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# PROJECTIVE PLANE MOTIONS WITH STRAIGHT TRAJECTORIES

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Let $\bar{P}(P)$ be the projective plane over $\mathbf{R}$, realized as the set of all directions of a 3-dimensional vector space $\bar{V}(V)$, respectively. Let us fix a base $\bar{\mathscr{A}}_0 = \{\bar{\mathbf{f}}_1, \bar{\mathbf{f}}_2, \bar{\mathbf{f}}_3\}$ ($\mathscr{A}_0 = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$) in $\bar{V}(V)$. By a frame in $\bar{P}$ or $P$ we mean any base $\bar{\mathscr{A}}$ or $\mathscr{A}$ such that $\bar{\mathscr{A}} = \bar{\mathscr{A}}_0 \,.\, \gamma_2$, or $\mathscr{A} = \mathscr{A}_0 \,.\, \gamma_1$, where $\det \gamma_1 = \det \gamma_2 = 1$.

A projective plane motion is a one-parametric system of matrices $g(t)$, where $\det g(t) = 1$, and it represents a one-parametric system of maps from $\bar{P}$ into $P$ governed by the rule $g(t)\,(\bar{\mathscr{A}}_0) = \mathscr{A}_0\, g(t)$. A moving frame of a projective motion $g(t)$ is, by definition, any pair $(\bar{\mathscr{R}}(t), \mathscr{R}(t))$ of frames such that $g(t)\,(\bar{\mathscr{R}}(t)) = \mathscr{R}(t)$, where the image of any frame $\bar{\mathscr{A}} = \bar{\mathscr{A}}_0 \,.\, \gamma_2$ by a projective map $g$ is defined by the rule $g(\bar{\mathscr{A}}) = \mathscr{A}_0 g \gamma_2$.

Let further $(\bar{\mathscr{R}}(t), \mathscr{R}(t))$ be any moving frame of a projective plane motion $g(t)$. We define

$$\mathscr{R}' = \mathscr{R} \,.\, \varphi\,, \quad \bar{\mathscr{R}}' = \bar{\mathscr{R}} \,.\, \psi \tag{1}$$

and

$$\omega = \varphi - \psi\,, \quad \eta = \varphi + \psi\,.$$

Then $\operatorname{Tr} \varphi = \operatorname{Tr} \psi = 0$.

Let now $\bar{X} \in \bar{P}$ be a fixed point with coordinates $\mathscr{X}$ in $\bar{\mathscr{R}}$. Then $\bar{X}' = \lambda(t)\, \bar{X}$ and so for $\bar{X} = \bar{\mathscr{R}} \,.\, \mathscr{X}$ we get $\lambda \bar{\mathscr{R}} \mathscr{X} = \bar{\mathscr{R}}(\psi \mathscr{X} + \mathscr{X}')$ and hence $\mathscr{X}' = (\lambda E - \psi)\, \mathscr{X}$. The trajectory $X(t)$ of the point $\bar{X}$ is $X(t) = \mu(t)\, \mathscr{R}\mathscr{X}$. For the derivatives we get

$$X' = \mathscr{R}(\mu' \mathscr{X} + \mu \mathscr{X}' + \mu\varphi \mathscr{X}) = \mathscr{R}[\varphi\mu + \mu' E + \mu(-\psi + \lambda E)]\, \mathscr{X} =$$
$$= \mathscr{R}[(\mu' + \lambda)\, E + \mu\omega]\, \mathscr{X}\,.$$

Similarly

$$X'' = \mathscr{R}[\mu(\varphi\omega - \omega\psi + \omega') + (\tau + \mu' + \lambda\mu)\, \omega + (\lambda\tau + \tau)\, E]\, \mathscr{X}\,,$$

where $\tau = \mu' + \lambda$.

In what follows we shall investigate projective plane motions whose all trajectories are straight lines. Such motions must satisfy the condition

$$|X(t), X'(t), X''(t))| = 0 \quad \text{for all points } X \text{ and all} \quad t \in \mathbf{I} .$$

This condition is equivalent to the equation

$$(2) \qquad |\Omega_0 X, \Omega_1 X, \Omega_2 X| = 0 ,$$

where

$$(3) \qquad \Omega_0 = E , \quad \Omega_1 = \omega , \quad \Omega_2 = \varphi\omega - \omega\psi + \omega' ,$$

as we immediately see from the expression for $X'$ and $X''$.

If we change the moving frame of the motion by a matrix $h(t)$, we see that the matrices $\omega$ and $\eta$ will change to $\tilde{\omega}$ and $\tilde{\eta}$ according to the following rule: $\tilde{\omega} = h^{-1}\omega h$, $\tilde{\eta} = h^{-1}\eta h + 2h^{-1}h'$. From this expression we see that $\omega$ may be supposed to be in the real normal Jordan form. For each Jordan form of a $3 \times 3$ matrix we shall now solve the equation (2) taking into account the trace condition. Thus we arrive at

**Theorem 1.** *For each projective plane motion with straight line trajectories we can choose a moving frame in such a way that $\omega$ and $\eta$ will have one of the following forms*:

I $$\omega = \mathrm{Diag}\{\lambda_1, \lambda_2, \lambda_3\} , \quad \eta = 0 ;$$

II $$\omega = \mathrm{Diag}\{\lambda, \lambda, -2\lambda\} , \quad \eta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & 0 \end{pmatrix}$$

*where $b_{31}(t)$, $b_{32}(t)$ are arbitrary functions*;

III $$\omega = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & 0 & 0 \end{pmatrix}$$

*where $b_{31}(t)$ is an arbitrary function*;

IV $$\omega = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

V $$\omega = \begin{pmatrix} -2\sigma & 0 & 0 \\ 0 & \sigma & \gamma \\ 0 & -\gamma & \sigma \end{pmatrix}, \quad \eta = 0 ;$$

VI $$\omega = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \eta = \mathrm{Diag}\{0, 6, -6\} .$$

Proof. There are six possible normal Jordan forms of the matrix $\omega$ of $g(t)$ listed in the statement of Theorem 1. We have to solve the equation (2) in all these cases. As the computations are similar to one another, we shall present only two typical cases.

I $\omega = \mathrm{Diag}\{\lambda_1, \lambda_2, \lambda_3\}$, $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1$.

Denote $\eta = (b_{i,j})$, $i, j = 1, 2, 3$.
We can write $\Omega_1 = \mathrm{Diag}\{0, 1, \mu\}$, where $\mu = (\lambda_3 - \lambda_1)/(\lambda_2 - \lambda_1)$, because $\Omega_1 = 1/(\lambda_2 - \lambda_1)(\omega - \lambda_1 E)$ and (2) does not change by linear combinations. Using (3) we compute
(up to linear combinations of $E$ and $\Omega_1$)

$$\Omega_2 = \begin{pmatrix} 0 & b_{12} & \mu b_{13} \\ -b_{21} & 0 & (\mu - 1)\, b_{23} \\ -\mu b_{31} & (1 - \mu)\, b_{32} & \sigma \end{pmatrix}, \quad \text{where} \quad \sigma = 2\mu(\lambda_3 - \lambda_2) + 2\mu' .$$

The equation (2) will have the form

$$(2') \qquad \begin{vmatrix} x & 0 & b_{12}y + \mu b_{13}z \\ y & y & -b_{21}x + (\mu - 1)\, b_{23}z \\ z & \mu z & -\mu b_{31}x + (1 - \mu)\, b_{32}y + \sigma z \end{vmatrix} = 0 .$$

From (2′) we get the following conditions: $\mu \neq 0, 1$, $b_{i,j} = 0$ for $i \neq j$, $\sigma = 0$. Now we shall show that the matrix $\eta$ can be changed to $\tilde{\eta} = 0$ by a change of the moving frame.
The matrix $h$ of the transformation is

$h = \mathrm{Diag}\{\alpha, \beta, \gamma\}$, and so $\tilde{\eta} = h^{-1}\eta h + 2h^{-1}h' =$
$= \mathrm{Diag}\{b_{11} + 2\alpha^{-1}\alpha', b_{22} + 2\beta^{-1}\beta', b_{33} + 2\gamma^{-1}\gamma'\}$.

The resulting differential equations for $\alpha, \beta, \gamma$ have always a solution.

II $\omega = \mathrm{Diag}\{\lambda, \lambda, -2\lambda\}$.

Similarly as in I we have

$$\eta = (b_{i,j}),$$
$$\Omega_1 = \mathrm{Diag}\{0, 0, 1\},$$
$$\Omega_2 = \begin{pmatrix} 0 & 0 & b_{13} \\ 0 & 0 & b_{23} \\ -b_{31} & -b_{32} & 0 \end{pmatrix}.$$

The equation (2) has the form

$$(2^*) \qquad \begin{vmatrix} x & 0 & b_{13}z \\ y & 0 & b_{23}z \\ z & z & -b_{31}x - b_{32}z \end{vmatrix} = 0 .$$

From (2*) we get $b_{23} = b_{13} = 0$.

Now we change the matrix $\eta$ to

$$\tilde{\eta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b_{31} & b_{32} & 0 \end{pmatrix}.$$

The matrix of the transformation $h$ is

$$h = \begin{pmatrix} \alpha & \beta & 0 \\ \gamma & \delta & 0 \\ 0 & 0 & v \end{pmatrix}.$$

Let us denote

$$\varrho = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad C = (b_{31}, b_{32})$$

and similarly for $\tilde{B}$, $\tilde{C}$. Then

$$h = \begin{pmatrix} \varrho & 0 \\ 0 & v \end{pmatrix}, \quad \eta = \begin{pmatrix} B & 0 \\ C & b_{33} \end{pmatrix}, \quad \tilde{\eta} = \begin{pmatrix} \tilde{B} & 0 \\ \tilde{C} & \tilde{b}_{33} \end{pmatrix},$$

$$\tilde{\eta} = \begin{pmatrix} \tilde{\varrho} B \varrho & 0 \\ \varrho v^{-1} C & b_{33} \end{pmatrix} + 2 \begin{pmatrix} \tilde{\varrho}\varrho' & 0 \\ 0 & v^{-1}v' \end{pmatrix}.$$

The differential equations $\tilde{B} = \tilde{\varrho} B + 2\tilde{\varrho}\varrho' = 0$, $\tilde{b}_{33} = b_{33} + 2v^{-1}v'$ have always solutions for $\varrho$ and $v$.

**Theorem 2.** *Any projective plane motion with straight line trajectories is equivalent to one of the following motions (up to a parameter change):*

I $g(\mathrm{t}) = \mathrm{Diag}\,\{\mathrm{t}^2(\mathrm{t}^3 - C)^{-1/3}, \mathrm{t}^{-1}(\mathrm{t}^3 - C)^{-1/3}, \mathrm{t}^{-1}(\mathrm{t}^3 - C)^{2/3}\}$;

II
$$g(\mathrm{t}) = \begin{pmatrix} \mathrm{t} & 0 & 0 \\ 0 & \mathrm{t} & 0 \\ m(\mathrm{t}) & n(\mathrm{t}) & \mathrm{t}^{-2} \end{pmatrix},$$

where $m(\mathrm{t})$, $n(\mathrm{t})$ are arbitrary functions;

III
$$g(\mathrm{t}) = \begin{pmatrix} 1 & 0 & 0 \\ u(\mathrm{t}) & 1 & \mathrm{t} \\ 0 & 0 & 1 \end{pmatrix},$$

where $u(\mathrm{t})$ is an arbitrary function;

IV
$$g(\mathrm{t}) = \begin{pmatrix} 1 & \mathrm{t} & 0 \\ 0 & 1 & \mathrm{t} \\ 0 & 0 & 1 \end{pmatrix};$$

V
$$g(\mathrm{t}) = (\cos t)^{-1/3} \begin{pmatrix} \cos \mathrm{t} & 0 & 0 \\ 0 & \cos \mathrm{t} & -\sin \mathrm{t} \\ 0 & \sin \mathrm{t} & \cos \mathrm{t} \end{pmatrix};$$

VI
$$g(t) = \begin{pmatrix} t & -\frac{1}{3}t^{-2} & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-2} \end{pmatrix}.$$

Proof. We have to integrate the formulas (1).

(1) is a system of linear differential equations with variable coefficients. As the method of integration is similar for all cases I–VI we shall present detailed computations for the case II only.

The matrices $\varphi$, $\psi$ from (1) are

$$\varphi = \frac{1}{2}\begin{pmatrix} b_{11}+1 & b_{12} & b_{13} \\ b_{22} & b_{22}+1 & b_{23} \\ b_{31} & b_{32} & b_{33}-2 \end{pmatrix}, \quad \psi = \frac{1}{2}\begin{pmatrix} b_{11}-1 & b_{12} & b_{13} \\ b_{21} & b_{22}-1 & b_{23} \\ b_{31} & b_{32} & b_{33}+2 \end{pmatrix}$$

and we have two systems of differential equations

$$\begin{aligned} 2\boldsymbol{u}_1' &= \boldsymbol{u}_1 + b_{31}\boldsymbol{u}_3\,, & 2\bar{\boldsymbol{u}}_1' &= -\bar{\boldsymbol{u}}_1 + b_{31}\bar{\boldsymbol{u}}_3\,, \\ 2\boldsymbol{u}_2' &= \boldsymbol{u}_2 + b_{32}\boldsymbol{u}_3\,, & 2\bar{\boldsymbol{u}}_2' &= -\bar{\boldsymbol{u}}_2 + b_{32}\bar{\boldsymbol{u}}_3\,, \\ \boldsymbol{u}_3' &= -\boldsymbol{u}_3\,; & \bar{\boldsymbol{u}}_3' &= \bar{\boldsymbol{u}}_3\,. \end{aligned}$$

The solutions are

$$\begin{aligned} \boldsymbol{u}_1 &= e^{1/2t}\boldsymbol{K}_1 + \tfrac{1}{2}e^{1/2t}\boldsymbol{K}_3 I_1\,, \\ \boldsymbol{u}_2 &= e^{1/2t}\boldsymbol{K}_2 + \tfrac{1}{2}e^{1/2t}\boldsymbol{K}_3 I_2\,, \\ \boldsymbol{u}_3 &= e^{-t}\,.\,\boldsymbol{K}_3\,, \end{aligned}$$

where $\boldsymbol{K}_1$, $\boldsymbol{K}_2$, $\boldsymbol{K}_3$ are arbitrary constant vectors,

$$I_1 = \int e^{-3/2t}b_{31}\,dt\,, \quad I_2 = \int e^{-3/2t}b_{32}\,dt$$

and $b_{31}$, $b_{32}$ are arbitrary functions.

We have $\mathscr{R} = \mathscr{R}_0\,.\,\gamma_1$ for the bases $\mathscr{R}_0 = \{\boldsymbol{K}_1, \boldsymbol{K}_2, \boldsymbol{K}_3\}$, $\mathscr{R} = \{\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3\}$, where

$$\gamma_1 = \begin{pmatrix} e^{1/2t} & 0 & 0 \\ 0 & e^{1/2t} & 0 \\ \frac{1}{2}e^{1/2t}I_1 & \frac{1}{2}e^{1/2t}I_2 & e^{-t} \end{pmatrix}.$$

Similarly we get

$$\gamma_2 = \begin{pmatrix} e^{-1/2t} & 0 & 0 \\ 0 & e^{-1/2t} & 0 \\ \frac{1}{2}e^{-1/2t}\bar{I}_1 & \frac{1}{2}e^{-1/2t}\bar{I}_2 & e^{t} \end{pmatrix},$$

where $\bar{I}_1 = \int e^{3/2t}b_{31}\,dt$, $\bar{I}_2 = \int e^{3/2t}b_{32}\,dt$.

The projective plane motion in the case II has the expression

$$g(t) = \gamma_1\gamma_2^{-1} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & \\ \frac{1}{2}e^tI_1 - \frac{1}{2}e^{-2t}\bar{I}_1, & \frac{1}{2}e^tI_2 - \frac{1}{2}e^{-2t}\bar{I}_2, & e^{-2t} \end{pmatrix}.$$

Denote $m(t) = \frac{1}{2}e^t I_1 - \frac{1}{2}e^{-2t}\bar{I}_1$, $n(t) = \frac{1}{2}e^t I_2 - \frac{1}{2}e^{-2t}\bar{I}_2$.
Then the functions $m(t)$, $n(t)$ are again arbitrary functions.

Remark 1. From the expressions I–VI in Theorem 2 we see that every projective motion with straight line trajectories can be regarded as an affine motion after a suitable choice of the line at infinity. To show the geometrical meaning we give the corresponding affine expressions:

I
$$g(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & At + B \end{pmatrix}.$$

This is a centroaffine motion with two invariant directions. The points of the line $y = kx$ have parallel trajectories.

II
$$g(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m(t) & n(t) & t \end{pmatrix}.$$

The motion consists of collineations with a fixed center at infinity, the axis of the collineation changes. The trajectories are parallel.

III
$$g(t) = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & u(t) \\ 0 & 0 & 1 \end{pmatrix}.$$

The motion is equiaffine. Its properties are similar as in II, but the axes are parallel.

IV
$$g(t) = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & t & 1 \end{pmatrix}.$$

The motion is equiaffine. The trajectories of the points with $x =$ const. are parallel.

V
$$g(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -t \\ 0 & t & 1 \end{pmatrix}.$$

The motion is centroequiform.

VI a)
$$g(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 1 & t \end{pmatrix}.$$

The motion is centroaffine.

b)
$$g(t) = \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & t \end{pmatrix}.$$

The motion is affine. The trajectories of the points with $y$ = const. are parallel.

Remark 2. A projective motion has straight line trajectories if it satisfies the condition

$$\Omega_2 = \alpha\Omega_1 + \beta E, \tag{4}$$

where $\alpha$, $\beta$ are some functions of t. Let us find out conditions under which the motions I–VI satisfy (4). Computation shows that the motions I, IV, V, VI satisfy (4) without any restriction. In the case II we get $g(s)$ in the form

$$g(s) = \begin{pmatrix} s & 0 & 0 \\ 0 & s & 0 \\ sK_1 + s^{-2}K_2, & s\bar{K}_1 + s^{-2}\bar{K}_2, & s^{-2} \end{pmatrix}$$

For III we get

$$g(t) = \begin{pmatrix} 1 & 0 & 0 \\ Kt & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

## Reference


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## Souhrn


## PROJEKTIVNÍ ROVINNÉ POHYBY S PŘÍMKOVÝMI TRAJEKTORIEMI

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V článku jsou nalezeny všechny rovinné projektivní pohyby s přímkovými trajektoriemi a jejich maticová vyjádření. Je ukázáno, že každý takový pohyb patří do některé z podgrup projektivní grupy a je popsána jeho afinní nebo euklidovská realizace.



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