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A CONVERGENT NONLINEAR SPLITTING VIA ORTHOGONAL PROJECTION

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1. INTRODUCTION

In the whole paper, V is a real or complex Hilbert space. The following theorem was proved by Lučka [4, Lemma 4.2] in a different notation.

Theorem 1. *Let $A \in [V]$ and assume that $P_1, P_2 \in [V]$ are orthogonal projections of V on finite dimensional subspaces such that $P_1 P_2 = P_2 P_1 = P_1$. Suppose that the equation $w = AP_1 w + A(I - P_1)z$ has a unique solution w for any $z \in V$ and that a linear operator W_1 is defined by $W_1 z = w$ so that $W_1 \in [V]$ and $\|(I - P_1)W_1\| = a < 1$. Then $W_2 z = w = AP_2 w + A(I - P_2)z$ defines an operator $W_2 \in [V]$ which fulfils $\|(I - P_2) W_2\| \leqq a$.*

In the present paper, we extend this theorem to nonlinear operators A. The proofs use contraction arguments and no differentiability is required.

Consider an operator equation

$$x = Tx\,, \tag{1.1}$$

where T maps the Hilbert space V into itself. Given another Hilbert space V_a and mappings $p \in [V_a, V]$, $r \in [V, V_a]$ such that $rp = I$ on V_a, define an iterative method

$$rx_k + d_k = r\,T(x_k + pd_k)\,, \quad d_k \in V_a\,, \quad x_{k+1} = T(x_k + pd_k)\,, \tag{1.2}$$

which requires the solution of an operator equation in the space V_a. After multiplying the first equation by p and substituting into the second, this iterative method reduces to

$$x_{k+1} = T(x_k + P(x_{k+1} - x_k))\,, \tag{1.3}$$

where $P = pr$ is a projection in V. If the operator r is the adjoint of p, the projection P is orthogonal.

The iterative method (1.3) is essentially a nonlinear splitting for the equation (1.1),

$$x_{k+1} = F(x_{k+1}, x_k),$$

where

$$F(x, y) = T(Px + (I - P)\,y),$$

hence $Tx = F(x, x)$.

Such nonlinear splittings were studied by Wazewski [12] and by Kurpel' [2]. Another splitting was considered by Looze and Sandell [3]. There are relations to the aggregation method of approximate inversion of matrices by Fiedler and Pták [1]: in the linear case $Tx = Ax + f$, $A \in [V]$, the iterative method (1.3) is equivalent to a linear stationary iterative method using a preconditioning operator $(I - AP)^{-1}$; cf. also Pokorná and Prágerová [9].

The closely related iterative aggregation method [8] can be stated in the form (1.3) with P depending on x_k. For relations to multigrid methods see [7].

2. PRELIMINARIES

For Hilbert spaces U and V, $[U, V]$ denotes the space of all bounded linear operators mapping U into V. $[V]$ stands for $[V, V]$. If $T : V \to V$ is a mapping of V into itself, its pseudonorm is defined as the minimal Lipschits constant of T on V,

$$\|T\| = \sup\{\|Tx - T_y\|/\|x - y\|;\ x, y \in V, x \neq y\}.$$

If $T \in [V]$, the pseudonorm of T coincides with the usual norm induced by the norm $\|\cdot\|$ on V.

A linear operator $P \in [V]$ is an orthogonal projection if it is a projection, $P^2 = P$, and P equals its adjoint. Then [10] we have $\|P\| = 1$ and

$$\|Px\|^2 + \|(I - P)\,y\|^2 = \|Px + (I - P)\,y\|^2 \tag{2.1}$$

for all $x, y \in V$.

The Banach principle of contraction will be used in the following form: If $T : V \to V$ and $\|T\| < 1$, then for any $x_0 \in V$ the iterates $x_{k+1} = Tx_k$ converge to the unique solution of the equation $x = Tx$.

3. CONVERGENCE OF ITERATIONS

Theorem 2. *Let $T : V \to V$ and let $P \in [V]$ be a projection. Assume that the equation*

$$w = T(Pw + (I - P)\,z) \tag{3.1}$$

for any $z \in V$ has a unique solution $w \in V$ and following (3.1) define $W : V \to V$ by $Wz = w$. Suppose that

$$\|(I - P)\,W\| = a < 1. \tag{3.2}$$

Then the equation $x = Tx$ has a unique solution x^* and for any sequence of iterates $x_{k+1} = Wx_k$, $x_0 \in V$, we have

$$\|(I - P)(x_{k+1} - x^*)\| \leqq a\| (I - P)(x_k - x^*)\| . \tag{3.3}$$

If W is continuous at x^*, then $x_k \to x^*$.

If in addition $\|W\| = b < +\infty$, then

$$\|x_{k+1} - x^*\| \leqq b\|(I - P)(x_k - x^*)\| . \tag{3.4}$$

Proof. Consider a sequence $x_{k+1} = Wx_k$, $x_0 \in V$, and denote $z_k = (I - P) x_k$. By the definition of W and by (3.1),

$$W(I - P) = W, \tag{3.5}$$

hence $z_{k+1} = (I - P) Wz_k$. By the contraction principle, the assumption (3.2) implies that there exists a unique $z^* \in V$ such that

$$z^* = (I - P) Wz^* , \tag{3.6}$$

and it holds

$$\|z_{k+1} - z^*\| \leqq a\|z_k - z^*\| . \tag{3.7}$$

Denote $x^* = Wz^*$. Then $z^* = (I - P) x^*$ by (3.6), and by (3.5), $x^* = Wz^* = W(I - P) x^* = Wx^*$. From the definition of W we conclude that $x^* = Tx^*$. On the other hand, if $x = Tx$, then $x = Wx$ by the definition of W. Denote $z = (I - P) x$. Then $z = (I - P) W(I - P) x = (I - P) Wz$ as a consequence of (3.5), and since z is the unique solution of the equation (3.6), $z = z^*$. Consequently, $x = W(I - P) x = Wz = Wz^* = x^*$.

The estimate (3.3) follows from the inequality (3.7) and from the equation $z_k - z^* = (I - P)(x_k - x^*)$.

If the mapping W is continuous at the point x^*, then $z_k \to z^*$ implies $x_k \to x^*$, for $x_{k+1} = Wz_k$.

The estimate (3.4) is obtained from (3.5) and (3.3), because $x_{k+1} - x^* = Wx_k - Wx^* = W(I - P) x_k - W(I - P) x^*$.

4. THE NONLINEAR SPLITTING THEOREM

Theorem 3. *Let $P_1, P_2 \in [V]$ be projection operators, P_2 orthogonal, and assume that*

$$P_1P_2 = P_2P_1 = P_1 . \tag{4.1}$$

Let $T: V \to V$ and assume that for any $z \in V$ there exists a unique $w \in V$ such that

$$w = T(P_1w + (I - P_1) z) . \tag{4.2}$$

Following (4.2), *define* $W_1 : V \to V$ *by* $W_1 z = w$ *and assume that*

$$\|(I - P_1) W_1\| = a < 1 . \tag{4.3}$$

Then for any $z \in V$ *the equation*

$$w = T(P_2 w + (I - P_2) z) \tag{4.4}$$

has a unique solution $w \in V$. *Let* $W_2 : V \to V$ *be defined by* $W_2 z = w$ – *cf.* (4.4).

Then there exists a mapping $H : V \to V$ *such that*

$$W_2 = W_1 H, \ \|H\| \leqq (1 - a^2)^{-1/2}, \quad Hx^* = x^* , \tag{4.5}$$

where x^* *is the unique solution of the equation* $x = Tx$, *and the estimate*

$$\|(I - P_2) W_2\| \leqq a < 1 \tag{4.6}$$

holds.

Proof. The assumption (4.1) implies that

$$P_2 = P_2(P_1 + (I - P_1)) = P_1 + (I - P_1) P_2(I - P_1)$$

and

$$I - P_2 = (I - P_1)(I - P_2) .$$

It follows that for any $z, w \in V$,

$$P_2 w + (I - P_2) z = P_1 w + (I - P_1)(P_2(I - P_1) w + (I - P_2) z) .$$

Therefore, the equation (4.4) is equivalent to

$$w = W_1(P_2(I - P_1) w + (I - P_2) z) \tag{4.7}$$

in virtue of the definition of the mapping W_1.

Since P_2 is an orthogonal projection, we have $\|P_2\| = 1$. The equation

$$y = (I - P_1) W_1(P_2 y + (I - P_2) z) \tag{4.8}$$

has a unique solution y for any $z \in V$ by virtue of the Banach contraction principle and the assumption (4.3). Define $Y : V \to V$ by $Yz = y$ – cf. (4.8).

We show that for any $z \in V$ the equation (4.4) possesses the unique solution $w \in V$ determined by

$$w = W_1(P_2 Yz + (I - P_2) z) . \tag{4.9}$$

If (4.9) holds, then by the definition of the mapping Y, $Yz = (I - P_1) w$, and substituting Yz into (4.9) we find that w is a solution of (4.7), hence a solution of (4.4). If $\tilde{w}$ is an arbitrary solution of the equation (4.4), then by (4.7), $\tilde{y} = (I - P_1) \tilde{w}$ satisfies (4.8), hence $\tilde{y} = Yz$. By (4.7), $w = \tilde{w}$.

Let $y_1 = Yz_1$ and $y_2 = Yz_2$. The equation (4.8) implies that

$$\|y_1 - y_2\| \leqq \|(I - P_1) W_1\| \ \|(P_2 y_1 + (I - P_2) z_1) - (P_2 y_2 + (I - P_2) z_2)\| .$$

Using the equation (2.1) with $P = P_2$ and the assumption (4.3), we obtain

$$\|y_1 - y_2\|^2 \leqq a^2(\|y_1 - y_2\|^2 + \|z_1 - z_2\|^2),$$

hence

$$\|Y\| \leqq a(1 - a^2)^{-1/2}. \tag{4.10}$$

Since $W_1 = W_1(I - P_1)$ and $(I - P_1) P_2 P_1 = 0$, the equation (4.9) and the definition of W_2 imply $W_2 = W_1 H$, where

$$Hz = P_2(Yz + P_1 x^*) + (I - P_2) z, \tag{4.11}$$

where x^* is the unique solution of the equation $x = Tx$, the existence and uniqueness of which is implied by Theorem 2 and the assumption (4.3).

Since $Yx^* = (I - P_1) x^*$ in virtue of (4.8), the definition of Y and $W_1 x^* = x^*$, we conclude that $Hx^* = x^*$.

Let $u_j = Hz_j$, $y_j = Yz_j$, $j = 1, 2$. Since P_2 is an orthogonal projection, we get by (2.1), (4.11) and $\|P_2\| = 1$,

$$\|u_1 - u_2\|^2 \leqq \|y_1 - y_2\|^2 + \|z_1 - z_2\|^2,$$

and using the inequality (4.10),

$$\|u_1 - u_2\|^2 \leqq (a^2(1 - a^2)^{-1} + 1)\| z_1 - z_2\|,$$

which implies that

$$\|H\| \leqq (a^2(1 - a^2)^{-1} + 1)^{1/2} = (1 - a^2)^{-1/2},$$

and concludes the proof of the proposition (4.5).

It remains to prove the estimate (4.6). Let $w_j = W_2 z_j$, $j = 1, 2$. The pairs (w_j, z_j) satisfy the equation (4.7) and applying $I - P_1$ to (4.7) we get for $j = 1, 2$,

$$(I - P_1) w_j = (I - P_1) W_1(P_2(I - P_1) w_j + (I - P_2) z_j). \tag{4.12}$$

Since the assumption (4.1) implies that

$$(I - P_1) = P_2(I - P_1) + (I - P_2),$$

the left hand side of the equation (4.12) can be written as

$$(I - P_1) w_j = P_2(I - P_1) w_j + (I - P_2) w_j.$$

Using (2.1) we obtain from (4.12) and the assumption (4.3)

$$\|P_2(I - P_1)(w_1 - w_2)\|^2 + \|(I - P_2)(w_1 - w_2)\|^2 \leqq$$
$$\leqq a^2(\|P_2(I - P_1)(w_1 - w_2)\|^2 + \|(I - P_2)(z_1 - z_2)\|^2).$$

Since $a < 1$, it follows that

$$\|(I - P_2)(w_1 - w_2)\|^2 \leqq a^2\|(I - P_2)(z_1 - z_2)\|^2.$$

Taking into account that $\|I - P_2\| = 1$, we obtain the estimate (4.6).

The following corollaries are obtained by combining the results of Theorem 2 and Theorem 3.

Corollary 1. *Let the assumptions of Theorem 2 hold and let W_1 be continuous at the point x^*. Then W_2 is continuous at x^* and for any $x_0 \in V$ the iterations*

$$x_{k+1} = T(P_2 x_{k+1} + (I - P_2)\, x_k) \tag{4.13}$$

converge to x^*.

Corollary 2. *Let the assumptions of Theorem 2 hold and let $\|W_1\| = b < +\infty$. Then the estimate*

$$\|x_{k+1} - x^*\| \leqq a^k\, b(1 - a^2)^{-1/2}\, \|x_0 - x^*\| \tag{4.14}$$

holds for the iterations (4.13).

Corollary 3. *Let $\|T\| = a < 1$ and let P_2 be an orthogonal projection. Then the iterations* (4.13) *converge to x^* and the inequality* (4.14) *holds with $b = a$.*

Proof. Use Theorem 3 for $P_1 = 0$, $W_1 = T$, and Corollary 2.

Corollary 4. *Let $A \in [V]$, $\|A\| = a < 1$. Denote*

$$W(P) = A(I - PA)^{-1}\,(I - P)$$

for an orthogonal projection $P \in [V]$. Then for any orthogonal projection P,

$$r(W(P)) \leqq \|(I - P)\, W(P)\| \leqq a\,,$$

where r denotes the spectral radius.

5. CONCLUDING REMARKS

It is easily seen that if the equation (4.2) or (4.4) possesses a unique solution, so does the respective correction equation (1.2) under the assumptions stated in the introduction.

The correction equation (1.2) is usually solved only approximately, which gives rise to further problems, see e.g. [3], where this question was tackled for another splitting, and also [2, 4], where a number of examples and applications of the present method can be found.

Theorem 3 yields a comparison of estimates of the rate of convergence similarly as the classical theorems about block iterative methods and their generalizations [11].

Corollary 4 was used in a local convergence proof for the iterative aggregation method [8]. For its extensions see [7].

The results presented here are contained in the author's thesis [6]. Theorem 3 extends a similar result by Lučka [5, Lemma 4.3], which was brought to our attention in the proofstage.

References

[1] *M. Fiedler, V. Pták:* On aggregation in matrix theory and its applications to numerical inverting of large matrices. Bull. Acad. Pol. Sci. Math. Astr. Phys. 11 (1963) 757—759.

[2] *N. S. Kurpel':* Projection-iterative Methods of Solution of Operator Equations (Russian). Naukova Dumka, Kiev 1968.

[3] *D. P. Looze, N. R. Sandell, Jr.:* Analysis of decomposition algorithms via nonlinear splitting functions. J. Optim. Theory Appl. 34 (1981) 371—382.

[4] *A. Ju. Lučka:* Projection-iterative Methods of Solution of Differential and Integral Equations (Russian). Naukova Dumka, Kiev 1980.

[5] *A. Ju. Lučka*: Convergence criteria of the projection-iterative method for nonlinear equations (Russian). Preprint 82.24, Institute of Mathematics AN Ukrain. SSR, Kiev 1982.

[6] *J. Mandel*: Convergence of some two-level iterative methods (Czech). PhD Thesis, Charles University, Prague 1982.

[7] *J. Mandel*: On some two-level iterative methods. In: Defect Correction - Theory and Applications (K. Böhmer, H. J. Stetter. editors), Computing Supplementum Vol. 5, Springer-Verlag, Wien, to appear.

[8] *J. Mandel, B. Sekerka:* A local convergence proof for the iterative aggregation method. Linear Algebra Appl. 51 (1983), 163—172.

[9] *O. Pokorná, I. Prágerová*: Approximate matrix invertion by aggregation. In: Numerical Methods of Approximation Theory, Vol. 6 (L. Collatz, G. Meinhardus, H. Werner, editors), Birghäuser Verlag, Basel—Boston—Stuttgart 1982.

[10] *A. E. Taylor:* Introduction to Functional Analysis. J. Wiley Publ., New York 1958.

[11] *R. S. Varga:* Matrix Iterative Analysis. Prentice Hall Inc., Englewood Cliffs, New Jersey 1962.

[12] *T. Wazewski:* Sur une procédé de prouver la convergence des approximations successives sans utilisation des séries de comparaison. Bull. Acad. Pol. Sci. Math. Astr. Phys. 8 (1960) 47—52.

Souhrn

KONVERGENTNÍ NELINEÁRNÍ ROZŠTĚPENÍ POMOCÍ ORTOGONÁLNÍ PROJEKCE

JAN MANDEL

V práci se studuje konvergence iterací v Hilbertově prostoru V

$$x_{k+1} = W(P)\, x_k\,, \quad W(P)\, z = w = T(Pw + (I - P)\, z)\,,$$

kde T zobrazuje V do sebe a P je projekce. Iterace konvergují k jedinému řešení rovnice $x = Tx$, jestliže operátor $W(P)$ je spojitý a Lischitzova konstanta zobrazení $(I - P)\, W(P)$ je menší než jedna. Ukazuje se, že tyto podmínky jsou splněny, jestliže

T je kontrakce v normě a projekce P je ortogonální. Splňuje-li operátor $W(P_1)$ výše uvedené předpoklady a P_2 je ortogonální projekce taková, že $P_1P_2 = P_2P_1 = P_1$, pak je operátor $W(P_2)$ definován a rovněž splňuje tyto předpoklady.

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