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_Aplikace matematiky_, Vol. 31 (1986), No. 6, 417–419

Persistent URL: [http://dml.cz/dmlcz/104220](http://dml.cz/dmlcz/104220)

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NON-POLYCONVEXITY OF THE STORED ENERGY FUNCTION
OF A SAINT VENANT-KIRCHHOFF MATERIAL

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(Received October 15, 1984)

Summary. A direct proof of the non-polyconvexity of the stored energy function of a Saint Venant-Kirchhoff material is given by means of a simple counter-example.

Keywords: polyconvexity, stored energy function, Saint Venant-Kirchhoff material.

In his famous paper [1] dealing with existence theorems in nonlinear elasticity, John Ball introduced the notion of polyconvexity and proved the existence of an equilibrium state — understood as a minimizer of the total energy function — for hyperelastic materials whose stored energy function is polyconvex, subjected to conservative applied forces. It is well known, for instance, that Ogden’s materials are polyconvex materials [2], but as far as we know there has been no direct proof of the non-polyconvexity of the usual Saint Venant-Kirchhoff model. The purpose of this note is to provide such a direct proof by constructing an easy counter-example. However, there exists an “indirect” proof, where the non-polyconvexity is a consequence of the non weak lower semi-continuity of the associated functional (cf. Nečas [4]).

Let $M^3$ be the set of real matrices of order 3 and let $M^3_+$ be the subset of matrices with determinant $> 0$. Let us recall [1], [2] that a real-valued function $W$ defined on $M^3_+$ is polyconvex if and only if there exists a convex function $g$ defined on $M^3 \times M^3 \times \mathbb{R}^+$ such that

$$
\forall F \in M^3_+, \quad W(F) = g(F, \text{adj } F, \text{det } F)
$$

where $\text{adj } F = \text{det } F(F^{-1})$. Notice [1], [3] that $M^3 \times M^3 \times \mathbb{R}^+$ coincides with the convex hull of $\{(F, \text{adj } F, \text{det } F), F \in M^3_+\}$. The stored energy function of a Saint Venant-Kirchhoff material with Lamé’s coefficients $\lambda$ and $\mu$ is

$$
W(F) = a_1 \text{ tr } (F^T F) + a_2 \text{ tr } (F^T F)^2 + b_1 \text{ tr } \text{adj } (F^T F)
$$
where
\[
(3) \quad a_1 = - \frac{3\lambda + 2\mu}{4}, \quad a_2 = \frac{\lambda + 2\mu}{8}, \quad b_1 = \frac{\lambda}{4}.
\]

We want to decide whether \( W \) is polyconvex or not. It is well-known that for physical reasons \( \lambda \) and \( \mu \) are positive; therefore the first coefficient in \( W \) is negative and this is the first indication that \( W \) need not be polyconvex (note that if all coefficients were nonnegative polyconvexity would be immediate \([1], [3]\)).

**Theorem.** \( W \) is not polyconvex.

**Proof.** Let us construct a counter-example. Let \( \varepsilon \) be a positive number, and let \( F \) and \( F' \) be the following elements of \( M^3_+ \):
\[
F = \varepsilon I, \quad F' = \varepsilon \text{ diag}(1, 1, 3).
\]
One immediately obtains
\[
\det F = \varepsilon^3, \quad \text{adj} F = \varepsilon^2 I, \quad \det F' = 3\varepsilon^3, \quad \text{adj} F' = \varepsilon^2 \text{ diag}(3, 3, 1),
\]
\[
\frac{F + F'}{2} = \varepsilon \text{ diag}(1, 1, 2),
\]
\[
\frac{\det F + \det F'}{2} = 2\varepsilon^3, \quad \frac{\text{adj} F + \text{adj} F'}{2} = \varepsilon^2 \text{ diag}(2, 2, 1),
\]
so that the following relations are satisfied, (of course, they do not hold for arbitrary \( F \) and \( F' \) in \( M^3_+ \)):
\[
(4) \quad \frac{F + F'}{2} \in M^3_+, \quad \frac{\text{adj} F + \text{adj} F'}{2} = \frac{\text{adj} F + \text{adj} F'}{2}, \quad \frac{\det F + \det F'}{2} = \frac{\det F + \det F'}{2}.
\]
If \( W \) were polyconvex, equations (1) and (4) would lead to
\[
(5) \quad W\left(\frac{F + F'}{2}\right) \leq \frac{1}{2}(W(F) + W(F')).
\]
For the sake of brevity, let us write
\[
F^TF = \varepsilon^2 I, \quad F'^TF' = \varepsilon^2 J, \quad \left(\frac{F + F'}{2}\right)^T \left(\frac{F + F'}{2}\right) = \varepsilon^2 K
\]
with \( J = \text{ diag}(1, 1, 9), \quad K = \text{ diag}(1, 1, 4) \).

Then using expression (2), where the first term is homogeneous of degree 1 and the remaining terms are homogeneous of degree 2 with respect to \( F^TF \), we derive from
inequality (5)
\[ a_1 \text{ tr } K e^2 + (a_2 \text{ tr } K^2 + b_1 \text{ tr } \text{adj } K) e^4 \leq \]
\[ \frac{1}{2}(a_1(\text{tr } I + \text{tr } J) e^2 + (a_2(\text{tr } I + \text{tr } J^2) + b_1(\text{tr } I + \text{tr } \text{adj } J)) e^4) \]
and this inequality amounts to
\[ a_1 + (25a_2 + 2b_1) e^2 \geq 0 \]
which (recall that \( a_1 \) is negative) cannot be true for \( e \) small enough.

References


Souhrn

NEPOLYKONVEXITA FUNKCE VNITŘNÍ ENERGIE
SAINT VENANTOVA-KIRCHHOFFOVA MATERIÁLU

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Je podán protipříklad dokazující, že funkce vnitřní energie Saint Venantova-Kirchhoffova materiálu není polikonvexní.

Резюме

НЕ-ПОЛИВЫПУКЛОСТЬ ВНУТРЕННЕЙ ФУНКЦИИ МАТЕРИАЛА СЕН ВЭНАН-
-КИРХГОФА

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Дается пример материала Сен Вэнан-Кирхгофа, функция внутренней энергии которого
не является поливыпуклой.

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