of the lithospheric plate during the period $\langle t_1, t_2 \rangle$ is uniform, b) the term $\varrho c_e T_{,t}$, because the heat conduction in the lithospheric plate is slow, so that during the period considered it is stationary, c) the term $\varrho \beta_{ij} T_0 e_{ij,t}$, because the variability of the sources in the lithosphere and asthenosphere, and of the body and surface forces in time is slow so that our geodynamic problem during a short (from the geological point of view) period in the first approximation can be approximated by the steady-state problem. d) We shall limit ourselves to obducting plates only.*)

We shall deal with the quasi-steady-state problem consisting of the equilibrium equation

(1.4) $$(c_{ijkl}\, e_{kl}(\boldsymbol{u}) + \beta_{ij}(T - T_0))_{,j} + f_i = 0$$

and of the heat conduction equation

(1.5) $$(\varkappa_{ij} T_{,j})_{,i} + W = 0\,.$$

The problem is indeed not coupled, because (1.5) does not contain $\boldsymbol{u}$, which makes it possible to solve (1.5) for T and then (1.4) in which the coupled term $(\beta_{ij}(T - T_0))_{,j}$ will correct the vector of the body forces f_i. Therefore we can consider the both problems separately.

Boundary conditions

So far we have discussed only the description of the behavior inside the blocks. The interaction between the colliding blocks and the environment is modelled by the boundary conditions for the displacement vector $\boldsymbol{u}$ and the temperature T. We consider the following three types of boundary conditions:

– On the Earth's surface Γ_τ the surface forces as well as the temperature are prescribed, i.e.

(1.6a, b) $$\tau_{ij} n_j = \bar{P}_{0i}\,, \quad T = \bar{T}_0 \quad \text{on} \quad \Gamma_\tau\,.$$

– The boundary Γ_α represents the contact between the colliding lithospheric plates and between the investigated (obducting) plate and the asthenosphere (see Fig. 1). The conditions of Signorini type describe the situation of friction-free contact of two bodies and the fact that heat propagates from the asthenosphere into the lithosphere:

(1.7a, b) $$u_n \leqq 0\,, \quad \tau_n \leqq 0\,, \quad u_n \tau_n = 0\,, \quad \tau_s = 0 \quad \text{on} \quad \Gamma_\alpha\,,$$
$$T \leqq T_2\,, \quad q \leqq 0\,, \quad (T - T_2)\, q = 0$$

where $q = \varkappa_{ij} T_{,i} n_j$ is the heat flow, $\boldsymbol{n}$ is the outer normal to the boundaty, $\boldsymbol{s}$ is the unit tangential vector.

*) In Fig. 1 the obducting plate is represented by the region G. On the other hand, the model problem discussed also describes a subducting lithospheric plate.

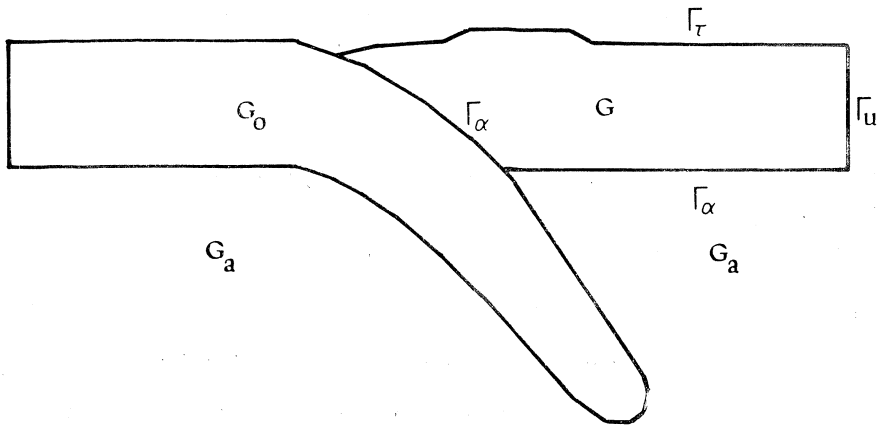

Fig. 1. Plate tectonic collision model: G — obducting lithospheric plate; G_0 — subducting lithospheric plate; G_a — asthenosphere; Γ_α — contact boundary.

– On the boundary Γ_u the displacement vector $\boldsymbol{u}$ and the temperature T are prescribed*), i.e.

(1.8a, b) $$\boldsymbol{u} = \bar{\boldsymbol{u}}_0\,, \quad T = \bar{T}_1 \quad \text{on} \quad \Gamma_u\,.$$

The aim of the paper is to suggest the mathematical analysis of the present geodynamical problem and to prove that this model problem, from the mathematical point of view, is correct. As was seen above these problems can be solved separately. The variational formulation of both problems will be given. The existence and unicity of the solution as well as the convergence of the finite element approximations to the exact solution are proved. The proofs are analogous to those of [3] and [4], therefore in such cases we only refer to them.

2. THE SIGNORINI PROBLEM IN ELASTICITY

As the problem solved is quasi-coupled, we can solve both problems separately. Let us start with the Signorini problem in elasticity.

Formulation of the problem

Let $G \subset R^2$ be a bounded plane region with Lipschitz boundary ∂G, occupied by an obducting plate at the moment $t = t_0$, $t \in \langle t_1, t_2 \rangle$ (see Fig. 1). Let $x = (x_1, x_2)$

*) The vector function $\boldsymbol{u}_0$ is derived from our knowledge of the motion of the lithospheric plate at the moment $t = t_0$. The vector function $\boldsymbol{P}_0$ describes the surface loads caused by the effect of the weight of the atmosphere, oceans, etc. The temperatures T_0 and T_1 describe the spreading of the temperature on the Earth's surface Γ_τ as well as the spreading of the temperature with depth on the boundary Γ_u, T_2 is the temperature of the asthenosphere.

be Cartesian coordinates. Let $\boldsymbol{n} = (n_1, n_2)$ denote the unit outward normal to the boundary ∂G. Let $\boldsymbol{u} = (u_i(x)) \in W^1 = [H^1(G)]^2$ be the displacement vector, let $e_{ij}(\boldsymbol{u})$ be the small strain tensor defined by

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) . \tag{2.1}$$

Let the stress-strain relation, the so-called Duhamel-Neumann's law (the generalized Hook's law), be defined by

$$\tau_{ij} = c_{ijkl}\, e_{kl}(\boldsymbol{u}) - \beta_{ij}(T - T_0) \tag{2.2}$$

(in particular, for isotropic media,

$$\tau_{ij} = \lambda e_{kk}(\boldsymbol{u})\, \delta_{ij} + 2\mu\, e_{ij}(\boldsymbol{u}) - (3\lambda + 2\mu)\, \alpha_t \delta_{ij}(T - T_0) , \tag{2.2a}$$

where α_t is a coefficient of linear thermal expansion), where $\tau_{ij} = \tau_{ij}(x)$ is a stress tensor, $T_0 = T_0(x)$ is the input temperature at which the materials of the plate are in a strainless and stressless state, $\beta_{ij}(x) \in C^1(\bar{G})$ a coefficient of thermal expansion, and let $c_{ijkl}(x) \in C^1(\bar{G})$ satisfy

$$c_{ijkl} = c_{jikl} = c_{klij} \tag{2.3}$$

and

$$c_{ijkl} e_{ij} e_{kl} \geqq c_0 e_{ij} e_{ij} , \quad c_0 = \text{const.} > 0 , \quad \forall e_{ij} = e_{ji} . \tag{2.4}$$

The stress tensor τ_{ij} satisfies the equilibrium conditions

$$\tau_{ij,j} + f_i = 0 \quad \text{in} \quad G , \tag{2.5}$$

where $\boldsymbol{f} \in [L^2(\bar{G})]^2$ is the vector of the body forces. Further, we define the stress vector $\boldsymbol{\tau}$ on the boundary ∂G by

$$\tau_i = \tau_{ij}(x)\, n_j$$

and its normal component

$$\tau_n = \tau_i n_i$$

and tangential component

$$\tau_s = \tau_i s_i ,$$

where $\boldsymbol{s} = (s_1, s_2) = (-n_2, n_1)$ is the unit tangential vector. We define the normal and tangential displacement components by

$$u_n = u_i n_i , \quad u_s = u_i s_i .$$

Let the boundary ∂G consist of several disjoint parts, $\partial G = \bar{\Gamma}_\tau \cup \bar{\Gamma}_u \cup \bar{\Gamma}_\alpha$. Let us assume that $\boldsymbol{f} \in [L^2(G)]^2$, $\boldsymbol{P}_0 \in [L^2(\Gamma_\tau)]^2$. According to [1] we have the following problem:

Find a vector function $\boldsymbol{u} \in [H^1(G)]^2$ satisfying

$$(c_{ijkl}\, e_{kl}(\boldsymbol{u}))_{,j} + F_i = 0 \quad i = 1, 2 \quad \text{in} \quad G , \tag{2.6$_0$}$$

where $F_i = f_i - (\beta_{ij}(T - T_0))_{,j} \in L_2(G)$, together with the following boundary conditions:

$$\tau_{ij} n_j = \bar{P}_{0i} \quad \text{on} \quad \Gamma_\tau , \tag{2.7$_0$}$$

$$u_n \leqq 0 , \quad \tau_n \leqq 0 , \quad u_n \tau_n = 0 , \quad \tau_s = 0 \quad \text{on} \quad \Gamma_\alpha , \tag{2.8$_0$}$$

$$u_i = \bar{u}_{0i} \quad \text{on} \quad \Gamma_u . \tag{2.9$_0$}$$

Remark. The coupling term satisfies the condition $(\beta_{ij}(T - T_0))_{,j} \in L^2(G)$. Really, if $\beta_{ij} \in C^1(\bar{G})$ and $T, T_0 \in H^1(G)$, then $\beta_{ij}(T - T_0) \in H^1(G)$ and then $(\beta_{ij}(T - T_0))_{,j} \in L^2(G)$.*)

Variational Formulation: Weak Solution

We shall transform the problem to one with homogeneous boundary conditions. In the equations let us replace $\boldsymbol{u}$ by $\boldsymbol{u} + \boldsymbol{w}$, where $\boldsymbol{w}$ is a sufficiently smooth vector function in $\bar{G} = G \cup \partial G$ satisfying (2.9$_0$) and $\boldsymbol{w} = 0$ on Γ_α. According to this transformation the boundary conditions, surface $(\boldsymbol{P}_0)$ and body $(\boldsymbol{F})$ forces will be changed.

Thus we obtain the following equivalent formulation of our problem (we will use the same symbol $\boldsymbol{u}$ and $\boldsymbol{F}$ as above):

$$(c_{ijkl} e_{kl})_{,j} + F_i = 0 , \quad i = 1, 2 \quad \text{in} \quad G , \tag{2.6}$$

where $F_i = f_i - (\beta_{ij}(T - T_0))_{,j} + (c_{ijkl}\, e_{kl}(\boldsymbol{w}))_{,j}$,

$$\tau_{ij} n_j = \bar{P}_i \quad \text{on} \quad \Gamma_\tau , \tag{2.7}$$

where $\bar{P}_i = \bar{P}_{0i} - c_{ijkl}\, e_{kl}(\boldsymbol{w})\, n_j$,

$$u_n \leqq 0 , \quad \tau_n \leqq 0 , \quad u_n \tau_n = 0 , \quad \tau_s = 0 \quad \text{on} \quad \Gamma_\alpha , \tag{2.8}$$

$$\boldsymbol{u} = 0 \quad \text{on} \quad \Gamma_u . \tag{2.9}$$

Let $\boldsymbol{F} \in [L^2(G)]^2$, $\boldsymbol{P} \in [L^2(\Gamma_\tau)]^2$. Let us define the space of virtual displacements as

$$V = \{\boldsymbol{v} \in W^1 \mid \boldsymbol{v} = 0 \text{ on } \Gamma_u\} \tag{2.10}$$

and the set of admissible virtual displacements as

$$K = \{\boldsymbol{v} \in V \mid v_n \leqq 0 \text{ on } \Gamma_\alpha\} . \tag{2.11}$$

Multiplying (2.6) by v_i, integrating over G, using the divergence theorem and boundary conditions, we obtain Euler's equation $\delta L = 0$ for the functional

$$L(\boldsymbol{v}) = \tfrac{1}{2} B(\boldsymbol{v}, \boldsymbol{v}) - S(\boldsymbol{v}) , \tag{2.12}$$

*) For our consideration also the conditions $c_{ijkl} \in L^\infty(G)$, $\varkappa_{ij} \in L^\infty(G)$, $\beta_{ij} \in L^\infty(G)$ can be used. Then $(\beta_{ij}(T - T_0)), j \in H^{-1}(G)$.

where $B(\boldsymbol{u}, \boldsymbol{v})$ is the bilinear form defined by

$$B(\boldsymbol{u}, \boldsymbol{v}) = \int_G c_{ijkl}\, e_{ij}(\boldsymbol{u})\, e_{kl}(\boldsymbol{v})\, \mathrm{d}G\,, \tag{2.13}$$

and

$$S(\boldsymbol{v}) = \int_G F_i v_i\, \mathrm{d}G + \int_{\Gamma_\tau} \bar{P}_i v_i\, \mathrm{d}S\,, \quad \boldsymbol{v} \in K\,. \tag{2.14}$$

We have obtained the variational formulation of our problem:

Find $\boldsymbol{u} \in K$ such that

$$L(\boldsymbol{u}) \leqq L(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in K\,. \tag{2.15}$$

Theorem 2.1. *A vector function $\boldsymbol{u} \in K$ satisfies (2.15) if and only if*

$$B(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) \geqq S(\boldsymbol{v} - \boldsymbol{u}) \quad \forall \boldsymbol{v} \in K\,.$$

Sketch of the proof. The set K is nonempty, closed and convex in V. The functional $L(\boldsymbol{v})$ is convex. For completing the proof, see [8].

Definition 2.1. *A function $\boldsymbol{u} \in K$ satisfying (2.15) will be called a weak solution of (2.6)–(2.9).*

Remark. It can be proved that any classical solution of our problem represented by the solution of (2.6)–(2.9) is a weak solution. On the other hand, if the weak solution is smooth enough, then it represents a classical solution of our problem.

We have the following result:

Theorem 2.2. *Let (2.3), (2.4) hold, then there exists a unique solution of the problem (2.15).*

Proof. The set K is closed and convex in W^1 hence it is weakly closed, as a closed (and convex) ball in a Hilbert space is weakly closed. It is known that if the functional L is coercive and weakly lower semicontinuous then it has a minimum in a Hilbert space. Let us verify these assumptions:

a) Coerciveness: From the assumption (2.4) we have

$$c_{ijkl}\, e_{ij}(\boldsymbol{u})\, e_{kl}(\boldsymbol{u}) \geqq \mu_0\, e_{ij}(\boldsymbol{u})\, e_{ij}(\boldsymbol{u})\,,$$

and using Korn's inequality,

$$L(\boldsymbol{v}) \geqq \tfrac{1}{2}\mu_0 \int_G e_{ij}(\boldsymbol{v})\, e_{ij}(\boldsymbol{v})\, \mathrm{d}G - \int_{\Gamma_\tau} \bar{P}_i v_i\, \mathrm{d}S - \int_G F_i v_i\, \mathrm{d}G \geqq c\|\boldsymbol{v}\|^2_{W^1} - c_1\|\boldsymbol{v}\|_{W^1} \quad \forall \boldsymbol{v} \in V\,.$$

b) weakly lower semi-continuity: For two points $v + u$ and u, using (2.4) and Korn's inequality we have

$$DL(v + u, v) - DL(u, v) = \int_G c_{ijkl}\, e_{ij}(v)\, e_{kl}(v)\, dG \geqq$$

$$\geqq \mu_0 \int_G e_{ij}(v)\, e_{ij}(v)\, dG \geqq \|v\|^2_{W^1} \quad \forall v \in V.$$

To prove the uniqueness we shall assume that there exist two weak solutions u_1 and u_2. Then $\tilde{u} = u_1 - u_2 \in K$ and

$$B(u_1, u_2 - u_1) \geqq S(u_2 - u_1), \quad B(u_2, u_1 - u_2) \geqq S(u_1 - u_2),$$

hence

$$B(u_2 - u_1, u_1 - u_2) \geqq 0,$$

so that

$$c\|\tilde{u}\|_{W^1} \leqq B(\tilde{u}, \tilde{u}) \leqq 0$$

and then $\tilde{u} = 0$, which completes the proof.

Numerical Solution

The finite element method (FEM) will be used for numerical solution. We will assume that the domain G is a bounded domain with a polygonal boundary ∂G. Let the domain G be "triangulated", i.e. $\bar{G} = G \cup \partial G$ is covered by a finite number of triangles T_h, forming a triangulation $\mathscr{T}_h$. We further assume that the end points $\bar{\Gamma}_u \cap \bar{\Gamma}_\tau, \bar{\Gamma}_u \cap \bar{\Gamma}_\alpha, \bar{\Gamma}_\tau \cap \bar{\Gamma}_\alpha$ coincide with the vertices of T_h. The family $\{\mathscr{T}_h\}$, $0 < h \leqq \leqq h_0$ of triangulations is assumed to be regular. Let V_h be the set of linear finite elements, i.e. the space of all continuous vector functions in $\bar{G}$ which are piecewise linear over $\mathscr{T}_h$. Let us define the set

$$K_h = \{v \mid v \in V_h,\ v_n \leqq 0 \text{ on } \Gamma_\alpha\}; \tag{2.16}$$

then $K_h \subset K$ for $\forall h$.

Definition 2.2. *A function* $u_h \in K_h$ *satisfying*

$$L(u_h) \leqq L(v) \quad \forall v \in K_h \tag{2.17}$$

is called a finite element approximation of (2.6)–(2.9).

Theorem 2.3. *There exists a unique finite element approximation* (2.17).

Proof. Let $L(u_h)$ be the functional defined by (2.12). As K_h is closed and convex, it is weakly closed. Further the proof is similar to that of Theorem 2.2.

Now our aim is to prove the convergence of the FEM approximation $\boldsymbol{u}_h$ to the exact solution $\boldsymbol{u}$, and to give an estimate of the rate of convergence of $\|\boldsymbol{u} - \boldsymbol{u}_h\|$. We shall use the following lemma.

Lemma 2.1. Let $F(v)$ be a functional defined on a closed convex subset M of a reflexive Banach space B. Assume that F is twice differentiable in B (in the Gateaux sense) and the second differential satisfies the inequalities

$$(2.18) \quad c_0\|z\|^2 \leqq D^2F(u; z, z) \leqq c\|z\|^2 \quad \forall u \in M\,, \quad \forall z \in B\,, \quad c_0 > 0\,, \quad c > 0\,,$$

i.e. it is positive definite and continuous. Let $M_h \subset M$ be a closed convex set. Let the minimizing elements of $F(v)$ over M and M_h be denoted by u and u_h, respectively. Assume that there $w_h \in M_h$ exists such that $2u - w_h \in M$. Then

$$(2.19) \qquad \|u - u_h\| \leqq (c/c_0)^{1/2} \|u - w_h\|\,.$$

Proof. see [4].

In our case $B \equiv V$, $M \equiv K$, $M_h \equiv K_h$. Then we find $\boldsymbol{w}_h \in K_h$ such that $2\boldsymbol{u} - \boldsymbol{w}_h \in K$ and $\boldsymbol{w}_h$ is sufficiently close to $\boldsymbol{u}$. Then the solution $\boldsymbol{u}_h$ is of the same order of accuracy as $\boldsymbol{w}_h$.

To prove the convergence of the finite element approximations we cannot a priori assume the solution $\boldsymbol{u}$ to be regular. To prove the convergence we shall need the following theorem.

Theorem 2.4. *Let us assume that there is only a finite number of "endpoints", $\bar{\Gamma}_\alpha \cap \bar{\Gamma}_\tau$, $\bar{\Gamma}_u \cap \bar{\Gamma}_\tau$, $\bar{\Gamma}_u \cap \bar{\Gamma}_\alpha$. Then the set $K \cap [C^\infty(\bar{G})]^2$ is dense in K.*

For the proof see [3].

Our aim now is to prove the convergence of finite element approximations without regularity of the solution $\boldsymbol{u}$.

Theorem 2.5. *Let V be a Hilbert space defined by (2.10), $K \subset V$ a convex closed subset defined by (2.11), $K_h \subset K$ a closed convex subset defined by (2.16). Let $L(\boldsymbol{v})$ be the functional defined on V by (2.12). Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ denote the minimizing elements of $L(\boldsymbol{v})$ over the sets K and K_h, respectively. Then*

$$(2.20) \qquad \lim_{h\to 0} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{W^1} = 0\,.$$

The proof with the aid of Theorem 2.4 is parallel to that given in [3].

To give an estimate of the rate of convergence of $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{W^1}$, we shall used Falk's technique discussed in [6].

Lemma 2.2. *For $\boldsymbol{u} \in K$, $\boldsymbol{u}_h \in K_h$ we have*

(2.21) $\|u - u_h\|_{W^1} \leqq C_0\{B(u_h - u, v_h - u) + B(u, v - u_h) + B(u, v_h - u) - - (F, v - u_h) - (F, v_h - u)\}^{1/2}, \quad \forall v \in K, \quad v_h \in K_h,$

where $C_0 > 0$ is a constant.

Proof. The proof follows from the conditions

$$B(u, v - u) - (F, v - u) \geqq 0 \qquad \forall v \in K,$$
$$B(u_h, v_h - u_h) - (F, v_h - u_h) \geqq 0 \quad \forall v_h \in K_h.$$

Adding these inequalities, adding and subtracting the terms $B(u, u_h) - B(u_h, u)$ to the resulting inequality and performing some modifications we obtain

$$\begin{aligned} B(u - u_h, u - u_h) &\leqq B(u, v - u_h) + B(u_h, v_h - u) + (F, u_h - v_h) + (F, u - v) = \\ &= B(u, v - u_h) + B(u, v_h - u) + B(u, v_h - u) + \\ &+ B(u_h, v_h - u) + (F, u_h - v) + (F, u - v) = \\ &= B(u_h - u, v_h - u) + B(u, v - u_h) + B(u, v_h - u) + \\ &+ (F, u_h - v) + (F, u - v_h). \end{aligned}$$

Then

$$C_1\|u - u_h\|^2_{W^1} \leqq B(u - u_h, u - u_h) \leqq B(u_h - u, v_h - u) + B(u, v - u_h) + + B(u, v_h - u) + (F, u_h - v) + (F, u - v_h).$$

This immediately implies our assertion (2.21). Q.E.D.

Corollary. *Let $K_h \subset K$. Then substituting $v = u_h$ in* (2.21) *we obtain*

(2.22) $\|u - u_h\|_{W^1} \leqq C_0\{B(u_h - u, v_h - u) + B(u, v_h - u) + (F, u - v_h)\}^{1/2} \quad \forall v_h \in K_h.$

As Γ_α is a polygonal boundary we can prove the following estimate:

Theorem 2.6. *Let Γ_α be polygonal. Let the solution u fulfil $u \in K \cap W^2$ and $u|_{\Gamma_\alpha} \in [H^2(\Gamma_\alpha)]$. Then*

(2.23) $$\|u - u_h\|_{W^1} = O(h).$$

Proof. Using Lemma 2.2 we estimate (2.21). This estimate can be applied provided the solution u is sufficiently regular. In (2.21) the two terms $B(u, v - u_h) - (F, v - u_h)$ and $B(u, v_h - u) - (F, v_h - u)$ are estimated by using Green's theorem and later by using a suitable choice of $v_h \in K_h$, $v \in K$. Then Green's theorem implies

$$B(u, v_h - u) - (F, v_h - u) = \int_G -(c_{ijkl}e_{kl})_{,j}(v_h - u)_i \, dG + \int_{\partial G} \tau_{ij}(u)\, n_j(v_h - u)_i \, dS -$$

$$- \int_{\Gamma_\tau} \bar{P}_i (v_h - u)_i \, dS - \int_G F_i (v_h - u)_i \, dG =$$

$$= \int_G \left(-(c_{ijkl} e_{kl})_{,j} - F_i \right) (v_h - u)_i \, dG + \int_{\Gamma_\alpha \cup \Gamma_\tau} \tau_{ij}(u) \, n_j (v_h - u)_i \, dS -$$

$$- \int_{\Gamma_\tau} \bar{P}_i (v_h - u)_i dS = \int_{\Gamma_\alpha} \tau_{ij}(u) \, n_j (v_h - u)_i \, dS = \int_{\Gamma_\alpha} \tau_n(u) \, (v_h - u)_n \, dS \leqq 0 \, .$$

Thus

$$-(c_{ijkl} e_{kl})_{,j} = F_i \, , \quad i = 1, 2, \quad \text{a.e. in } G \, .$$

In virtue of (2.22) and of this fact we have

(2.24)

$$\| u - u_h \|_{W^1} \leqq C_0 \left\{ B(u_h - u, \, v_h - u) + \int_{\partial G} \tau_{ij}(u) \, n_j (v_h - u)_i \, dS + \int_{\Gamma_\tau} \bar{P}_i (v_h - u)_i dS \right\}^{1/2} \leqq$$

$$\leqq C_0 \left\{ B(u_h - u, \, v_h - u) + \int_{\Gamma_\alpha} \tau_n(u) \, (v_{hn} - u_n) \, dS \right\}^{1/2} \leqq$$

$$\leqq C_0 \{ \| u_h - u \|_{W^1} \, \| v_h - u \|_{W^1} + C_2 \| v_{hn} - u_n \|_{[L_2(\Gamma_\alpha)]^2} \}^{1/2} \leqq$$

$$\leqq C_0 \{ \tfrac{1}{2} \varepsilon^{-1} \| v_h - u \|^2_{W^1} + C_1 \| v_{hn} - u_n \|_{[L_2(\Gamma_\alpha)]^2} + \tfrac{1}{2} \varepsilon \| u_h - u \|^2_{W^1} \}^{1/2} \, ,$$

where $\varepsilon > 0$ is an arbitrary given number. Now we must estimate the relevant norms in (2.24).

Let $v_h = u_{LI}$, where $u_{LI} \in V_h$ is the Lagrange interpolation of u on the triangulation $\mathcal{T}_h$. But $(u_{LI})_n \leqq 0$ on Γ_α so that $u_{LI} \in K$. Since $u_{LI} \in V_h$, then $u_{LI} \in K_h$. Hence

$$\| u_{LI} - u \|_{W^1} \leqq C_r h \| u \|_{W^2} \, , \tag{2.25}$$

$$\| (u_{LI})_n - u_n \|_{[L_2(\Gamma_\alpha)]^2} \leqq C_s h^2 \sum \| u_n \|_{[H^2(\Gamma_\alpha)]^2} \tag{2.26}$$

and

$$\| u - u_h \|_{W^1} \leqq C_0 \{ \tfrac{1}{2} \varepsilon \| u_h - u \|^2_{W^1} + \tfrac{1}{2} \varepsilon^{-1} \| v_h - u \|^2_{W^1} + C_2 \| v_{hn} - u_n \|_{[L_2(\Gamma_\alpha)]^2} \}^{1/2} \leqq$$

$$\leqq \bar{C}_0 \{ \varepsilon h \| u \|^2_{W^1} + \varepsilon^{-1} \, h^2 \| u \|^2_{W^1} + h^2 \sum \| u_n \|_{[H^2(\Gamma_\alpha)]^2} \}^{1/2} = O(h) \, ,$$

where $\bar{C}_0 = C_0 [\max (\tfrac{1}{2} C_1 C_r, \, \tfrac{1}{2} C_1 C_r, \, C_2 C_s)]$ and ε is chosen sufficiently small, which completes the proof.

3. THE SIGNORINI PROBLEM IN THERMICS

The studied problem (see Section 1) is quasi-coupled, so that both problems are solved separately. In this section we will discuss the problem in thermics only.

Formulation of the Problem

Let $G \subset R^2$ be the same plane region with a Lipschitz boundary ∂G, occupied by an obducting plate at the moment $t = t_0$, $t \in \langle t_1, t_2 \rangle$, and defined in Section 2. Let the boundary ∂G consist of several disjoint parts $\partial G = \bar{\Gamma}_\tau \cup \bar{\Gamma}_u \cup \bar{\Gamma}_\alpha$. Let $T \in H^1(G)$ be the temperature, $\varkappa_{ij} = \varkappa_{ij}(x) \in C^1(\bar{G})$ the thermal conductivity, $W = W(x) \in L^2(\bar{G})$ the heat sources in the lithospheric plate.

Then we solve the following problem:

Find a function $T \in H^1(G)$ that satisfies the equation

$$-(\varkappa_{ij}(x)\, T_{,j})_{,i} = W \quad \text{in} \quad G\,, \tag{3.1}$$

with the boundary conditions

$$T = \bar{T}_0 \qquad \text{on} \quad \Gamma_\tau\,, \tag{3.2}$$

$$T = \bar{T}_1 \qquad \text{on} \quad \Gamma_u\,, \tag{3.3}$$

$$T \leqq T_2\,, \quad q \leqq 0\,, \quad (T - T_2)\, q = 0 \quad \text{on} \quad \Gamma_\alpha\,, \tag{3.4}$$

where $\varkappa_{ij}(x) \in C^1(\bar{G})$, $W \in L^2(\bar{G})$ and

$$\exists c > 0\,, \quad c = \text{const.}\,, \quad \forall \xi \in R^2 \\ \varkappa_{ij}(x)\, \xi_i \xi_j \geqq c\|\xi\|^2 \quad \text{a.e.} \quad \text{in } G\,, \tag{3.5}$$

and where $\bar{T}_0$, $\bar{T}_1$, T_2 are the given functions on Γ_τ, Γ_u, and Γ_α, respectively, with the properties that $\bar{T}_0 = \bar{T}_1$ for $x \in \bar{\Gamma}_\tau \cap \bar{\Gamma}_u$, $\bar{T}_0 = T_2$ for $x \in \bar{\Gamma}_\tau \cap \bar{\Gamma}_\alpha$, $\bar{T}_1 = T_2$ for $x \in \bar{\Gamma}_u \cap \bar{\Gamma}_\alpha$ and $q = \varkappa_{ij} T_{,i} n_j$ is the heat flow.

Variational Formulation. Weak Solution

We shall transform the problem to a problem with homogeneous boundary conditions. Let us replace T by $T + z$, where z is a sufficiently smooth function satisfying (3.2), (3.3) and $z = 0$ on Γ_α. This transformation changes the functions in the boundary conditions. It can be shown that the heat sources are $Q = W + (\varkappa_{ij} z_{,j})_{,i} \in L^2(\bar{G})$.

Then we solve the following problem (the same symbol T will be used as above):

$$-(\varkappa_{ij}(x)\, T_{,j})_{,i} = Q \qquad \text{in} \quad G\,, \tag{3.6}$$

$$T = 0 \qquad \text{on} \quad \Gamma_\tau\,, \tag{3.7}$$

$$T = 0 \qquad \text{on} \quad \Gamma_u\,, \tag{3.8}$$

$$T \leqq T_2\,, \quad q \leqq 0\,, \quad (T - T_2)\, q = 0 \quad \text{on} \quad \Gamma_\alpha\,, \tag{3.9}$$

where $Q = W + (\varkappa_{ij} z_{,j})_{,i}$.

Let $Q \in L^2(\bar{G})$, $\varkappa_{ij}(x) \in C^1(\bar{G})$. Let us set

(3.10a, b) $$^1V = \{v \mid v \in H^1(G),\quad v = 0 \text{ on } \Gamma_u \cup \Gamma_\tau\},$$
$$^1K = \{v \mid v \in {}^1V,\ v \leqq T_2 \text{ on } \Gamma_\alpha\},$$

where we assume as above that the domain G is a polygonal one. We have

(3.11) $$L(T) = 1/2\, B(T, T) - S(T),$$

where

(3.12) $$B(T, v) = \int_G \varkappa_{ij}(x)\, T_{,j} v_{,i}\, \mathrm{d}G \quad \forall T, v \in {}^1K,$$

(3.13) $$S(v) = \int_G Qv\, \mathrm{d}G \quad \forall v \in {}^1K.$$

We will consider the following variational problem:

Find $T \in {}^1K$ such that

(3.14) $$L(T) = \min_{v \in {}^1K} L(v).$$

Theorem 2.7. *A function $T \in {}^1K$ is a variational solution of our problem if and only if*

$$B(T, v - T) \geqq S(v - T) \quad \forall v \in {}^1K.$$

Proof. The set ^1K is a nonempty, closed and convex subset of ^1V. The functional $L(T)$ is convex. To complete the proof, see [8].

Remark. It can be shown that any "classical" solution of (3.6)–(3.9) is also a weak solution. On the other hand, if the solution is smooth enough, then it is also a classical solution of our problem.

Theorem 2.8. *There exists a unique solution of the problem* (3.14).

Proof. The set ^1K is closed and convex in $H^1(G)$ so that it is weakly closed. We shall prove the coercivity and weakly lower-semicontinuity and the assumptions for the existence and uniqueness of the problem. Let us prove: a) Coercivity: as (3.5) holds we have $|B(v, v)| \geqq C\|v\|_1^2$, where $C > 0$. b) Weakly lower semi-continuity: For two points $v + T$ and v we have

$$DL(v + T, v) - DL(T, v) = \int_G \varkappa_{ij}(x)\, v_{,j} v_{,i}\, \mathrm{d}G \geqq k_0 \int_G v_{,j} v_{,i}\, \mathrm{d}G \geqq c_1 \|v\|_1^2.$$

To prove the uniqueness we will assume that there exist two weak solutions ^1T and ^2T. Then $\bar{T} = {}^1T - {}^2T \in {}^1V$ and

$$\int_G \varkappa_{ij}(x)\, \bar{T}_{,j} v_{,i}\, \mathrm{d}G \leqq 0.$$

For $v = \overline{T}$, by virtue positivity of $\varkappa_{ij}(x)$ and Friedrich's inequality, it follows that $\|\overline{T}\|_1 \leqq 0$, i.e. ${}^1T = {}^2T$, which completes the proof.

Numerical Solution

The finite element method will be used. Let $\mathcal{T}_h$ be a system of regular triangulations as above. Let ${}^1V_h$ be the space of linear finite elements, i.e. the space of all continuous functions in $\overline{G}$, which are piecewise linear over $\mathcal{T}_h$. We define

$$(3.15) \qquad {}^1K_h = \{v \mid v \in {}^1V_h,\ v \leqq T_2 \text{ on } \Gamma_\alpha\}\,.$$

Let ${}^1K_h \subset {}^1K$ for $\forall h$.

Definition 2.3. *Let ${}^1K_h$ be the set defined above. Let $T_h \in {}^1K_h$. Then T_h is a finite element approximation of our problem if*

$$(3.16) \qquad L(T_h) = \min_{v \in {}^1K_h} L(v)\,.$$

Theorem 2.9. *There exists a unique solution of the finite element approximation* (3.16).

Proof is analogous to that of Theorem 2.3.

Lemma 2.3. We have

$$(3.17) \quad \|T - T_h\|_1^2 \leqq C\{B(T_h - T,\ v_h - T) + B(T, v - T_h) + B(T, v_h - T) - \\ - (Q, v - T_h) - (Q, v_h - T)\}^{1/2} \quad \forall v \in {}^1K\,, \quad v_h \in {}^1K_h\,, \quad C = \text{const.} > 0\,.$$

Proof is analogous to that of Lemma 2.2.

To estimate the rate of convergence of $\|T - T_h\|$ we establish

Theorem 2.10. *Let $T_2 \in H^2(\Gamma_\alpha) \cap H^1(\Gamma_\alpha)$, $T \in {}^1K \cap H^2(G)$ and $T|_{\Gamma_\alpha} \in H^2(\Gamma_\alpha)$. Let ${}^1K_h \subset {}^1K$. Then*

$$(3.18) \qquad \|T - T_h\|_1 = O(h)\,.$$

Proof. Using Green's formula we have

$$(3.19) \quad B(T, v_h - T) - (Q, v_h - T) = \int_G (-\varkappa_{ij}(x)\, T_{,j})_{,i}\, (v_h - T)\, \mathrm{d}G + \\ + \int_{\Gamma_\alpha} T_{,n}(v_h - T)\, \mathrm{d}S - \int_G Q(v_h - T)\, \mathrm{d}G \geqq 0\,, \quad \forall v_h \in {}^1K_h\,,$$

where $T_{,n} = \varkappa_{ij}(x)\, T_{,j} n_i$, n_i are components of the unit outward normal to ∂G. Almost everywhere in G we have

$$(3.20) \qquad -(\varkappa_{ij}(x)\, T_{,j})_{,i} = Q\,.$$

According to (3.17), (3.19) and (3.20), we obtain after some modifications

(3.21)

$$\|T - T_h\|_1^2 \leqq C\left\{B(T_h - T, v_h - T) + \int_{\Gamma_\alpha} T_{,n}(v - T_h)\,dS + \int_{\Gamma_\alpha} T_{,n}(v_h - T)\,dS\right\}^{1/2} \leqq$$

$$\leqq C\left\{\|T_h - T\|_1\,\|v_h - T\|_1 + \int_{\Gamma_\alpha} T_{,n}(v - T_h)\,dS + C_2\|v_h - T\|_{L_2(\Gamma_\alpha)}\right\}^{1/2} \leqq$$

$$\leqq C\left\{\tfrac{1}{2}\varepsilon\|T - T_h\|_1^2 + \tfrac{1}{2}\varepsilon^{-1}\|T - v_h\|_1^2 + \int_{\Gamma_\alpha} T_{,n}(v - T_h)\,dS + \right.$$

$$\left. + C_2\|v_h - T\|_{L_2(\Gamma_\alpha)}\right\}^{1/2}, \quad \text{where} \quad \varepsilon > 0 \quad \text{is arbitrary}.$$

Let $v_h = T_{LI}$, $T_{LI} \in {}^1V_h$ be the Lagrange interpolation of T on the triangulation $\mathcal{T}_h$. As $(T_{LI})_n \leqq T_2$ on Γ_α, we have $T_{LI} \in {}^1K$ and as $T_{LI} \in {}^1V_h$, we also have $T_{LI} \in {}^1K_h$. Hence

$$\|T_{LI} - T\|_1 \leqq {}^1C_r\,h\|T\|_2\,, \tag{3.22}$$

$$\|(T_{LI})_n - T_n\|_{L_2(\Gamma_\alpha)} \leqq {}^1C_s h^2 \sum \|T_n\|_{H^2(\Gamma_\alpha)}\,. \tag{3.23}$$

To estimate the third member on the right hand side in the last inequality in (3.23) the technique of [7] will be used. We define

$$w = \sup(T_h, T_2) \quad \text{on} \quad \Gamma_\alpha\,,$$
$$w = 0 \qquad \text{on} \quad \partial G - \Gamma_\alpha\,.$$

Then $w \in H^1(\partial G)$, $w \leqq T_2$ on Γ_α and there exists a function $v \in H^1(G)$ such that $v = w$ on ∂G. Then $v \in {}^1K$, $T_h - w = 0$ for $T_h \leqq T_2$ and $T_h - w = T_h - T_2$ for $T_h > T_2$. Let s_i be the vertices of $\mathcal{T}_h$ on Γ_α. Then, as $T_h(s_i) \leqq T_2(s_i)$, $i = 1, \ldots, N$, we have $T_h \leqq (T_2)_{LI}$ on Γ_α, where $(T_2)_{LI}$ is the linear Lagrange interpolation of T_2 on Γ_α. Let

$${}^1\Gamma_\alpha = \{x \in \Gamma_\alpha \mid T_h(x) > T_2(x)\}\,.$$

Then

$$\int_{\Gamma_\alpha} (w - T_h)^2\,dS = \int_{{}^1\Gamma_\alpha} (T_2 - T_h)^2\,dS \leqq \int_{{}^1\Gamma_\alpha} ((T_2)_{LI} - T_2)^2\,dS = 0(h^4)\,,$$

as on ${}^1\Gamma_\alpha$, $0 < T_h(x) - T_2(x) \leqq ((T_2(x))_{LI} - T_2(x))$ holds. Hence

$$\|T - T_h\|_1 \leqq C_0\left\{\varepsilon/2\,\|T - T_h\|_1^2 + \tfrac{1}{2}\varepsilon^{-1}\|T - v_h\|_1^2 + \int_{\Gamma_\alpha} T_{,n}(v - T_h)\,dS + \right.$$

$$\left. + C_2\|v_h - T\|_{L_2(\Gamma_\alpha)}\right\}^{1/2} \leqq C_0\{\tfrac{1}{2}\varepsilon^1 C_{r1} h\|T\|_2 + \tfrac{1}{2}\varepsilon^{-1}\,{}^1C_{r2}\,h\|T\|_2 + O(h^4) +$$

$$+ C_2 {}^1C_s h^2 \sum \|T_n\|_{H^2(\Gamma_\alpha)}\} = O(h),$$

which completes the proof.

Theorem 2.11. *Let* $T_2 = 0$ *on* Γ_α, $T \in {}^1K \cap H^2(G)$, $T|_{\Gamma_\alpha} \in H^2(\Gamma_\alpha)$. *Then* $\|T - T_h\|_1 = O(h)$.

Proof is similar to that of Theorem 2.10.

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Souhrn

O JEDNOM TYPU SIGNORINIHO ÚLOHY BEZ TŘENÍ V LINEÁRNÍ TERMOELASTICITĚ

JIŘÍ NEDOMA

V článku je vyšetřována úloha Signoriniho typu v teorii termoelasticity pro případ ustáleného stavu. Úloha je modelovou úlohou z geodynamiky, jejíž fyzikální analysa je založena na hypotéze o tektonice litosferických desek a teorii termoelasticity.

Je diskutována existence a jednoznačnost řešení Signoriniho úlohy bez tření pro případ ustáleného stavu v teorii termoelasticity a její numerické řešení metodou konečných elementů. Je ukázáno, že konvergence přibližného řešení k přesnému je řádu $O(h)$, za předpokladu, že řešení je dostatečně regulární.

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