Fazlollah M. Reza

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A MULTIPLICATION THEOREM FOR TWO-VARIABLE POSITIVE REAL MATRICES

FAZLOLLAH M. REZA

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1. INTRODUCTION

The theory of electrical networks relies heavily on the use of a Positive Real Function $Z(s)$ of the complex frequency s. The mathematical properties of these functions along with a variety of generating theorems (analysis), and decomposition methods (synthesis) have been well developed by circuit theorists in the second quarter of this century.

A function $Z(s)$ is said to be PRF (short for Positive Real Function) if it satisfies the requirements (a) and (b):

(a) $Z(s)$ is real for real s,

(b) $\operatorname{Re} Z(s) \geqq 0$ for $\operatorname{Re} s \geqq 0$,

and $Z(s)$ is analytic within the r.h.p.; its singularities on the imaginary axis are simple poles with positive residues. The electrical networks $\{N\}$ with the PRF characteristics consist of ordinary linear reciprocal passive resistors, inductors and capacitors. The corresponding impedance functions for these elementary building blocs are, respectively, $\{R, Ls, (Cs)^{-1}\}$. In the terminology of electrical networks $Z(s)$ is known as an impedance and $[Z(s)]^{-1}$ as an admittance function. The condition (a) is imposed because of the fact that the assigned values R, L, and C are real numbers representing real physical entities. The condition (b) implies that the family $\{N\}$ is a passive system, that is, for $\operatorname{Re} s \geqq 0$ they absorb electrical energy, $\operatorname{Re} Z(s) \geqq 0$. The family $\{N\}$ described above is generally referred to as the family of linear reciprocal passive electrical systems. When such a system is under the application of electrical sources at n points of entries, then the system will be referred to as an n-port. The study of passive n-ports implies that $[Z(s)]$ is a PR matrix for $\operatorname{Re} s \geqq 0$, i.e. ${}^t x\, Z(s)\, x$ is a PRF for any constant vector x.

There has been a considerable amount of interplay between the theory of electrical networks and the theory of functions of a complex variable. Results of mutual interest to the function theorists and circuit theorists are abundant, particularly in the literature of the circuit theory from 1925–1975.

In the third quarter of this century the design of multi-dimensional filters along with some other technological advancements have brought forth the extension of the theory of PRF to multi-frequency cases, that is, $Z(s_1, s_2, \ldots, s_n)$. Accordingly, in the past two decades we have seen a noticeable growth toward the study of multi-dimensional electrical networks. References to this subject may be found in [1]. But, in spite of the recent explosion of publications in this area, our in-depth knowledge of the field is rather limited. Perhaps a natural limitation is induced by the time gap necessary for absorbing and applying the more recent mathematical results of the functions of several complex variables such as those referred to in [2] and [3].

This note is motivated by the need for constructing PRF in several variables as advocated by the authors of [4], [5] and [6]. The result reported here is hoped to be an interesting, and a useful, multiplication-division theorem.

A simple frequency transformation is reviewed in Section 2. In Section 3, a basic multiplication-division theorem is presented. The theorem is further generalized in Section 4 to arbitrary two-variable n-ports, which embraces a generalization of the results of [4].

2. A TWO-VARIABLE FREQUENCY TRANSFORMATION

The following two-variable frequency transformation from the $\{p_1, p_2\}$-plane to the $\{s_1, s_2\}$-plane is quite useful in the study of two-variable systems, [4]:

$$(1) \qquad \begin{aligned} s_1 &= \sqrt{(p_1 \, p_2)}\,, \qquad & p_1 &= s_1 \, s_2\,, \\ s_2 &= \sqrt{(p_1/p_2)}\,, \qquad & p_2 &= s_1/s_2\,. \end{aligned}$$

The simplicity of this bilateral transformation follows from the fact that the proper choice of the square root signs in (1), based on analytic continuation of the incurring variables, leads to a mapping correspondence between the regions M and N:

$$\{M\colon \operatorname{Re} p_1 > 0,\ \operatorname{Re} p_2 > 0\} \to \{N\colon \operatorname{Re} s_1 > 0,\ \operatorname{Re} s_2 > 0\}\,.$$

In fact,

$$(2a) \qquad \operatorname{Re} p_k \geqq 0 \to \left|\operatorname{Arg} \sqrt{p_k}\right| \leqq \frac{\pi}{4}\,; \quad k = 1, 2\,.$$

These conditions in view of (1) imply

$$(2b) \qquad \left|\operatorname{Arg} s_k\right| \leqq \frac{\pi}{2}\,, \quad k = 1, 2\,.$$

Moreover, for $\operatorname{Re} p_1 = \operatorname{Re} p_2 = 0$ we observe:

"For any set of real pair $\{\omega_1, \omega_2\}$ the transformation of equation (1), $\{p_1 = j\omega_1,\ p_2 = j\omega_2\}$ leads to a pair $\{s_1, s_2\}$ where one of the two variables is a real and the other a pure imaginary number."

The validity of this simple observation becomes evident by looking at the table below compiled for $\{p_1 = j\omega_1, p_2 = j\omega_2\}$.

$p_1 = j\omega_1$ \ $p_2 = j\omega_2$	$\omega_2 > 0$	$\omega_2 < 0$
$\omega_1 > 0$	$\text{Re } s_1 = 0$ $\text{Im } s_2 = 0$	$\text{Im } s_1 = 0$ $\text{Re } s_2 = 0$
$\omega_1 < 0$	$\text{Im } s_1 = 0$ $\text{Re } s_2 = 0$	$\text{Re } s_1 = 0$ $\text{Im } s_2 = 0$

This mapping information will be directly applied for proving the main theorem of Section 3.

3. TWO-VARIABLE MULTIPLICATION-DIVISION THEOREM

Theorem 1. *Let $Z_1(s_1)$ and $Z_2(s_2)$ be two driving-point impedance functions (PRF). Then the two-variable functions (3) and (4) are driving-point impedance functions (PRF) in $\{p_1, p_2\}$,*

(3) $$Z(p_1, p_2) = Z_1(\sqrt{(p_1 p_2)}) \,.\, Z_2(\sqrt{(p_1/p_2)})\,,$$

(4) $$Z(p_1, p_2) = Z_1(\sqrt{(p_1 p_2)}) \,/\, Z_2(\sqrt{(p_1/p_2)})\,.$$

These functions represent linear reciprocal passive two-variable networks which are not necessarily rational systems.

Proof. (a) All positive real values of $\{s_1, s_2\}$ lead to positive real values of $\{p_1, p_2\}$ and $Z(p_1, p_2)$.

(b) $Z(p_1, p_2)$ remains analytic inside the region M due to the analyticity of $Z_1(s_1)$ and $Z_2(s_2)$ inside the region N.

(c) For an arbitrary set of real pairs $\{\sigma_1, \sigma_2\}$, let

(5) $$Z_1(j\sigma_1) = R_1(\sigma_1) + j\,X_1(\sigma_1)\,,$$

(6) $$Z_2(j\sigma_2) = R_2(\sigma_1) + j\,X_2(\sigma_2)\,.$$

In each of the four cases of the table above, we calculate $\text{Re } Z(j\omega_1, j\omega_2)$. For the product function one finds

1. $\omega_1\omega_2 > 0$; $s_1 = j\sigma_1$, $s_2 = j\sigma_2$,

(7) $$\text{Re } Z(j\omega_1, j\omega_2) = R_1(\sigma_1)\, Z_2(\sigma_2) \geqq 0\,.$$

Note that $Z(p_1, p_2)$ derived in (3) or (4) is not an arbitrary PRF, but a function constrained within the class of PRF restricted by transformation (1).

2. $\omega_1\omega_2 < 0$; $s_1 = \sigma_1$, $s_2 = j\sigma_2$,

$$\operatorname{Re} Z(j\omega_1, j\omega_2) = Z_1(\sigma_1)\, R_2(\sigma_2) \geqq 0\,. \tag{8}$$

Thus the functions (3) and (4) are two-variable PRF. These functions are not necessarily rational. In the particular case of reactive Z_1 and Z_2, that is $\operatorname{Re} Z_1(j\sigma_1) = \operatorname{Re} Z_2(j\sigma_2) \equiv 0$, we find:

Corollary 1. *Let $Z_1(s_1)$ and $Z_2(s_2)$ be two driving-point reactance functions, then $Z(p_1, p_2)$ represented by (3) or (4) is a rational two-variable reactance function.*

Proof. Equations (7) and (8) imply that for any real pair $\{\omega_1, \omega_2\}$,

$$\operatorname{Re} Z(j\omega_1, j\omega_2) \equiv 0\,. \tag{9}$$

To show the rationality of $Z(p_1, p_2)$ we may examine the familiar Foster type expansion of Z_1 and Z_2. The rationality of an arbitrary typical term becomes now evident for their product or their ratio:

$$\frac{A_k s_1}{s_1^2 + a_k^2}\,\frac{B_i s_2}{s_2^2 + b_i^2} = \frac{A_k B_i p_1}{p_1^2 + a_k^2(p_1/p_2) + b_i^2 p_1 p_2 + a_k^2 b_i^2} \tag{10}$$

4. GENERALIZATION OF TWO-VARIABLE PR MATRICES (MULTIPORTS)

Theorem 2. *Let $[Z_1(s_1)]$ and $[Z_2(s_2)]$ be two $n \times n$ PR matrices representing two linear reciprocal passive (time-invariant) n-ports. The following two matrix functions are admissible two-variable PR matrices representing linear reciprocal passive n-ports:*

$$2[Z(p_1, p_2)] = [Z_1(\sqrt{(p_1 p_2)})]\,[Z_2(\sqrt{(p_1/p_2)})] + [Z_2(\sqrt{(p_1/p_2)})]\,[Z_1(\sqrt{(p_1 p_2)})] \tag{11}$$

and

$$2[Z(p_1, p_2)] = [Z_1(\sqrt{(p_1 p_2)})]\,[Z_2(\sqrt{(p_1/p_2)})]^{-1} + [Z_2(\sqrt{(p_1/p_2)})]^{-1}\,[Z_1(\sqrt{(p_1 p_2)})]\,, \tag{12}$$

provided Z_1 and Z_2 are invertible.

Proof. The proof is analogous to that of Theorem 1. For part (c), note that in contrast with the scalar case, the incurring product matrices in (13) are not necessarily PR matrices:

$$[R_1(\sigma_1)]\,[Z_2(\sigma_2)]\,, \quad [Z_1(\sigma_1)]\,[R_2(\sigma_2)]\,. \tag{13}$$

The product of two symmetric non-negative matrices need not be a symmetric matrix. The difficulty may be overcome either by assuming commutativity of $[Z_1]$

and $[Z_2]$, or introducing the product "$Z_1Z_2 + Z_2Z_1$" as proposed in [7]. In the latter case one finds a PR matrix in stead of (7).

$$(14) \qquad [R_1(\sigma_1)]\,[Z_2(\sigma_2)] + [Z_2(\sigma_2)]\,[R_1(\sigma_1)] \geqq 0\,.$$

Evidently the matrices of (11) and (12) are symmetric. In fact,

$$(15) \qquad [Z_1Z_2 + Z_2Z_1]^t = [Z_2^tZ_1^t + Z_1^tZ_2^t] = [Z_2Z_1 + Z_1Z_2]$$

in view of the reciprocity assumption (t stands for transpose). The validity of equation (12) follows by replacing Z_2 by Z_2^{-1} in (15).

Corollary 2. *Same as Corollary* 1 *for two-variable rational n-ports.*

5. FURTHER GENERALIZATIONS AND COMMENTS

Extensions of the main theorem of this paper in several directions are aparent. The material of Section 4 already indicates how the theorem can be generalized for multiports, that is, PRF matrices of two variables $\{s_1, s_2\}$. Moreover, according to [7], if inductive (RL) and capacitive (RC) linear reciprocal passive systems are represented via their PR impedance matrices

$$\{Z_{1L}(s_1),\ Z_{1C}(s_1),\ Z_{2L}(s_2),\ Z_{2C}(s_2)\}$$

then

$$\{Z_{1L}(s_1)\,.\,Z_{1C}(s_1) + Z_{1C}(s_1)\,Z_{1L}(s_1)\}$$

is a PR matrix. Thus, according to our main results, the matrix $Z(p_1, p_2)$ is a PR matrix representing a multiport,

$$(16) \qquad Z(p_1, p_2) = Z_{1L}(\sqrt{(p_1p_2)})\,Z_{1C}(\sqrt{(p_1p_2)})\,Z_{2L}(\sqrt{(p_1/p_2)})\,Z_{2C}(\sqrt{(p_1/p_2)})\,.$$

The question has been raised as to the generalization of our main theorem to n-dimensional multiports, that is, PRF matrices of complex variables $\{s_1, s_2, \ldots, s_n\}$. Such a general result invites further investigation. Dr. J. Gregor of the Technical University of Prague has communicated in a private correspondence to the present author the possibility of such a generalization. His approach employs the well-known argument property of PRF, that is,

$$(17) \quad |\arg [Z_1(\sqrt{(p_1p_2)})\,.\,Z_2(\sqrt{(p_1/p_2)})]| = |\arg Z_1(\sqrt{(p_1p_2)}) + \arg Z_2(\sqrt{(p_1/p_2)})|\,,$$

$$(18) \quad |\arg Z_1(\sqrt{(p_1p_2)})| + |\arg Z_2(\sqrt{(p_1/p_2)})| = |\arg(\sqrt{(p_1p_2)})| + |\arg(\sqrt{(p_1, p_2)})|\,.$$

This latter method (J. Gregor) makes it also possible to use transformations replacing

(1) for three or more variables. For three variables one may write

$$p_1 = s_2 s_3 , \qquad p_2 = s_1 s_3 , \qquad p_3 = s_1 s_2 , \tag{19}$$

$$s_1 = \sqrt{(p_2 p_3/p_1)} , \quad s_2 = \sqrt{(p_1 p_3/p_2)} , \quad s_3 = \sqrt{(p_1 p_2/p_3)} , \tag{20}$$

$\operatorname{Re} s_i > 0 \; i = 1, 2, 3$ if $\operatorname{Re} p_i > 0 \; i = 1, 2, 3$.

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Souhrn

MULTIPLIKAČNÍ VĚTA PRO POSITIVNĚ REÁLNÉ MATICE DVOU PROMĚNNÝCH

FAZLOLLAH M. REZA

Je formulována věta o součinu a podílu dvou positivně reálných (PR) funkcí dvou proměnných. Věta je zobecněna na PR matice, jejichž prvky jsou funkce dvou proměnných. PR funkce a matice vystupují často při studiu elektrických n-pólů a vícedimensionálních systémů (včetně digitálních filtrů).

Author's address: Prof. Dr. *Fazlollah M. Reza,* Department of Electrical Engineering, Concordia University and Mc Gill University, Montreal, Quebec H 3 G 1 M 8, Canada.