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LEAST SQUARE METHOD FOR SOLVING CONTACT PROBLEMS 
WITH FRICTION OBEYING THE COULOMB LAW

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INTRODUCTION

Let us assume a structure consisting of two or more deformable bodies in mutual contact, involving friction on common surfaces. It is well-known that problems of such a kind can be formulated in terms of variational inequalities (see [1], [5]). One of the most classical models of friction, namely that obeying the Coulomb law, has been recently analyzed mathematically ([2]). In [4] the relation between the continuous problem and its discrete version, obtained by applying finite elements, is studied. The question of the numerical realization has still remained open. The aim of the present paper is to propose one possible way, based on the least square method. The original variational inequality formulation is replaced in finite dimension by a family of nonlinear equations, using the technique of the simultaneous penalization and regularization. These equations can be viewed as the state equations for a cost functional $J$, the global minimum of which will be searched. The paper is organized as follows: in Section 1, the continuous model is presented. Section 2 analyzes the finite element discretization of the continuous model, based on a mixed variational formulation introduced in [4]. The least square method is described in Section 3 and its relation to the method presented in Section 2 is established. Some remarks, concerning the numerical realization, especially how to calculate the gradient of $J$, are included in Section 4.

1. SETTING OF THE PROBLEM

Let an elastic body be represented by a polygonal domain $Q \subset \mathbb{R}_2$, the boundary $\partial Q$ of which consists of 3 disjoint and non-empty parts $\Gamma_u$, $\Gamma_p$ and $\Gamma_K$, i.e.:

$$\partial Q = \Gamma_u \cup \Gamma_p \cup \Gamma_K.$$ 

We suppose that $\Gamma_K$ (a contact part) is represented by one straight line segment parallel to the $x_2$ - axis (see Fig. 1).
On each part of $\partial \Omega$, different boundary conditions will be assumed. On $\Gamma_w$, the body is supposed to be fixed, i.e.:

$$u_i = 0 \quad \text{on} \quad \Gamma_w, \quad i = 1, 2.$$  

On $\Gamma_p$, surface tractions are prescribed:

$$\tau_{ij}(u) n_j = P_i \quad \text{on} \quad \Gamma_p, \quad i = 1, 2.$$  

Finally, along $\Gamma_K$ the body is unilaterally supported by a rigid foundation and the influence of friction is taken into account, i.e.

$$u_a \leq 0, \quad T_a(u) \leq 0, \quad u_a T_a(u) = 0 \quad \text{on} \quad \Gamma_K$$  

(unilateral conditions),

$$\begin{cases} |T_i(u)| \leq \mathcal{T} T_n(u) \\ \text{if} \quad |T_i(u)| < \mathcal{T} T_n(u) \quad \text{then} \quad u_i = 0 \\ \text{if} \quad |T_i(u)| = \mathcal{T} T_n(u) \quad \text{then there exists} \quad \lambda \geq 0 \quad \text{such that} \\ u_t = -\lambda T_i(u) \end{cases}$$

(Coulomb law of friction)

on $\Gamma_K$.

Symbol $\tau(u) = \{\tau_{ij}(u)\}_{i,j=1}^2$ denotes the stress tensor related to the linearized strain tensor $\varepsilon(u) = \{\varepsilon_{ij}\}_{i,j=1}^2$ by means of the linear Hooke’s law:

$$\tau_{ij}(u) = c_{ijkl} \varepsilon_{kl}(u), \quad \varepsilon_{kl}(u) = 1/2(\partial u_k/\partial x_i + \partial u_i/\partial x_k).$$

Elasticity coefficients $c_{ijkl}$ are supposed to be bounded and measurable in $\Omega$ (i.e. $c_{ijkl} \in L^2(\Omega)$), satisfying the usual symmetry conditions

$$c_{ijkl} = c_{jikl} = c_{klji} \quad \text{a.e. in} \quad \Omega$$

and the ellipticity condition:

$$\exists \tilde{\mathcal{A}} > 0 \quad \text{such that} \quad c_{ijkl} \tilde{\mathcal{A}}_{ij} \zeta_{kl} \geq \tilde{\mathcal{A}} \zeta_{ij} \zeta_{ij} \quad \forall \zeta_{ij} = \zeta_{ji} \in \mathbb{R}_+ \quad \text{a.e. in} \quad \Omega.$$
\( u_n, u_t \) are respectively, the normal and tangential components of the displacement field \( u = (u_1, u_2) \). Similarly, \( T_n(u) \), \( T_t(u) \) denote the normal and tangential components, respectively, of the stress vector \( T(u) = (\tau_{1j}(u) n_j, \tau_{2j}(u) n_j) \). Finally, \( \mathcal{F} \) is the coefficient of the Coulomb friction. By a classical solution of the Signorini problem with friction obeying the Coulomb law, we mean a displacement field \( u \) which is in the equilibrium state with a given body force \( F = (F_1, F_2) \), i.e. satisfies the equilibrium equations

\[
\frac{\partial \tau_{ij}}{\partial x_j} + F_i = 0 \quad \text{in} \quad \Omega, \quad i = 1, 2
\]

and the boundary conditions (1.1)–(1.4). Justification and derivation of (1.3) and (1.4) can be found in [1].

In order to give the weak form of the problem in question, we shall assume a simpler model involving friction, the so called model with a given friction. The classical formulation of such a problem can be formally obtained by replacing the unknown value \( |T_n(u)| \) by a known function (or more generally, functional) \( g \). Let us introduce the following sets:

\[
V = \{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_n \}, \\
\mathcal{V} = V \times V, \\
K = \{ v \in \mathcal{V} \mid v_n \leq 0 \text{ on } \Gamma_k \}, \\
H^{1/2}(\Gamma_k) = \{ \mu \in L^2(\Gamma_k) \mid \exists \mu \in V : \mu = v \text{ on } \Gamma_k \}, \\
H^{-1/2}(\Gamma_k) = (H^{1/2}(\Gamma_k))^* \quad \text{(the dual space to } H^{1/2}(\Gamma_k)), \\
H^{-1/2}_+(\Gamma_k) = \{ \mu^* \in H^{-1/2}(\Gamma_k) \mid \langle \mu^*, v \rangle \geq 0 \forall v \in V, v \geq 0 \text{ on } \Gamma_k \}.
\]

The symbol \( \langle , \rangle \) denotes the duality pairing between \( H^{-1/2}(\Gamma_k) \) and \( H^{1/2}(\Gamma_k) \).

Let \( g \in H^{-1/2}_+(\Gamma_k) \) be given. By a weak solution of the Signorini problem with a given friction we mean a function \( u = u(g) \in K \) such that

\[
(\mathcal{P}) \quad a(u, v - u) + \langle \mathcal{F} g, |v_t| - |u_t| \rangle \geq L(v - u) \quad \forall v \in K,
\]

where

\[
a(u, v) = \int_\Omega \tau_{ij}(u) \varepsilon_{ij}(v) \, dx, \\
L(v) = \int_\Omega F_i v_i \, dx + \int_{\Gamma_p} P_i v_i \, ds, \quad F \in (L^2(\Omega))^2, \quad P \in (L^2(\Gamma_p))^2.
\]

Using classical results of the calculus of variations one can easily prove the existence and the uniqueness of \( u \in K \), solving \( (\mathcal{P}) \). Applying Green’s formula to \( (\mathcal{P}) \) it is readily seen that \(-T_n(u(g)) \in H^{-1/2}_+(\Gamma_k) \). Hence a mapping \( \Phi : H^{-1/2}_+(\Gamma_k) \rightarrow H^{-1/2}_+(\Gamma_k) \) can be defined by

\[
(1.7) \quad \Phi(g) = -T_n(u).
\]
By a variational solution of the Signorini problem with Coulomb friction we mean any function $u \in K$ satisfying
$$\Phi(-T_n(u)) = -T_n(u),$$
i.e. $-T_n(u)$ is a fixed point of the mapping $\Phi$ in $H_+^{1/2}(T_K)$. The existence of such a fixed point has been studied in [2] in the case when $\Omega$ is an infinitely long strip and $\Gamma_p = \emptyset$ and in [3] for a bounded domain with a smooth boundary $\partial \Omega$.

2. FINITE ELEMENT DISCRETIZATION

An approximation of the Signorini problem with friction obeying the Coulomb law can be defined by means of finite elements. Let $\{\mathcal{T}_h\}$, $h \to 0^+$ be a regular family of triangulations of $\overline{\Omega}$, which is consistent with the decomposition of $\partial \Omega$ into $\Gamma_u$, $\Gamma_p$ and $\Gamma_K$. With any $\mathcal{T}_h$ the following finite dimensional spaces will be associated:
$$V_h = \{v_h \in C(\Omega) \mid v_{hi} \in P_1(T), v_h = 0 \text{ on } \Gamma_u\}, \quad V_h = V_h \times V_h,$$
i.e. $V_h$ contains all piecewise linear functions over a given triangulation $\mathcal{T}_h$. Let $\{\mathcal{T}_H\}, H \to 0^+$ be a partition of $\Gamma_K$, nodes of which will be denoted by $b_1, \ldots, b_{m(H)}$. In the sequel we shall consider families of $\{\mathcal{T}_H\}$ satisfying
$$\min H_i \geq \beta,$$
where $H_i = \text{length of } \overline{b_ib_{i+1}}$, $H = \max H_i$. Let
$$L_H = \{\mu_H \in L^2(\Gamma_K) \mid \mu_{Hi} \in P_0(b_i, b_{i+1}), \ i = 1, \ldots, m(H)\}, \quad A_H = \{\mu_H \in L_H \mid \mu_H \geq 0 \text{ on } \Gamma_K\},$$
i.e. $A_H$ contains all non-negative, piecewise-constant functions over $\mathcal{T}_H$. Analogously to the continuous case, we start with the approximation of the auxiliary problem ($\mathcal{P}$).

Let $g_H \in A_H$ be given. We look for a pair $\{u_h, \lambda_H\} \in V_h \times A_H$, satisfying
$$\begin{align*}
(\mathcal{P})_{hh} & \left\{ a(u_h, v_h - u_h) + \langle \lambda_H, v_{hi} - u_{hi} \rangle + \langle \mathcal{F} g_H, |v_{hi}| - |u_{hi}| \rangle \geq L(v_h - u_h) \quad \forall v_h \in V_h, \right.
\left.
\langle \lambda_H - \lambda_H, u_{hi} \rangle \leq 0 \quad \forall \mu_H \in A_H. \right\}
\end{align*}$$
The symbol $\langle , \rangle$ denotes the scalar product in $L^2(\Gamma_K)$.

Remark 2.1. $\lambda_H \in A_H$ satisfying $(\mathcal{P})_{hh}$ is the Lagrange multiplier associated with the unilateral boundary condition on $\Gamma_K$. $-\lambda_H$ plays the role of the approximate normal stress along $\Gamma_K$. 215
Next, we shall suppose that the following condition is satisfied:

\((S)\) \quad \mu_H \in L_H \quad \langle \mu_H, z_h \rangle = 0 \quad \forall z_h \in V_h \quad \Rightarrow \quad \mu_H = 0 .

An equivalent form of \((S)\) is

\[ \exists \beta > 0 \quad \forall \mu_H \in L_H : \sup_{z \in V_h} \frac{\langle \mu_H, z_h \rangle}{\|z_h\|_{1,\Omega}} \geq \beta . \]

One can easily verify that under the condition \((S)\), there exists a unique solution \(\{u_h, \hat{\lambda}_H\}\) of \((P)_{hH}\).

Interpretation of \((P)_{hH}\)

Let

\[ K_{hH} = \{v_h \in V_h \mid \langle \mu_H, v_h \rangle \leq 0 \quad \forall \mu_H \in A_H\} . \]

\(K_{hH}\) contains all functions from \(V_h\), the mean value of the normal component \(v_{hn}\) of which is non-positive on any \(\overline{b_i b_{i+1}}\), \(i = 1, \ldots, m(H)\).

Substituting \(\mu_H = 0, 2\hat{\lambda}_H\) into the second relation of \((P)_{hH}\), we have

\[ \langle \hat{\lambda}_H, u_{hn} \rangle = 0 , \quad \langle \mu_H, u_{hn} \rangle \leq 0 \quad \forall \mu_H \in A_H , \]

i.e. \(u_h \in K_{hH}\) and

\[(2.1) \quad a(u_h, v_h - u_h) + \langle \mathcal{F} g_H, |v_h| - |u_h| \rangle \geq L(u_h) \quad \forall \v_h \in K_{hH} . \]

Let \(\Phi_H : A_H \to A_H\) be a mapping defined as follows:

\((P)_{hH}\)

\[ \Phi_H(g_H) = \hat{\lambda}_H . \]

\(\Phi_H\) can be viewed as an approximation of the mapping \(\Phi\) defined by (1.7). The main result of this section is

**Theorem 2.1.** For any \(\mathcal{F} \in C(\Gamma, H)\), \(\mathcal{F} \geq 0\) there exists at least one solution of \((P)_{hH}\).

**Proof.** i) \(\Phi_H\) is a continuous mapping from \(A_H\) into itself (see [4], Th. 2.3). ii) We shall show that

\[ \Phi_H(B_r \cap A_H) \subset B_r \cap A_H \]

for any \(r \geq r_0\), where \(r_0\) does not depend on \(\mathcal{F}\). \(B_r\) denotes the ball with the center at the origin and the radius equal to \(r\) measured in a suitable topology (see (2.6) below).

Substituting \(v_h = 0, 2u_h\) into (2.1) we get

\[ (2.2) \quad a(u_h, u_h) + \langle \mathcal{F} g_H, |u_h| \rangle = L(u_h) , \]

hence

\[ (2.3) \quad \|u_h\|_{1,\Omega} \leq \frac{1}{\nu} (\|F\|_{0,\Omega} + \|P\|_{0,\Omega}) \]

by virtue of Korn's inequality.
Let
\begin{equation}
\mathcal{V}_h = \{ v_h \in V_h \mid v_h = (v_{h1}, 0) \}.
\end{equation}

As \( f \) is, we have \( v_{h1} = v_{h1}, v_{ht} = 0 \) if \( v_h \in \mathcal{V}_h \) and
\[
a(u_h, v_h) + \langle \lambda_H, v_{h1} \rangle = L(v_h) \quad \forall v_h \in \mathcal{V}_h.
\]

Hence
\begin{equation}
\sup_{v_h} \frac{\langle \lambda_H, v_{h1} \rangle}{\|v_{h1}\|_{1, \Omega}} \leq M \|u_h\|_{1, \Omega} + \left( \|F\|_{0, \Omega} + \|P\|_{0, r_p} \right).
\end{equation}

Let us introduce the following notation:
\begin{equation}
\|\mu_H\|_{1/2, h} = \sup_{z \in \mathcal{V}_h} \frac{\langle \mu_H, z_h \rangle}{\|z_h\|_{1, \Omega}}, \quad \mu_H \in L_H.
\end{equation}

If the condition \((S)\) is satisfied, then (2.6) defines a norm on \( L_H \). Moreover,
\begin{equation}
\exists \gamma > 0 \quad \forall \mu_H \in L_H : \|\mu_H\|_{1/2, h} \leq \gamma \|\mu_H\|_{1/2}.
\end{equation}

The constant \( \gamma \) in general depends on \( h, H \). (2.5) and (2.6) result in
\[
\|\lambda_H\|_{1/2, h} \leq (M/\gamma + 1) \left( \|F\|_{0, \Omega} + \|P\|_{0, r_p} \right).
\]

Let us set
\[
r_0 = (M/\gamma + 1) \left( \|F\|_{0, \Omega} + \|P\|_{0, r_p} \right).
\]

Then \( \Phi_H(B_r \cap A_H) \subset B_r \cap A_H \) for any \( r \geq r_0 \). Using the Schauder fixed-point theorem we arrive at the assertion.

It can be shown that
\[
\|\lambda_H - \tilde{\lambda}_H\|_{1/2} = \|\Phi_H(g_H) - \Phi_H(\tilde{g}_H)\|_{1/2} \leq q \|g_H - \tilde{g}_H\|_{1/2},
\]
where \( q = C(H) \left[ \mathcal{F} \right], \left[ \mathcal{F} \right] = \max_{r \in K} \mathcal{F}(x) \) and \( C(H) \to +\infty \) if \( H \to 0^+ \) (for the proof see [4]). If
\begin{equation}
\left[ \mathcal{F} \right] < 1/C(H),
\end{equation}
then \( \Phi_H \) is contractive and its unique fixed-point can be found by the method of successive approximations. Unfortunately, to keep \( q \in (0, 1) \), \( \left[ \mathcal{F} \right] \) has to tend to zero whenever \( H \to 0^+ \). This is the reason for which the method of successive approximations need not be successful, in general. Below we present an alternative approach, based on the smoothening of \((\mathcal{P})_{ht}\) combined with the least square method.

### 3. Least Square Method for Numerical Solution of \((\mathcal{P})_{ht}\)

Let \( \beta : C^1 \to R_1 \) be a function such that
- \( \beta(x) \geq 0 \) \( \forall x \in R_1 \) and \( \beta(x) = 0 \) if and only if \( x \leq 0 \)
- \( \beta \) is monotone on \( R_1 \).

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For any \( u_h, v_h \in V_h \) let
\[
(\bar{\beta}(u_h), v_h) = \sum_{i=1}^{M(H)} \beta(\bar{u}_{h1}^i) \bar{v}_{h1}^i H_i = \sum_{i=1}^{M(H)} \beta(\bar{u}_{h1}^i) \bar{v}_{h1}^i H_i
\]
and
\[
f_i(g_H, v_h) = \langle \mathcal{F} g_H, \sqrt{(v_{h1}^2 + \varepsilon^2)} \rangle, \quad \varepsilon > 0.
\]
Here \( \bar{u}_{h1}^i \) denotes the mean value of \( u_{h1} \) on \( b_i b_{i+1} \):
\[
\bar{u}_{h1}^i = \int_{b_i b_{i+1}} u_{h1} \, ds.
\]

**Lemma 3.1.** The following identity holds:
\[
(\bar{\beta}(u_h), v_h) = \langle \omega_H, v_{h1} \rangle,
\]
where \( \omega_H \in A_H \) is defined by
\[
\omega_H|_{b_i b_{i+1}} = \beta(\bar{u}_{h1}^i) \chi_i,
\]
with \( \chi_i \) being the characteristic function of \( b_i b_{i+1} \).

**Proof.**
\[
(\bar{\beta}(u_h), v_h) = \sum_{i=1}^{M(H)} \beta(\bar{u}_{h1}^i) \bar{v}_{h1}^i H_i = \sum_{i=1}^{M(H)} \beta(\bar{u}_{h1}^i) \int_{b_i b_{i+1}} v_{h1} \, ds = \int_{\Gamma_K} \omega_H v_{h1} \, ds.
\]

**Lemma 3.2.** The following equivalence holds:
\[
u_h \in V_h, \quad (\bar{\beta}(u_h), v_h) = 0 \quad \forall v_h \in V_h \iff u_h \in K_{hH}.
\]

**Proof.** Let \( u_h \in V_h \) be such that
\[
(\bar{\beta}(u_h), v) = 0 \quad \forall v_h \in V_h.
\]
From this and Lemma 3.1 one has
\[
\langle \omega_H, v_{h1} \rangle = \langle \omega_H, v_{h1} \rangle = 0 \quad \forall v_{h1} \in V_h,
\]
so that \( \omega_H = 0 \) on \( \Gamma_K \) due to the condition (S). Definitions of \( \omega_H \) and \( K_{hH} \) yield the assertion of the lemma.

Let \( \varepsilon > 0 \) be a parameter tending to zero and let us consider the following penalized-regularized problem:
\[
(\mathcal{P}_\varepsilon) \begin{cases} \text{find } u^\varepsilon_h \in V_h \text{ such that } \\ a(u^\varepsilon_h, v) + 1/\varepsilon(\bar{\beta}(u^\varepsilon_h), v_h) + j_\varepsilon(g_H, u^\varepsilon_h) v_h = L(v_h) \quad \forall v_h \in V_h, \\ \\ j_\varepsilon(g_H, u^\varepsilon_h) v_h = \int_{\Gamma_K} \mathcal{F} g_H \frac{u^\varepsilon_{h1} v_{h1}}{\sqrt{(u^\varepsilon_{h1}^2 + \varepsilon^2)}} \, ds. \end{cases}
\]

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It is readily seen that for any $\varepsilon > 0$ there exists a unique solution $u^\varepsilon_h$ of $(\mathcal{P})_\varepsilon$. Now, we shall define a mapping $\Psi^\varepsilon_H : A_H \to A_H$ by means of
\begin{equation}
(3.1) \quad \Psi^\varepsilon_H(g_H)|_{b_{b^0,i+1}} = \frac{1}{\varepsilon} \beta(u^\varepsilon_H)|_{b_{b^0,i+1}}.
\end{equation}

Remark 3.1. The function $- \Psi^\varepsilon_H(g_H)$ will play again the role of the approximate normal stress along $\Gamma_K$. A function $u^\varepsilon_h$ satisfying $(\mathcal{P})_\varepsilon$, can be obtained by solving a nonlinear system of algebraic equations.

Analogously to the approach used in the last section, we shall consider the problem of finding a fixed point of the mapping $\Psi^\varepsilon_H$ in $A_H$, i.e.: find $\lambda^\varepsilon_H \in A_H$ such that
\[ (\mathcal{P})_\varepsilon \quad \quad \Psi^\varepsilon_H(\lambda^\varepsilon_H) = \lambda^\varepsilon_H. \]

Next, we shall study
i) the existence of $\lambda^\varepsilon_H$;
ii) the relation between the solutions of $(\mathcal{P})_\varepsilon$ and $(\mathcal{P})$ if $\varepsilon \to 0^+$. 

**Theorem 3.1.** For any $\mathcal{F} \in C(\Gamma_K)$, $\mathcal{F} \geq 0$ and $\varepsilon > 0$ there exists at least one solution of $(\mathcal{P})_\varepsilon$.

**Proof** is analogous to that of Theorem 2.1. From the definition of $j_\varepsilon(v_h)$ and $(\beta(u^\varepsilon_h), v_h)$ it follows that
\[ a(u^\varepsilon_h, u^\varepsilon_h) \leq a(u^\varepsilon_h, u^\varepsilon_h) + 1/\varepsilon(\beta(u^\varepsilon_h), u^\varepsilon_h) + j'_\varepsilon(g_H, u^\varepsilon_h) u^\varepsilon_h = L(u^\varepsilon_h), \]
from which
\[ \|u^\varepsilon_h\|_{1,\Omega} \leq 1/\varepsilon(\|F\|_{0,\Omega} + \|P\|_{0,r_F}) \]

independently of $\varepsilon > 0$. Let us substitute a function $v_h \in V_h$ into $(\mathcal{P})_\varepsilon$. As $v_{h1} = 0$, we immediately get
\[ a(u^\varepsilon_h, v_h) + \langle 1/\varepsilon \omega^\varepsilon_h, v_{h1} \rangle = L(v_h) \quad \forall v_h \in V_h, \]
so that
\begin{equation}
(3.2) \quad \|1/\varepsilon \omega^\varepsilon_h\|_{1/2,h} = \sup_{v_h \in V_h} \frac{\langle 1/\varepsilon \omega^\varepsilon_h, v_{h1} \rangle}{\|v_{h1}\|_{1,\Omega}} \leq M\|u^\varepsilon_h\|_{1,\Omega} + \|F\|_{0,\Omega} + \|P\|_{0,r_F} \leq r_0. \end{equation}

Hence the mapping $\Psi^\varepsilon_H$ maps a set $A_H \cap B_r$ into itself, where
\[ B_r = \{ \mu_H \in L_H \mid \|\mu_H\|_{-1/2,h} \leq r, \ r \geq r_0 \}. \]

It remains to verify that $\Psi^\varepsilon_H$ is continuous.

Let $g_H, \tilde{g}_H \in A_H$ be given and let $u^\varepsilon_h, z^\varepsilon_h \in V_h$ be the corresponding solutions of the penalized — regularized problems:
\[ a(u^\varepsilon_h, v_h) + 1/\varepsilon(\beta(u^\varepsilon_h), v_h) + j'_\varepsilon(g_H, u^\varepsilon_h) v_h = L(v_h), \]
\[ a(z^\varepsilon_h, v_h) + 1/\varepsilon(\beta(z^\varepsilon_h), v_h) + j'_\varepsilon(g_H, z^\varepsilon_h) v_h = L(v_h). \]
Substituting $z_h - u_h, u_h - z_h$ into the first and the second equation, respectively, and summing up these equations one has

\begin{align}
& a(u_h - z_h, u_h - z_h) + \int_{r_h} ^{\gamma} \mathcal{F}(g_H - \bar{g}_H) \frac{z_{ht}^e}{\sqrt{\left(\frac{z_{ht}^e}{\bar{g}_H^e} + \bar{c}\right)^2}} (z_{ht}^e - u_{ht}) \, ds + \\
& + \int_{r_h} ^{\gamma} \mathcal{F}(g_H - \bar{g}_H) \frac{u_{ht}^e}{\sqrt{\left(\frac{u_{ht}^e}{\bar{g}_H^e} + \bar{c}\right)^2}} (z_{ht}^e - u_{ht}) \, ds \leq \int_{r_h} ^{\gamma} \mathcal{F}(g_H - \bar{g}_H) \frac{z_{ht}^e}{\sqrt{\left(\frac{z_{ht}^e}{\bar{g}_H^e} + \bar{c}\right)^2}} (z_{ht}^e - u_{ht}) \, ds \\
& \leq \int_{r_h} ^{\gamma} \mathcal{F}(g_H - \bar{g}_H) \frac{z_{ht}^e}{\sqrt{\left(\frac{z_{ht}^e}{\bar{g}_H^e} + \bar{c}\right)^2}} (z_{ht}^e - u_{ht}) \, ds \leq c[\mathcal{F}] \|g_H - \bar{g}_H\|_{0,R} \|u_h - z_h\|_{1,0},
\end{align}

the monotonicity of $\beta$ and $j'_i(g_H, u_h)$ being taken into account. From (3.2) it follows that

\begin{align}
\|u_h^* - z_h^*\|_{1,0} \leq c[\mathcal{F}] \|g_H - \bar{g}_H\|_{0,R}.
\end{align}

Using the same approach as at the beginning of the proof, we get

\begin{align}
\|1/\varepsilon \omega_h^e - 1/\varepsilon \omega_h^e\|_{-1/2,\varepsilon} \leq M \|u_h^* - z_h^*\|_{1,0},
\end{align}

where

\begin{align}
\omega_h^e|_{b_i,b_{i+1}} = \beta(\bar{u}_h^{i}) x_i,
\omega_h^e|_{b_i,b_{i+1}} = \beta(\bar{z}_h^{i}) x_i,
\end{align}

and $\bar{u}_h^{i}, \bar{z}_h^{i}$ are the mean values of $u_h^{i}, z_h^{i}$ on $b_i,b_{i+1}$, respectively. Combining (3.3) with (3.4) we finally get

\begin{align}
\|1/\varepsilon \omega_h^e - 1/\varepsilon \omega_h^e\|_{-1/2,\varepsilon} \leq c[\mathcal{F}] M \|g_H - \bar{g}_H\|_{0,R}
\end{align}

which yields the continuity of the mapping $\Psi^e_H$. The existence of a fixed point of $\Psi^e_H$ in $A_H \cap B_r, r \geq r_0$, is then a direct consequence of the Schauder theorem.

A natural question arises if there is any relation between $(P)$ and $(P)_e$. The answer is given by

**Theorem 3.2.** Let \{\lambda_H^e, \varepsilon \to 0+\} be fixed points of the mappings $\Psi^e_H$ in $A_H \cap B_r, r \geq r_0$,

\begin{align}
\lambda_H^e|_{b_i,b_{i+1}} = 1/\varepsilon \beta(\bar{u}_h^{i}) x_i.
\end{align}

Then there exist subsequences \{u_h^e\} \subset \{u_h\}, \{\lambda_H^e\} \subset \{\lambda_H\} and elements $u_h^*, \lambda_h^*$ such that

\begin{align}
u_h^e \to u_h^*,
\lambda_H^e \to \lambda_H^*, \varepsilon \to 0+.
\end{align}

At the same time $\lambda_h^*$ is a fixed point of $\Phi_H$ and $u_h^*$ is a solution of $(P)_{hH}$ with $g_H = \lambda_H^*$. **
Proof. Let $\lambda^e_H \in \Lambda_H \cup B_\varepsilon$ be fixed points of $\Psi^e_H$.

$$\lambda^e_H |_{b_i, b_{i+1}} = 1/\varepsilon \beta(\bar{\alpha}_{2}\bar{a}) \chi_i.$$  

Here $u^e_h \in V_h$ denotes the solution of $(\mathcal{P})_e$ with $g_H$ equal to $\lambda^e_H$. Hence, $u \in V_h$ satisfies (3.3) with $g_H$ replaced by $X_E H$. Therefore there exist subsequences $\{u^e_h\} \subset \{u^e_h\}$, $\{\lambda^e_H\} \subset \{\lambda^e_H\}$ and elements $u^*_h \in V_h$, $\lambda^*_H \in \Lambda_H$ such that (3.5) is satisfied. Let us write $v_h - u^e_h$ instead of $v_h$ in $(\mathcal{P})_e$:

$$a(u^e_h, v_h - u^e_h) + \langle \lambda^e_H, v_{ha} - u^e_{ha}\rangle + j^e(\lambda^e_H, u^e_h)(v_h - u^e_h) = L(v_h - u^e_h)$$

$\forall v_h \in V_h$.

Passing to the limit for $\varepsilon' \to 0^+$ we have

$$a(u^e_h, v_h - u^e_h) \to a(u^*_h, v_h - u^*_h),$$

$$\langle \lambda^e_H, v_{ha} - u^e_{ha}\rangle \to \langle \lambda^*_H, v_{ha} - u^*_v\rangle,$$

$$j^e(\lambda^e_H, u^e_h)(v_h - u^e_h) \to \int_{I_K} \mathcal{F}_{I_H}^\star \frac{u^e_{ha}}{\sqrt{(u^e_{ha})^2 + \varepsilon'^2}} (v_{ha} - u_{ha}) ds \to$$

$$\int_{I_K} \mathcal{F}_{I_H}^\star \text{sign } u^*_h(v_{ha} - u^*_h) ds - \langle \mathcal{F}_{I_H}^\star, \text{sign } u^*_h v_{ha}\rangle -$$

$$- \langle \mathcal{F}_{I_H}^\star, |u^*_h| \rangle \leq \langle \mathcal{F}_{I_H}^\star, |v_{ha}| - |u^*_h| \rangle.$$  

These limits yield

(3.6)  

$$a(u^*_h, v_h - u^*_h) + \langle \lambda^*_H, v_{ha} - u^*_v\rangle + \langle \mathcal{F}_{I_H}^\star, |v_{ha}| - |u^*_h| \rangle \geq$$

$$\geq L(v_h - u^*_h) \quad \forall v_h \in V_h.$$  

Now we prove that

$$\langle \mu_H - \lambda^*_H, u^*_h\rangle \leq 0 \quad \forall \mu_H \in \Lambda_H.$$  

First of all,

$$0 \leq 1/\varepsilon' \sum_{i=1}^{M(H)} \beta(\bar{\alpha}_{2}\bar{a}) i \leq c,$$

so that

(3.7)  

$$\sum_{i=1}^{M(H)} \beta(\bar{\alpha}_{2}\bar{a}) \to 0, \quad \varepsilon' \to 0^+.$$

On the other hand,

(3.8)  

$$\sum_{i=1}^{M(H)} \beta(\bar{\alpha}_{2}\bar{a}) \to \sum_{i=1}^{M(H)} \beta(\bar{\alpha}_{2}\bar{a}).$$

Comparing (3.7) with (3.8) we see that

$$\sum_{i=1}^{M(H)} \beta(\bar{\alpha}_{2}\bar{a}) = 0,$$

i.e. $u^*_h \in K_{H}$.
Let $\mu_H \in A_H$ be arbitrary. Then

\begin{equation}
\langle \mu_H, u_{hn}^* \rangle = \sum_{i=1}^{M(H)} \int_{b_i b_{i+1}} \mu_H u_{hn}^* \, ds = \sum_{i=1}^{M(H)} \mu_H \bar{u}_{hn}^{*i} H_i \leq 0
\end{equation}

as follows from the definition of $A_H$ and the fact that $\bar{u}_{hn}^{*i} \leq 0$. Finally, let us show that $\langle \lambda_H^*, u_{hn}^* \rangle = 0$. Indeed,

\begin{equation}
\langle \lambda_H^*, u_{hn}^* \rangle = \lim_{\varepsilon \to 0^+} \langle \lambda_H^{\varepsilon}, u_{hn}^{\varepsilon} \rangle = \lim_{\varepsilon \to 0^+} \sum_{i=1}^{M(H)} \int_{b_i b_{i+1}} 1/\varepsilon \beta(\bar{u}_{hn}^{\varepsilon i}) \bar{u}_{hn}^{\varepsilon i} ds \geq 0.
\end{equation}

At the same time $\langle \lambda_H^*, u_{hn}^* \rangle$ has to be non-positive as follows from (3.9). From (3.9) and (3.10) we finally get

\begin{equation}
\langle \mu_H - \lambda_H^*, u_{hn}^* \rangle \leq 0 \quad \forall \mu_H \in A_H.
\end{equation}

(3.6) and (3.11) yield the assertion of the theorem.

4. NUMERICAL REALIZATION OF $(P)_e$

Taking into account the results of the last section we see that the problem of finding a fixed point of the mapping $\Phi_H$ in $A_H$ can be replaced by the same problem for a mapping $\Psi_H^e$. Both problems are close in a certain sense (see Theorem 3.2). The last square method will be used for numerical realization of $(P)_e$.

Let $J : A_H \to R_1$ be the functional given by

\begin{equation}
J(g_H) = \frac{1}{2} \| \Psi_H^e(g_H) - g_H \|^2_{C_r \Omega_K},
\end{equation}

where $\Psi_H^e(g_H) \in A_H$ is defined by

$$
\Psi_H^e(g_H)|_{b_i b_{i+1}} = 1/\varepsilon \beta(\bar{u}_{hn}^{\varepsilon i}) \chi_i,
$$

and $u_h^* \in V_h$ is the solution of

\begin{equation}
a(u_h^*, v_h) + 1/\varepsilon \beta(u_h^*), v_h) + \int_{\Omega} f(g_H, u_h^*) v_h = L(v_h) \quad \forall v_h \in V_h.
\end{equation}

The problem $(P)_e$ can be now equivalently stated as the problem of finding global minimizers of $J$ in $A_H$ (at which $J$ is equal to zero).

Remark 4.1. This formulation of $(P)_e$ can be expressed in terms of the optimal control theory: $J$ is a cost functional, $u_h^*$ is the state variable defined by the state equation (4.2) and $g_H \in A_H$ is the control of our problem.

For the numerical realization of the minimization of $J$ over $A_H$, different optimization procedures may be used. Most of them require the knowledge of the gradient of $J$. This is why we sketch how to calculate it. To simplify notations, we omit the symbols $\varepsilon, h, H$ and we shall write $u, g, \ldots$ instead of $u_h^*, g_H^e, \ldots$. 
Let \( \varphi \in L_H \) be given. Then

\[
(4.3) \quad J'(g) \varphi = (g - \Psi(g), \varphi)_{0,r_K} - (g - \Psi(g), \Psi'(g) \varphi)_{0,r_K} =
\]
\[
= (g - \Psi(g), \varphi)_{0,r_K} - (g - \Psi(g), \Psi'_a(u(g)) \varphi)_{0,r_K} =
\]
\[
= (g - \Psi(g), \varphi)_{0,r_K} - (g - \Psi(g), \Psi'_a(u(g)) \omega)_{0,r_K},
\]
where \( \omega \equiv u'_a \varphi \). The symbols \( \Psi'_a \), \( \Psi'_u \) etc. denote the differentiation of \( \Psi \) with respect to \( g, u \) etc. respectively.

Writing the state equation (4.2) for \( g \) and \( g + \varphi \) we immediately get

\[
(4.4) \quad a(\omega, \varphi) + 1/c(\beta'(u) \omega, \varphi) + j'_a(g, u) \varphi + \langle j'_{aw}(g, u) \omega, \varphi \rangle = 0 \quad \forall \varphi \in V.
\]

Here

\[
\langle \beta'(u) s, t \rangle = \sum_{i=1}^{M(H)} \beta'(u_n) s_n^i t_n^i \quad \forall s, t \in V_h,
\]
\[
\langle j'_{aw}(g, u) \omega, \varphi \rangle = \int_{r_K} \mathcal{F} g \frac{\varepsilon^2 \omega_1}{(u_t^2 + \varepsilon^2) \sqrt{(u_t^2 + \varepsilon^2)}} \varphi_1 \, ds.
\]

Let \( q \in V_h \) be the solution of the adjoint equation

\[
(4.5) \quad a(q, \omega) + 1/c(\beta'(u) q, \omega) + j'_a(g, u) q + \langle j'_{aw}(g, u) \omega, q \rangle =
\]
\[
= - (\Psi(g) - g, \Psi'_a(u) \omega)_{0,r_K} \quad \forall \omega \in V_h.
\]

Inserting \( \omega \) into (4.5) instead of \( \varphi \) and comparing it with (4.4) we see that

\[
-(\Psi(g) - g, \Psi'_a(u) \omega)_{0,r_K} = a(q, \omega) + 1/c(\beta'(u) q, \omega) +
\]
\[
+ \langle j'_{aw}(g, u) q, \omega \rangle = a(q, \omega) + 1/c(\beta'(u) \omega, q) + \langle j'_{aw}(g, u) \omega, q \rangle =
\]
\[
= -j'_a(q, u) q.
\]

Hence

\[
J'(g) \varphi = (g - \Psi(g), \varphi)_{0,r_K} - j'_a(q, u) q \equiv
\]
\[
= \int_{r_K} (g - \Psi(g)) \varphi \, ds - \int_{r_K} \mathcal{F} \frac{\varphi_1}{\sqrt{(u_t^2 + \varepsilon^2)}} \varphi_1 \, ds,
\]

where \( q \in V \) is the unique solution of (4.5).

References

Souhrn

METODA NEJMENŠÍCH ČTVERCŮ PRO ŘEŠENÍ KONTAKTNÍCH ÚLOH S COULOMBOVSKÝM TŘENÍM

JAROSLAV HASLINGER

Předložená práce se zabývá numerickou realizací kontaktních úloh s coulombovským třením. Původní úloha je formulována jako problém nalezení pevného bodu jistého operátoru, generovaného variační nerovnicí. Tato nerovnice je pomocí penalizační a regularizační metody transformována na systém variačních nelineárních rovnic, které generují jiné operátory, jež jsou však v jistém smyslu blízké k výše vzpomenutému. Problém nalezení pevných bodů těchto operátorů se řeší pomocí metody nejmenších čtverců, v níž příslušné rovnice vystupují coby stavové rovnice a odpovídající kvadratická odchylka hraje úlohu kriteriální funkce.

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