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INTERNAL FINITE ELEMENT APPROXIMATIONS IN THE DUAL
VARIATIONAL METHOD FOR SECOND ORDER ELLIPTIC
PROBLEMS WITH CURVED BOUNDARIES

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1. INTRODUCTION

Internal finite element approximations of the dual variational formulation for
second order elliptic boundary value problems in $\mathbb{R}^2$ have been restricted, to
the authors' knowledge, to domains with polygonal boundaries. It is the aim
of the present paper to extend the results to domains with piecewise smooth ($C^2$)
curved boundaries.

The space of divergence-free vector functions with vanishing normal flux on some
part of the boundary is approximated by (internal) subspaces of finite elements,
having the same properties. We also satisfy the requirement to save the order of
approximation which belongs to polygonal domains. Thus we construct the
so-called conforming dual finite element approximations. If also a conforming
primal approximation is available, one obtains a posteriori error estimates and
two-sided bounds of energy [4, 7, 12, 14].

Using the concept of stream functions [5], some a priori $L^2$-error estimates are
deduced, provided the exact solution is regular enough. We also prove the convergence
of the proposed method to a non-regular solution.

Let us introduce some notations. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz
boundary $\partial \Omega$ (see [12]). The outward unit normal to $\partial \Omega$ will be denoted by $n$. The
usual norm and semi-norm in the Cartesian product of the Sobolev spaces $(W^{k,p}(\Omega))^r$,
r = 1, 2, ..., are denoted by $\| \cdot \|_{k,p,\Omega}$ and $| \cdot |_{k,p,\Omega}$, respectively. We shall omit the sub-
script $p$ in the case $p = 2$ and we put $H^{k}(\Omega) = W^{k,2}(\Omega)$. Further, let $L^\infty(\Omega) =
= W^{0,\infty}(\Omega)$ and let the notation $(\cdot, \cdot)_{0,\partial}$ be used for the usual scalar product in the
space $(L^2(\Omega))^r = (H^0(\Omega))^r$. By $P_j(\Omega)$ we denote the space of polynomials of order
at most $j$ defined on $\Omega$. Let $C^k(\Omega)$ denote the space of continuous functions, the deri-
vatives of which up to the order $k$ are continuous in $\Omega$. 

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Next, we define the operator \( \text{curl} : H^1(\Omega) \to (L^2(\Omega))^2 \) by the relation

\[
\text{curl} \, v = \left( \frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1} \right)^T, \quad v \in H^1(\Omega).
\]

Let us emphasize that all statements will always hold only for a sufficiently small discretization parameter \( h \). Moreover, notations \( C, C_1, C_2, \ldots \) are reserved for the so-called generic constants.

Many boundary value problems of mathematical physics can be formulated in the following classical way: Find \( u \) such that

\[
\begin{align*}
-\text{div} (A \, \text{grad} \, u) &= f \quad \text{in} \quad \Omega, \\
u &= \bar{u} \quad \text{on} \quad \Gamma_u, \\
(A \, \text{grad} \, u)^T v &= g \quad \text{on} \quad \Gamma_g,
\end{align*}
\]

where \( \Gamma_u, \Gamma_g \) are disjoint and open in \( \partial \Omega \) (one of them can be empty),

\[
M_1 \cup \Gamma_u \cup \Gamma_g = \partial \Omega,
\]

and \( M_1 \) is a finite set of those points, where one type of the boundary condition changes into another. Further, \( f \in L^2(\Omega), \bar{u} \in H^1(\Omega), g \in L^2(\Gamma_g) \) and \( A \in (L^\infty(\Omega))^4 \) is supposed to be a symmetric and uniformly positive definite \( 2 \times 2 \) matrix. In the case \( \Gamma_u = \emptyset \), we moreover assume that

\[
\int_\Omega f \, dx + \int_{\partial \Omega} g \, ds = 0.
\]

Let us recall that the dual variational formulation of the problem (1.1) consists in finding \( \bar{p} \) which minimizes the functional

\[
J(q) = \frac{1}{2} b(q, q) - l(q)
\]

over the space

\[
Q = \{ q \in (L^2(\Omega))^2 \mid (q, \text{grad} \, v)_{0,\Omega} = 0 \quad \forall v \in V \},
\]

where

\[
V = \{ v \in H^1(\Omega) \mid v = 0 \quad \text{on} \quad \Gamma_u \},
\]

\[
b(q, q') = (A^{-1}q, q')_{0,\Omega}
\]

is a symmetric and \( Q \)-elliptic bilinear form and

\[
l(q) = b(\bar{p}, q) - (q, \text{grad} \, \bar{u})_{0,\Omega},
\]

where \( \bar{p} \in (L^2(\Omega))^2 \) satisfies the equation

\[
(\bar{p}, \text{grad} \, v)_{0,\Omega} = (f, v)_{0,\Omega} + \int_{\Gamma_g} g v \, ds \quad \forall v \in V.
\]

For other details see [7, 9, 16]. Let us note that if \( q \in Q \cap (H^1(\Omega))^2 \), then \( \text{div} \, q = 0 \) in \( \Omega \) and \( q^T v = 0 \) on \( \Gamma_g \). The space \( Q \) can be characterized also as follows. If the sets \( \Gamma_u \)
and $\Gamma_\psi$ are connected, then
\begin{equation}
Q = \text{curl } W,
\end{equation}
where
\begin{equation}
W = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_\psi \}.
\end{equation}
The general case is described in the following theorem (see [9], p. 59).

**Theorem 1.1.** Let $\partial \Omega_1, \ldots, \partial \Omega_H$ be the components of $\partial \Omega$ and let $m$ be the number of those components for which $\partial \Omega_i \cap \Gamma_\psi \neq \emptyset$. Let $n$ be the number of the components of $\Gamma_\psi$. Then, if $m \geq 2$ or $n \geq 2$, there exist functions $a^1, \ldots, a^{m-1}, \beta^1, \ldots, \beta^{n-1} \in \mathbb{E}(Q)$ such that
\begin{equation}
Q = \mathbb{L}(\text{curl } W \cup \{ a^1, \ldots, a^{m-1}, \beta^1, \ldots, \beta^{n-1} \}),
\end{equation}
where $Q$ and $W$ are defined by (1.4) and (1.8), respectively, and $\mathbb{L}$ denotes the linear hull.

The details on the functions $a^i, \beta^j$ will be presented in Remark 2.1.

In Chapters 2 and 4 some subspaces $Q_h$ of the space $Q$ will be constructed. A function $p_h$ minimizing the functional (1.3) over $Q_h$ will be called internal approximation of the solution of the dual problem. An algorithm for finding $p_h$ will be presented in Chapter 5.

## 2. PIECEWISE CONSTANT EQUILIBRIUM FINITE ELEMENT SPACES

In this chapter we introduce subspaces $Q_h \subset Q$ consisting of constant elements and we derive their approximation properties. We shall deal only with the problems from the class $\mathcal{C}^{(2)}$ according to the following definition.

**Definition.** A couple $(\Omega, \Gamma_\psi)$ is said to be from the class $\mathcal{C}^{(2)}$, if
1. $\Omega \subset \mathbb{R}^2$ is a bounded domain with a Lipschitz boundary, which consists of a finite number of arcs from the class $C^{(2)}$. The set of the end points of these arcs will be denoted by $M_2$.
2. The part $\Gamma_\psi$ of the boundary $\partial \Omega$ consists of a finite number of convex and concave arcs. The set of the end points of these arcs will be denoted by $M_3$.

An arc $\Gamma \subset \partial \Omega$ is said to be convex (concave), if there exists a convex domain $\Omega_0 \subset \Omega$ ($\Omega_0 \subset \mathbb{R}^2 - \overline{\Omega}$) such that $\Gamma \subset \partial \Omega_0$.

Note that for $(\Omega, \Gamma_\psi) \in \mathcal{C}^{(2)}$ the set
\begin{equation}
M = M_1 \cup M_2 \cup M_3
\end{equation}
is finite.

Let us describe the way of triangulation of a domain from the class $\mathcal{C}^{(2)}$. First we establish an approximation $\Gamma_{gh}$ of the part $\Gamma_\psi$, such that $\Gamma_{gh} \subset \overline{\Omega}$. Denote by
\( \Gamma_g^0, \ldots, \Gamma_g^{n-1} \) all components of \( \Gamma_g \). Every component (curve) \( \Gamma_g^i \) will be now approximated by a polygonal curve \( \Gamma_{gh}^i \subset \bar{\Omega} \) consisting of a finite number of line segments the length of which does not exceed \( h \). Each of those segments is a chord or a tangent of a convex or of a concave arc, respectively, which is contained in \( \Gamma_g^i \) (see Figs. 1 and 3). If \( \Gamma_g^i \) is a closed curve, we require \( \Gamma_{gh}^i \) to be also a closed curve. Moreover, we require that \( M_1 \cup M_3 \subset \bar{\Gamma}_{gh} \cap \bar{\Gamma}_g \), where

\[
\Gamma_{gh} = \bigcup_{i=0}^{n-1} \Gamma_{gh}^i .
\]

The subdomain of \( \Omega \), bounded by \( \Gamma_{gh} \) and \( \Gamma_{gh} \), will be denoted by \( \Omega_h \), and we define

\[
D_h = \Omega - \bar{\Omega}_h .
\]

Now \( \mathcal{T}_h \) will denote the triangulation of the domain \( \Omega_h \) generated in a standard way, assuming that the triangles adjacent to \( \Gamma_u \) may have at most one curved side (i.e. the inner triangles are “straight ones” only).

Furthermore, we shall always assume the validity of the so-called consistence condition of a triangulation, i.e. the interior of any side of any triangle \( K \in \mathcal{T}_h \) is disjoint with the set \( M \) (see (2.1)). Each segment from \( \Gamma_{gh} \cap \Gamma_g \) coincides with a side of one triangle \( K \).

Moreover, we assume that all triangulations belong to a regular family of triangulations \( \mathfrak{M} \). (A family of triangulations \( \mathfrak{M} \) is said to be regular if

(i) there exists a constant \( \varkappa > 0 \) such that for any \( \mathcal{T}_h \in \mathfrak{M} \) and any \( K \in \mathcal{T}_h \) there exists a circle \( B_K \) with a radius \( \varrho_K \) such that \( B_K \subset K \) and

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\[ (2.2) \quad x h_K \leq \ell_K, \]

where \( h_K = \text{diam} \, K \),

(ii) for any \( \varepsilon > 0 \) there exists \( \mathcal{T}_h \in \mathcal{M} \) such that

\[ h = \max_{K \in \mathcal{T}_h} h_K \leq \varepsilon. \]

Finally, we can define the space of equilibrium finite elements as follows:

\[ (2.3) \quad Q_h = \{ q \in Q \mid q|_{\partial h} = 0, \, q|_{K} \in (P_0(K))^2 \, \forall K \in \mathcal{T}_h \}. \]

We shall now examine the approximation properties of the space \( Q_h \subseteq Q \). If we introduce the space

\[ W_h = \{ v \in W \mid v|_{\partial h} = 0, \, v|_{K} \in P_1(K) \, \forall K \in \mathcal{T}_h \}, \]

then a linear approximation operator \( r_h : W \cap H^2(\Omega) \to W_h \) will be determined by the relations

\[ (r_h v)(x) = v(x) \]

for all nodal points \( x \) of the triangulation \( \mathcal{T}_h \) such that \( x \notin \Gamma_{gh} \). Note that \( r_h v = 0 \) on \( \Gamma_{gh} \) and the function \( r_h v \in W_h \) is therefore uniquely determined. We have the well-known lemma for \( \Gamma_{gh} = \emptyset \) (see e.g. [6], p. 41):

**Lemma 2.1.** Let \( (\Omega, \Gamma_{gh}) \in \mathcal{C}^{(2)} \) and let \( \Gamma_u = \partial \Omega \). Then

\[ \|v - r_h v\|_{1,\Omega} \leq C h \|v\|_{2,\Omega} \quad \forall v \in H^2(\Omega). \]

**Proof.** There exists (see [11], p. 80) a linear continuous extension operator \( E : H^2(\Omega) \to H^2(\mathbb{R}^2) \) such that \( Ev|_{\partial \Omega} = v \) and

\[ (2.4) \quad \|Ev\|_{2,\mathbb{R}^2} \leq C_1 \|v\|_{2,\Omega}. \]

Let \( \mathcal{T}_h \) be a subset of all curved triangles from \( \mathcal{T}_h \). Consider \( K \in \mathcal{T}_h \) with vertices \( A_1, A_2, A_3 \), where \( A_2A_3 \) lies on \( \partial \Omega \) (see Fig. 2), and let \( F(K) \) denote the straight
triangle with vertices $A_1, A_2, A_3$ such that $A_i, i = 2, 3,$ are midpoints of the segments $A_1A_i$. Since $\partial \Omega$ is piecewise from $C^{(2)}$, we get

$$\tag{2.5} K \subset F(K) \forall K \in \bar{\mathcal{T}}_h$$

for $h$ small enough. Let us put $F(K) = K$ for $K \in \mathcal{T}_h - \bar{\mathcal{T}}_h$ and for any triangle let us define the function $\tilde{r}_k Ev = P_i(F(K))$ by the relation

$$\tilde{r}_k Ev|_K = r_h v|_K.$$ 

Then (see [1])

$$\| Ev - \tilde{r}_k Ev\|_{1,F(K)} \leq C_2 h |Ev|_{2,F(K)} \forall K \in \mathcal{T}_h.$$ 

Hence by (2.4) and (2.5), we have

$$\| v - r_h v\|_{0,\Omega} = \sum_{k \in \mathcal{T}_h} \| v - r_h v\|_{1,k} \leq \sum_{k \in \mathcal{T}_h} \| Ev - \tilde{r}_k Ev\|_{1,F(K)} \leq C_2 h^2 \sum_{k \in \mathcal{T}_h} |Ev|_{2,F(K)}^2 \leq C_3 h^2 |Ev|_{2,R^2} \leq C_4 h^2 \| v\|_{0,\Omega}.$$ 

**Theorem 2.1.** Let $(\Omega, \Gamma_\alpha) \in \mathcal{C}^{(2)}$, $\Gamma_\alpha = \partial \Omega$ and let the domain $\Omega$ be simply connected. Then there exists a linear operator $R_h : Q \cap (H^1(\Omega))^2 \to Q_h$ such that

$$\| q - R_h q\|_{0,\Omega} \leq Ch \| q\|_{1,\Omega}.$$ 

**Proof.** Let $q \in Q \cap (H^1(\Omega))^2$ be arbitrary. Since $\Omega$ is simply connected, there exists $v \in H^1(\Omega)$ (see [5], p. 25, the so-called stream function) such that $(v, 1)_{0,\Omega} = 0$ and

$$\| q - R_h q\|_{0,\Omega} = \| q - \text{curl} v\|_{0,\Omega} = \| v - r_h v\|_{1,\Omega} \leq C_1 h \| v\|_{2,\Omega}.$$ 

In our case, however, we have even $v \in H^2(\Omega)$. Setting

$$R_h q = \text{curl} r_h v,$$

we see that $R_h q \in Q_h$ and by Lemma 2.1, (2.7) and (2.8) we obtain

$$\| q - R_h q\|_{0,\Omega} = \| q - \text{curl} (v - r_h v)\|_{0,\Omega} = \| v - r_h v\|_{1,\Omega} \leq C_1 h \| v\|_{2,\Omega}.$$ 

Using the Poincaré inequality

$$\| v\|_{1,\Omega} \leq C_2 |v|_{1,\Omega},$$ 

we get

$$\| v\|_{2,\Omega} = \| v\|_{1,\Omega}^2 + |v|_{2,\Omega} \leq (C_2^2 + 1) (\| v\|_{1,\Omega}^2 + |v|_{2,\Omega}) = (C_2^2 + 1) \| q\|_{1,\Omega}.$$ 

Combining this relation and (2.9), we arrive at the assertion of the lemma. 

The case $\Gamma_\alpha = \partial \Omega$ for a multiply connected domain will be considered later in Theorem 2.3.

Now we shall deal with the case $\Gamma_\alpha = \emptyset$. Then the Friedrichs inequality holds:

$$\| v\|_{1,\Omega} \leq C |v|_{1,\Omega} \forall v \in W.$$ 

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Let \( \varepsilon > 0 \) be fixed. By \( G \subset \Omega \) we shall denote the \( \varepsilon \)-strip of that part of \( \Gamma_g \) which is curved (see Fig. 3), i.e.
\[
 G = \{ y \in \Omega \mid \exists x \in \Gamma_g - \Gamma'_g : \text{dist} (x, y) < \varepsilon \},
\]
where
\[
 \Gamma'_g = \{ x \in \Gamma_g \mid \exists \text{ a straight segment } S \subset \Gamma_g : x \in S \}.
\]
(For instance, \( G = \emptyset \) if \( \Gamma_g \) is polygonal.) First of all we prove an auxiliary lemma.

**Lemma 2.2.** Let \((\Omega, \Gamma_g) \in C^{(2)}\) and let \( \Gamma_g \neq \emptyset \). Then
\[
 \| v - r_h v \|_{1, \Omega} \leq C h (\| v \|_{2, \Omega} + \| v \|_{1, \infty, G}) \quad \forall v \in \tilde{W},
\]
where
\[
 \tilde{W} = \{ v \in W \cap H^2(\Omega) \mid v|_G \in W^{1, \infty}(G) \}.
\]

**Proof.** Let \( E_h \) denote the union of all triangles \( K \in \mathcal{T}_h \) at least one vertex of which lies on \( \Gamma_{gh} - \Gamma_g \) (i.e. at the points of intersection of the tangents — see Fig. 3). Consequently, \( E_h \subset G \) holds for sufficiently small \( h \). Further, for \( v \in \tilde{W} \) we have
\[
 (2.12) \quad \| v - r_h v \|_{1, \Omega}^2 = \| v - r_h v \|_{1, D_h}^2 + \| v - r_h v \|_{1, \Omega_h - E_h}^2 + \| v - r_h v \|_{1, E_h}^2.
\]

Fig. 3.

The first term on the right-hand side of (2.12) can be estimated as follows:
\[
 (2.13) \quad \| v - r_h v \|_{1, D_h}^2 = \| v \|_{1, D_h}^2 \leq 2 \| v \|_{1, \infty, D_h}^2 \mes D_h \leq Ch^2 \| v \|_{1, \infty, G}^2,
\]
since (see [15], Chap. 1.6)
\[
 \mes D_h \leq C_1 h^2.
\]
The second term can be estimated in the same manner as in (2.6), i.e.
\[
 (2.14) \quad \| v - r_h v \|_{1, \Omega_h - E_h}^2 \leq \sum_{K \in \mathcal{T}_h \setminus \mathcal{K}_E} \| v - r_h v \|_{1, K}^2 \leq Ch^2 \| v \|_{2, \Omega}^2.
\]
Thus, it remains to deal with the third term. Let \( K \subset E_h \) be an arbitrary triangle and let \( v_K \in P_1(K) \) be the linear interpolation of the function \( v|_K \). Then we have
\[
 (2.15) \quad \| v - v_K \|_{1, K} \leq Ch_K \| v \|_{2, K}.
\]
Next we have to estimate the difference

\[ w_K = v_K - (r_h v)|_K. \]

Let \( x \) be an arbitrary vertex of the triangle \( K \). If \( x \in \Gamma_{gh} \) (see Fig. 3), then

\[ \text{dist}(x, \Gamma_y) \leq C h^{2}, \]

and therefore

\[ |w_K(x)| \leq C h^{2} |v|_{1, \infty, G}, \]

since \( w_K(x) = v_K(x) = v(x) \). On the other hand, when the vertex \( x \notin \Gamma_{gh} \), we get even \( w_K(x) = 0 \). As \( w_K \in P_1(K) \), we see that (2.17) holds for all \( x \in K \) and using (2.2), we obtain the estimate

\[ |w_K|^2_{1, K} \leq 2 \text{mes } K \left( \frac{C h^{2} |v|_{1, \infty, G}}{\varepsilon h_K} \right)^2. \]

From this and (2.15) it follows that

\[ |v - r_h v|^2_{1, K} \leq |v - v_K|^2_{1, K} + |w_K|^2_{1, K} \leq C h^{2} (\|v\|_{2, K} + \text{mes } K |v|^2_{1, \infty, G}). \]

Hence

\[ |v - r_h v|^2_{1, K} = \sum_{K \in \mathcal{T}_h} |v - r_h v|^2_{1, K} \leq C h^{2} (\|v\|_{2, \Omega} + \text{mes } E_h |v|^2_{1, \infty, G}). \]

From here, from (2.12), (2.13), (2.14) and the Friedrichs inequality (2.11) we obtain the assertion of the lemma.

**Theorem 2.2.** Let \( (\Omega, \Gamma_y) \in \mathcal{G}(\Omega) \), let \( \Gamma_y \neq \emptyset \) and let \( \Gamma_u, \Gamma_g \) be connected sets. Then there exists a linear operator \( R_h : \overline{Q} \to Q_h \) such that

\[ \|q - R_h q\|_{0, \Omega} \leq Ch \left( \|q\|_{1, \Omega} + \|q\|_{0, \infty, G} \right), \]

where

\[ \overline{Q} = \{ q \in Q \cap (H^1(\Omega))^2 : q|_\partial \in (L^\infty(G))^2 \} . \]

**Proof.** Let \( q \in \overline{Q} \) be arbitrary. By (1.7) there exists \( v \in W \) such that \( q = \text{curl } v \). In our case, we have even \( v \in \overline{W} \). Setting again \( R_h q = \text{curl } r_h v \), we see that \( R_h q \in Q_h \) and from Lemma 2.2 and (2.11) we obtain

\[ \|q - R_h q\|_{0, \Omega} \leq C_1 h^{2} (\|v\|_{2, \Omega} + |v|_{1, \infty, G})^2 \leq C_2 h^{2} (\|v\|_{1, \Omega}^2 + |v|_{2, \Omega}^2 + |v|_{1, \infty, G}^2) = C_2 h^{2} (\|q\|_{1, \Omega}^2 + \|q\|_{2, \infty, G}^2). \]

To generalize Theorems 2.1 and 2.2 even further, we extend the domain of the mapping \( r_h \). The extended map will be denoted by \( r_h \) as well. Let us define the space \( W' \) by

\[ W' = \{ v \in H^1(\Omega) \mid \text{ if } n \geq 2 \text{ there exist } c_1, \ldots, c_n \in \mathbb{R}^4 : v|_{r_{y,i}} = c_i, \]

\[ i = 1, \ldots, n - 1; \ v|_{r_{y,0}} = 0 \} . \]
Here we note that the distances of the components $T_{lg}$ are positive due to the finiteness of the set $M_x$. Hence, for sufficiently small $h$, we can define the finite element subspace $W'_h$ of the space $W'$ in the following way:

$$W'_h = \{ v \in W' \mid v|_{D_{hl}} \in P_0(D_{hl}) \forall i \in \{0, \ldots, n - 1\}, v|_K \in P_1(K) \forall K \in \mathcal{T}_h \},$$

where $D_{hl}$ is the union of all components $D$ of the set $D_h$ for which $D \cap T_{lg} \neq \emptyset$.

The operator $r_h : W' \cap H^2(\Omega) \to W'_h$ will be defined as follows. For $v \in W' \cap H^2(\Omega)$ and $i \in \{0, \ldots, n - 1\}$ we put

$$r_h v|_{D_{hl}} = v|_{T_{lg}},$$

and we demand as before that $(r_h v)(x) = v(x)$ for all nodal points $x$ of the triangulation $\mathcal{T}_h$ such that $x \notin T_{gh}$. The function $r_h v \in W'_h$ is uniquely determined by these relations, and the following lemma is valid.

**Lemma 2.3.** Let $(\Omega, \Gamma_g) \in \mathcal{C}^{(2)}$. Then

$$\| v - r_h v \|_{1,\Lambda} \leq Ch(\| v \|_{2,\Lambda} + \| v \|_{1,\infty,G}) \quad \forall v \in \tilde{W'},$$

where

$$\tilde{W'} = \{ v \in W' \cap H^2(\Omega) \mid v|_G \in W^{1,\infty}(G) \}.$$

**Proof.** The case $\Gamma_g = \emptyset$ was proved in Lemma 2.1. Consequently, let $\Gamma_g \neq \emptyset$. This case can be proved by an argument parallel to that of Lemma 2.2, since we immediately see that (2.12), (2.13), (2.14) and (2.15) hold for $v \in \tilde{W'}$, too. We show now that also (2.17) can be proved for $v \in \tilde{W'}$.

Since $r_h v = v$ on $\Gamma_g$ (see (2.20)) and since $r_h v$ is constant on any $D_{hl}$, $i = 0, \ldots, n - 1$, we get from (2.16) that

$$|v(x) - (r_h v)(x)| \leq Ch^2 |v|_{1,\infty,D_{hl}},$$

where $x \in \Gamma_{gh}$ is a vertex of a triangle $K \subset E_h$ (see Fig. 3). However, $v(x) = v_K(x)$ and thus

$$|w_K(x)| = |v_K(x) - (r_h v)(x)| \leq Ch^2 |v|_{1,\infty,G}.$$

But this estimate is true also for a vertex $x \notin \Gamma_{gh}$, because $w_K(x) = 0$. Since $w_K \in P_1(K)$, (2.17) holds for all $x \in K$. The rest of the proof is the same as in Lemma 2.2. According to (2.20), we get $v - r_h v \in W$ and therefore we can apply the Friedrichs inequality (2.11).

We shall use the preceding lemma in proving Theorem 2.3, which generalizes Theorems 2.1 and 2.2. Before that we introduce two important remarks.

**Remark 2.1.** The functions $\alpha^j, \beta^j$ in Theorem 1.1 can be chosen for example in the following way (see [9], p. 59). If $n \geq 2$, then we define

$$\beta^j = \text{curl } \bar{w}^j, \quad j = 1, \ldots, n - 1,$$

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where \( \tilde{w}^j \in H^1(\Omega) \) are arbitrary fixed functions satisfying

\[
\tilde{w}^j = \delta_{ij} \text{ on } \Gamma_g^i, \quad i = 0, \ldots, n - 1, \quad j = 1, \ldots, n - 1,
\]

(\( \delta_{ij} \) is Kronecker's symbol). Taking \( \tilde{w}^j \in C^\infty(\Omega) \), we get \( \beta^j \in (C^\infty(\Omega))^2 \).

Further, let \( m \geq 2 \) and let \( \partial \Omega_i \cap \Gamma_u \neq \emptyset \) for \( i = 0, \ldots, m - 1 \) (otherwise we change the notation of the components of \( \partial \Omega \)). Then we can define the functions \( \alpha^j, j = 1, \ldots, m - 1, \) by

\[
(2.22) \quad \alpha^j = \text{curl } w^j \text{ on } S_j, \\
\quad \alpha^j = 0 \text{ on } \Omega - S_j,
\]

where \( S_j \subset \Omega \) is an arbitrary simply connected domain with a Lipschitz boundary such that

\[
(2.23) \quad \partial S_j \cap \partial \Omega_k = \emptyset \quad \forall k \in \{0, \ldots, H\} - \{j - 1, j\}
\]

and the sets

\[
(2.24) \quad \partial S_j^1 = \partial S_j \cap \partial \Omega_{j-1}, \quad \partial S_j^3 = \partial S_j \cap \partial \Omega_j
\]

are nonempty, connected and included in \( \Gamma_u \) (see Fig. 4). Next, \( \partial S_j^2 \) and \( \partial S_j^4 \) are components of the set \( \partial S_j - (\partial S_j^1 \cup \partial S_j^2) \), and \( w^j \in H^1(S_j), j = 1, \ldots, m - 1, \) are arbitrary fixed functions satisfying

\[
(2.25) \quad w^j = 1 \text{ in a neighbourhood of the component } \partial S_j^2, \\
\quad w^j = 0 \text{ in a neighbourhood of the component } \partial S_j^4.
\]

One can see, using (2.22), that \( \alpha^j \in (C^\infty(\Omega))^2 \) follows for \( w^j \in C^\infty(S_j) \). Moreover, (2.23)–(2.25) imply that

\[
(2.26) \quad \text{dist (supp } \alpha^j, \Gamma_g) > 0, \quad j = 1, \ldots, m - 1,
\]

and \( \varepsilon \) (in the definition of the set \( G \)) can be chosen so small that

\[
(2.27) \quad \alpha^j|_G = 0, \quad j = 1, \ldots, m - 1.
\]
One can easily verify that the functions $a_1, \ldots, a_w, p_1, \ldots, p_n$ are linearly independent. Henceforth, we shall assume these functions to be fixed in the space $(C^\infty(\Omega))^2$.

Remark 2.2. Let $q \in Q \cap (H^1(\Omega))^2$. According to [5], p. 22, for this $q$ a stream function exists if and only if

$$\int_{\partial \Omega} q^T v \, ds = 0 \quad \text{for all} \quad i = 0, \ldots, H.$$  

Consequently, for $m \geq 2$ the stream function does not exist, in general. There exists, however, precisely one linear combination $\alpha = \sum_{j=1}^{m-1} c_j \alpha_j$ such that a stream function $v \in W'$ exists for the difference $q - \alpha$, i.e.

$$q - \alpha = \text{curl} \, v$$

(2.28) holds. The coefficients $c_1, \ldots, c_{m-1}$ are the solution of the linear system of equations

$$\sum_{j=1}^{m-1} c_j \int_{\partial \Omega_i} (\alpha_i) v \, ds = \int_{\partial \Omega_i} q^T v \, ds, \quad i = 0, \ldots, m - 1,$$

(2.29) which is uniquely solvable (see [9], p. 61). Therefore, we can consider the mapping $q \in Q \cap (H^1(\Omega))^2 \mapsto \alpha \in L(\alpha_1, \ldots, \alpha_{m-1})$. It is readily seen that the mapping is linear. We shall prove that it is continuous as well.

Let $v^i \in C^\infty(\Omega), i = 0, \ldots, m - 1$, be fixed chosen functions such that

$v^i|_{\partial \Omega_j} = \delta_{ij}, \quad i = 0, \ldots, m - 1, \quad j = 0, \ldots, H.$

Using Green's Theorem, we obtain for $i = 0, \ldots, m - 1$

$$\int_{\partial \Omega} q^T v \, ds = \int_{\partial \Omega} v^i q^T v \, ds = \|\text{grad} \, v^i, q\|_{0, \Omega} \leq \|\text{grad} \, v^i\|_{0, \Omega} \|q\|_{0, \Omega} \leq C \|q\|_{1, \Omega},$$

where $C$ can be taken independent of $i$, since $v^i$ are fixed. Consequently, making use also of (2.29) and of the equivalence of all norms in a finite-dimensional space, we are led to the continuity of the map $q \mapsto \alpha$, i.e.

$$\|\alpha\|_{1, \Omega} \leq C_1 \sum_{i=0}^{m-1} \int_{\partial \Omega} q^T v \, ds \leq C_2 \|q\|_{1, \Omega}.$$  

(2.30)

Recall (see [5], p. 25) that the stream function is determined except for a constant. Thus the function $v$ in (2.28) is uniquely determined in $W'$ if $\Gamma_\gamma \neq \emptyset$. In the case $\Gamma_\gamma = \emptyset$, we choose (the unique) $v$ such that

$$\langle v, 1 \rangle_{0, \Omega} = 0.$$  

(2.31)

Theorem 2.3. Let $(\Omega, \Gamma_\gamma) \in \mathfrak{G}(2)$. Then there exists a linear operator $R_h : \tilde{Q} \to Q_h$ such that

$$\|q - R_h q\|_{0, \Omega} \leq C h (\|q\|_{1, \Omega} + \|q\|_{0, \infty, \Omega}),$$

where $\tilde{Q}$ and $Q_h$ are defined by (2.18) and (2.3), respectively.

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Proof. Let \( q \in \overline{Q} \) be arbitrary. According to (2.27), we write

\[
q = \text{curl } v + \sum_{j=1}^{m-1} c^j \mathbf{a}^j,
\]

where \( v \in W', c', \ldots, c^{m-1} \in R^1 \) and \( \mathbf{a}^j \) are chosen in \( (C^\infty(\overline{D}))^2 \). Since \( q \in \overline{Q} \), we even have \( v \in \widehat{W}' \). First of all let us construct an approximation of the function \( \mathbf{a} \) by means of functions from \( Q_h \).

Consider \( w \in C^\infty(S) \), which satisfies (2.25), and define (for sufficiently small \( h \)) a function \( \pi_hw \in C^\infty(S) \) by

\[
\begin{align*}
\pi_hw|_{K \cap S} &\in P_1(K \cap S), \quad K \in \mathcal{T}_h, \quad K \cap S \neq \emptyset, \\
\pi_hw &= w \quad \text{on } \partial S^2 \cup \partial S^4, \\
(\pi_hw)(x) &= w(x)
\end{align*}
\]

for all nodal points \( x \) of the triangulation \( \mathcal{T}_h \) such that \( x \in S \). Let us put

\[
\begin{align*}
\Pi_h\mathbf{a} &= \text{curl } \pi_hw \quad \text{on } S, \\
\Pi_h\mathbf{a} &= 0 \quad \text{on } \Omega - S.
\end{align*}
\]

We can show that \( \Pi_h\mathbf{a} \in Q_h \). In fact, from (2.32) it follows that \( \pi_hw \) fulfils (2.25) for sufficiently small \( h \). Consequently, using Theorem 1.1 and Remark 2.1, we obtain \( \Pi_h\mathbf{a} \in Q \), since \( \pi_hw \in H^1(S) \). Further, with regard to (2.26) and (2.33) we get \( \Pi_h\mathbf{a}|_{D_h} = 0 \), and \( \Pi_h\mathbf{a}|_K \in (P_0(K))^2 \) follows from (2.32) and (2.33) for all \( K \in \mathcal{T}_h \). Consequently, \( \Pi_h\mathbf{a} \in Q_h \) (cf. (2.3)).

Using (2.22) and (2.33), we derive that

\[
\left\| \mathbf{a} - \Pi_h\mathbf{a} \right\|_{0,\Omega} = \left\| \text{curl } (w - \pi_hw) \right\|_{0,S} \leq \left\| w - \pi_hw \right\|_{1,S} \leq Ch \left\| w \right\|_{2,S}.
\]

The last inequality is standard and can be proved in a way parallel to that of (2.6) (taking into account the fact that \( w = \pi_hw \) in a certain neighbourhood of the components \( \partial S^2 \) and \( \partial S^4 \)).

The Friedrichs inequality together with the fact that \( w|_{\partial S^j} = 0 \) and with (2.34) and (2.22) yields

\[
\left\| \mathbf{a} - \Pi_h\mathbf{a} \right\|^2_{0,\Omega} \leq Ch^2 \left( \left\| w \right\|_{1,S}^2 + \left\| \mathbf{a} \right\|_{2,S}^2 \right) = Ch^2 \left\| \mathbf{a} \right\|^2_{1,S} = Ch^2 \left\| \mathbf{a} \right\|^2_{0,\Omega}.
\]

Putting

\[
\Pi_h\mathbf{a} = \sum_{j=1}^{m-1} c^j \Pi_h\mathbf{a}^j \quad \text{for } \mathbf{a} = \sum_{j=1}^{m-1} c^j \mathbf{a}^j, \quad c^j \in R^1,
\]

we can easily find that

\[
\left\| \mathbf{a} - \Pi_h\mathbf{a} \right\|_{0,\Omega} \leq Ch \left\| \mathbf{a} \right\|_{1,\Omega}
\]

holds for all \( \mathbf{a} \) from the finite-dimensional space \( \mathcal{L}((\mathbf{a}^1, \ldots, \mathbf{a}^{m-1})) \).
Now, for \( q \in \tilde{Q} \) we define
\[
R_h q = \text{curl} \, r_h v + \Pi_h \mathbf{a}
\]
on the basis of the relation (2.28). Then obviously \( R_h q \in Q_h \) and we may write, making use of (2.28), (2.36), Lemma 2.3, (2.35), (2.31), the Friedrichs inequality if \( \Gamma_g = \emptyset \) or the Poincaré inequality (2.10) if \( \Gamma_g = \emptyset \), (2.27) and (2.30),
\[
\|q - R_h q\|_{0, \Omega} \leq \|\text{curl} \,(v - r_h v)\|_{0, \Omega} + \|\mathbf{a} - \Pi_h \mathbf{a}\|_{0, \Omega} \leq C_1 h(\|v\|_{2, \Omega} + |v|_{1, \infty, \partial} + \|\mathbf{a}\|_{1, \partial}) \leq C_2 h(\|v\|_{1, \partial} + |v|_{2, \Omega} + \|\text{curl} \, v\|_{0, \partial} + \|\mathbf{a}\|_{1, \partial}) \leq C_4 h(\|q\|_{1, \partial} + \|q\|_{0, \infty, \partial}).
\]

3. ERROR ESTIMATES AND CONVERGENCE

In this chapter we shall estimate the difference \( p - p_h \), where \( p \) and \( p_h \) are the solution of the dual problem and its internal approximation (from the space \( Q_h \) - (2.3)), respectively (see Chap. 1).

**Theorem 3.1.** Let \( (\Omega, \Gamma_g) \in \mathcal{G}(2) \) and let \( p \in \bar{Q} \) (cf. (2.18)). Then
\[
\|p - p_h\|_{0, \partial} \leq C h(\|p\|_{1, \partial} + \|p\|_{0, \infty, \partial}).
\]

**Proof.** The well-known Céa's Lemma ([1], p. 104) yields
\[
\|p - p_h\|_{0, \partial} \leq C_1 \inf_{q_h \in Q_h} \|p - q_h\|_{0, \partial} \leq C_1 \|p - R_h p\|_{0, \partial}.
\]
The assertion follows then from Theorem 2.3.

When no regularity of the solution \( p \in Q \) is assumed, we obtain a convergence of \( p_h \) to \( p \) by virtue of the following density theorem.

**Theorem 3.2.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with a Lipschitz boundary and let \( \Gamma_u \) and \( \Gamma_g \) satisfy (1.2). Then the set \( Q \cap (C^\infty(\tilde{\Omega}))^2 \) is dense in \( Q \) (with respect to the \( \|\cdot\|_{0, \partial} \) norm).

**Proof.** Let \( q \in Q \) be arbitrary. Then by Theorem 1.1 we have
\[
q = \text{curl} \, w + \mathbf{a} + \mathbf{b},
\]
where \( w \in \bar{W} \) and \( \mathbf{a}, \mathbf{b} \in (C^\infty(\tilde{\Omega}))^2 \) - see Remark 2.1. According to [3], p. 618, there exists a sequence \( \{w_k\} \subset \bar{W} \cap C^\infty(\tilde{\Omega}) \) such that
\[
\|w - w_k\|_{1, \partial} \to 0 \quad \text{if} \quad k \to \infty.
\]
Hence, putting \( q_k = \text{curl} \, w_k + \mathbf{a} + \mathbf{b} \in (C^\infty(\tilde{\Omega}))^2 \), we obtain \( q_k \in Q \),
\[
\|q - q_k\|_{0, \partial} = \|\text{curl} \, (w - w_k)\|_{0, \partial} \leq \|w - w_k\|_{1, \partial} \to 0.
\]
Remark 3.1. Similar results obtained under a little stronger assumptions can be found in [8, 10].

Theorem 3.3. If $(\Omega, \Gamma_0) \in C^{(2)}$, then $\|p - p_h\|_{0,\Omega} \to 0$ for $h \to 0$.

Proof. Let $\epsilon > 0$ be given. By Theorem 3.2 there exists $q \in \mathcal{Q} \cap (C^\infty(\Omega))^2 \subset \bar{\Omega}$ such that $\|p - q\|_{0,\Omega} < \epsilon/2$ and by Theorem 2.3, $\|q - R_h q\|_{0,\Omega} < \epsilon/2$ for sufficiently small $h$. Thus on the basis of Céa’s Lemma, we get

$$\|p - p_h\|_{0,\Omega} \leq C \inf_{q_h \in \mathcal{Q}_h} \|p - q_h\|_{0,\Omega} \leq C(\|p - q\|_{0,\Omega} + \|q - R_h q\|_{0,\Omega}) \leq C \epsilon.$$  

4. EQUILIBRIUM FINITE ELEMENT SPACES GENERATED BY POLYNOMIALS OF HIGHER ORDERS

Let us consider again problems of the class $C^{(2)}$. Assume now that each of the smooth arcs belonging to $C^{(2)}$, from which the boundary $\partial \Omega$ is composed, has a parametric representation

$$x = \varphi(s), \quad y = \psi(s), \quad \varphi, \psi \in C^{(2)},$$

and the functions $\varphi, \psi$ are available.

Let $\mathcal{M}$ be a regular family of triangulations of the domain $\Omega$, including curved elements — triangles with one curved side along the curved part of $\partial \Omega$.

To define subspace $W_h^{*} \subset W$ (see (2.19)) generated by polynomials of higher orders, we can use the approach of Zlámal ([18], [17] p. 28). Let us introduce the mapping

$$x = x(\xi, \eta) = x_1 + \bar{x}_2 \xi + \bar{x}_3 \eta + \begin{pmatrix} 1 - \xi - \eta \end{pmatrix} \begin{pmatrix} 1 - \eta \end{pmatrix}^{-1} (\varphi(s_1 + \bar{s}_3 \eta) - x_1 - \bar{x}_3 \eta),$$

$$y = y(\xi, \eta) = y_1 + \bar{y}_2 \xi + \bar{y}_3 \eta + \begin{pmatrix} 1 - \xi - \eta \end{pmatrix} \begin{pmatrix} 1 - \eta \end{pmatrix}^{-1} (\psi(s_1 + \bar{s}_3 \eta) - y_1 - \bar{y}_3 \eta),$$

where

$$\bar{x}_j = x_j - x_1, \quad \bar{y}_j = y_j - y_1, \quad j = 1, 2,$$

$$\bar{s}_3 = s_3 - s_1,$$

which maps the closed triangle $\bar{\mathcal{K}}$ with the vertices $R_1 = (0, 0), R_2 = (1, 0), R_3 = (0, 1)$ in the $\xi, \eta$-plane onto a closed triangle $\mathcal{K} \in \mathcal{T}_h$ with vertices $P_j = (x_j, y_j), j = 1, 2, 3$, in the $x, y$-plane. Then the side $R_1 R_3$ is mapped onto the arc $P_1 P_3$, the sides $R_1 R_2$ and $R_2 R_3$ are linearly mapped onto the sides $P_1 P_2$ and $P_2 P_3$, respectively (see Fig. 5). Finally, $s_1, s_3$ are the values of the arc parameter corresponding to the vertices $P_1$ and $P_3$, respectively.

Zlámal has proved the following assertion (see [18]): Let the boundary $\partial \Omega$ belong piecewise to $C^{(2)}$. Then for sufficiently small $h$ and any triangle $K \in \mathcal{T}_h$ the mapping
(4.1) maps one-to-one the closed triangle $\hat{K}$ onto the closed triangle $K$ and the Jacobian $J(\xi, \eta)$ of (4.1) is different from zero on $\hat{K}$.

Choosing a polynomial $r(\xi, \eta)$ in $\hat{K}$ we define

$$p(x, y) = r(\xi(x, y), \eta(x, y)) \quad \text{on} \quad K,$$

where $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ is the inverse mapping to (4.1). The polynomials $r(\xi, \eta)$ are such that their values on each side of $\hat{K}$ are uniquely determined by some (nodal) parameters associated with some points (nodes) lying on this side. The trial functions are now defined on the whole domain $\Omega$ by the values of the nodal parameters at the nodes.

If $K$ runs through the partition $\mathcal{T}_h$ of $\hat{\Omega}$, we get all nodes of $\hat{\Omega}$. Evidently, the trial functions form a finite-dimensional subspace of $H^1(\Omega)$. As the boundary $\partial \Omega$ is mapped piecewise onto $\hat{K}$, the conditions $v|_{\Gamma_{\nu}} = c$, $i = 0, 1, \ldots, n - 1$, where $c_0 = 0$ (see (2.19)), are easy to satisfy by choosing the boundary nodal parameters in such a way that $r(0, \eta) \equiv c_i$. Thus we obtain a subspace $W'_h$ of $W'$.

Remark 4.1. If the boundary $\partial \Omega$ is piecewise polynomial, one can use the so-called isoparametric finite elements to construct the subspaces $W'_h$ of $W'$. For details we refer to the book [1], Chap. 4.

We may therefore consider an arbitrary finite element space $W'_h$ such that

$$W'_h \subset W'.$$

First of all let us suppose that the part $\Gamma_u$ of $\partial \Omega$ is contained in at most one component of the boundary $\partial \Omega$. Then by (2.19), (2.21) and by Theorem 1.1 for $m < 2$ we see that

$$Q = \text{curl} \, W'.$$
We can define the space of equilibrium finite elements as follows:

\[(4.4) \quad Q_h = \text{curl } W'_h.\]

The desired inclusion $Q_h \subset Q$ results from (4.2) and (4.3).

Let $r_h : W' \cap H^{k+1}(\Omega) \to W'_h$, $k \geq 1$, be an operator with the following approximation property:

$$\|v - r_h v\|_{k+1, \Omega} \leq C h^k \|v\|_{k+1, \Omega}.$$  

(See [17, 18], where such estimates have been proved for subspaces of curved finite elements.) Then we can define an operator $R_h : Q \cap (H^k(\Omega))^2 \to Q_h$ by

$$R_h q = \text{curl } (r_h v),$$  

where $q = \text{curl } v$, and $R_h$ has this approximation property:

\[(4.5) \quad \|q - R_h q\|_{0, \Omega} = \|\text{curl } (v - r_h v)\|_{0, \Omega} \leq C_1 h^k \|v\|_{k+1, \Omega} \leq C_2 h^k \|q\|_{k, \Omega}.$$  

The last inequality has been obtained by means of the Friedrichs or Poincaré inequality, respectively. Therefore, if the solution $p$ of the dual problem belongs to $Q \cap (H^k(\Omega))^2$, we get by Céa’s Lemma that

$$\|p - p_h\|_{0, \Omega} \leq C h^k \|p\|_{k, \Omega}.$$  

If the set $\bigcup W'_h$ is dense in the space $W'$ (with the $\|\cdot\|_{1, \Omega}$-norm), then the set $\bigcup Q_h$ is dense in $Q$ (with the $\|\cdot\|_{0, \Omega}$-norm) and the convergence $\|p - p_h\|_{0, \Omega} \to 0$ can be derived in a way analogous to that of Theorem 3.3.

When $\Gamma_g$ is contained in at least two components of $\partial \Omega$ (i.e. $m \geq 2$), then the space of the equilibrium finite elements can be defined as follows:

\[(4.6) \quad Q_h = \mathcal{L}(\text{curl } W'_h \cup \{x_1^* \ldots, x_{m-1}^*\}),\]  

where $x_i^*$ are determined by (2.22), where $w_i^* \in \{w|_{S_j} : w \in W'_h\}$ are functions satisfying (2.25). The definition of $Q_h$ is independent of the particular choice of $x_i^*$, since any other $\tilde{x}_i^*$ can be expressed as $\tilde{x}_i^* = x_i^* + \text{curl } w_h$ for convenient $w_h \in W'_h$. Now an approach parallel to that in the proof of Theorem 2.3 can be used to obtain the approximation property (4.5) of $Q_h$.

5. APPENDIX

The internal approximation $p_h$ of the dual problem can be easily found via the following theorem.

**Theorem 5.1.** Let $\Gamma_g \neq \emptyset$, let $\Gamma_u$ be contained in at most one component of $\partial \Omega$ and let $\{w_i\}_{i=1}^d$ be a basis of the space $W'_h \subset W'$. Then we have

$$p_h = \sum_{i=1}^d x_i \text{curl } w_i,$$

where

\[\begin{align}
\sum_{i=1}^d x_i \text{curl } w_i = & \sum_{i=1}^d x_i \text{curl } w_i \\
= & \sum_{i=1}^d x_i \text{curl } w_i .
\end{align}\]

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where \( x_1, \ldots, x_d \) is the solution of the system of linear algebraic equations with a positive definite matrix

\[
\sum_{j=1}^{d} b(\text{curl } w_i, \text{curl } w_j) x_j = l(\text{curl } w_i), \quad i = 1, \ldots, d,
\]

where \( b(\cdot, \cdot) \) and \( l(\cdot) \) are defined by (1.5) and (1.6), respectively.

Proof. Since \( \Gamma_0^d \neq 0 \), the kernel of the mapping \( \text{curl} : W_h' \to Q_h \) reduces to the zero element. Consequently, the relation \( \text{dim } W_h' = \text{dim } Q_h \) follows. Thus \( \{\text{curl } w_i\}_{i=1}^d \) generate a basis of the space \( Q_h \) and the ellipticity of the bilinear form \( b(\cdot, \cdot) \) implies that the matrix \( (b(\text{curl } w_i, \text{curl } w_j))_{i,j=1}^d \) of the system (5.1) is positive definite. The rest of the assertion is obvious.

Remark 5.1. Since \( \text{supp } \text{curl } w_i = \text{supp } \text{grad } w_i \), by a suitable labelling of the basis functions we can reach that the matrix of the system has a structure similar to that of the corresponding system of the primal finite element method. Moreover, if the material is isotropic and homogeneous (i.e. if \( A \) in (1.1) is an identity matrix), the relation

\[
(\text{curl } w_i, \text{curl } w_j)_{0,\Omega} = (\text{grad } w_i, \text{grad } w_j)_{0,\Omega}
\]

holds, i.e. the inner products in the matrix of the system (5.1) can be calculated in the same way as in the primal finite element method.

Remark 5.2. In the case \( \Gamma_0 = \emptyset \) we have \( \text{dim } W_h' = 1 + \text{dim } Q_h \) and for the choice of the basis functions in \( Q_h \) see the paper [9], p. 46. If \( \Gamma_0 \) is contained in at least two components of the boundary \( \partial \Omega \) and if \( \{q^j\}_{j=1}^r \) is a basis of the space \( \text{curl } W_h' \), then \( \{q^1, \ldots, q^r, \alpha_1^h, \ldots, \alpha_m^h\} \), where \( \alpha_j^h \) are the same as those in (4.6), will be a basis of the space \( Q_h \). For the details we refer to the paper [9].

References

Souhrn

VNITŘNÍ APROXIMACE KONEČNÝMI PRVKY
V DUÁLNÍ VARIAČNÍ METODĚ PRO ELIPTICKÉ PROBLÉMY
DRUHÉHO ŘÁDU SE ZAKŘIVENÝMI HRANICÍMI

IVAN HLAVÁČEK, Michal KŘÍŽEK

Na oblastech s po částech hladkou hranicí jsou zkonstruovány pomocí proudové funkce podprostory konečných prvků v prostoru vektorových funkcí, jejichž divergence je rovna nule a jejichž normálová komponenta je na části hranice rovnoměrně nulová. Pomočí těchto podprostorů je definována vnitřní aproximace duální úlohy pro eliptické rovnice 2. řádu. Je dokázána konvergence této metody (bez předpokladu na regularitu řešení) a pro dostatečně hladké řešení je dokázána i optimální rychlost konvergence. Vnitřní aproximaci lze získat řešením soustavy lineárních algebraických rovnic s pozitivně definitní maticí.

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