

Annie Raoult

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Aplikace matematiky, Vol. 31 (1986), No. 6, 417--419

Persistent URL: <http://dml.cz/dmlcz/104220>

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NON-POLYCONVEXITY OF THE STORED ENERGY FUNCTION
OF A SAINT VENANT-KIRCHHOFF MATERIAL

ANNIE RAOULT

(Received October 15, 1984)

Summary. A direct proof of the non-polyconvexity of the stored energy function of a Saint Venant-Kirchhoff material is given by means of a simple counter-example.

Keywords: polyconvexity, stored energy function, Saint Venant-Kirchhoff material.

In his famous paper [1] dealing with existence theorems in nonlinear elasticity, John Ball introduced the notion of polyconvexity and proved the existence of an equilibrium state — understood as a minimizer of the total energy function — for hyperelastic materials whose stored energy function is polyconvex, subjected to conservative applied forces. It is well known, for instance, that Ogden's materials are polyconvex materials [2], but as far as we know there has been no direct proof of the non-polyconvexity of the usual Saint Venant-Kirchhoff model. The purpose of this note is to provide such a direct proof by constructing an easy counter-example. However, there exists an “indirect” proof, where the non-polyconvexity is a consequence of the non weak lower semi-continuity of the associated functional (cf. Nečas [4]).

Let M^3 be the set of real matrices of order 3 and let M_+^3 be the subset of matrices with determinant > 0 . Let us recall [1], [2] that a real-valued function W defined on M_+^3 is polyconvex if and only if there exists a convex function g defined on $M^3 \times M^3 \times \mathbb{R}^{+*}$ such that

$$(1) \quad \forall F \in M_+^3, \quad W(F) = g(F, \text{adj } F, \det F)$$

where $\text{adj } F = \det F(F^{-1})$. Notice [1], [3] that $M^3 \times M^3 \times \mathbb{R}^{+*}$ coincides with the convex hull of $\{(F, \text{adj } F, \det F), F \in M_+^3\}$. The stored energy function of a Saint Venant-Kirchhoff material with Lamé's coefficients λ and μ is

$$(2) \quad W(F) = a_1 \text{tr}(F^T F) + a_2 \text{tr}(F^T F)^2 + b_1 \text{tr}(\text{adj}(F^T F))$$

where

$$(3) \quad a_1 = -\frac{3\lambda + 2\mu}{4}, \quad a_2 = \frac{\lambda + 2\mu}{8}, \quad b_1 = \frac{\lambda}{4}.$$

We want to decide whether W is polyconvex or not. It is well-known that for physical reasons λ and μ are positive; therefore the first coefficient in W is negative and this is the first indication that W need not be polyconvex (note that if all coefficients were nonnegative polyconvexity would be immediate [1], [3]).

Theorem. W is not polyconvex.

Proof. Let us construct a counter-example. Let ε be a positive number, and let F and F' be the following elements of M_+^3 :

$$F = \varepsilon I, \quad F' = \varepsilon \operatorname{diag}(1, 1, 3).$$

One immediately obtains

$$\det F = \varepsilon^3, \quad \operatorname{adj} F = \varepsilon^2 I, \quad \det F' = 3\varepsilon^3, \quad \operatorname{adj} F' = \varepsilon^2 \operatorname{diag}(3, 3, 1),$$

$$\frac{F + F'}{2} = \varepsilon \operatorname{diag}(1, 1, 2),$$

$$\det \frac{F + F'}{2} = 2\varepsilon^3, \quad \operatorname{adj} \frac{F + F'}{2} = \varepsilon^2 \operatorname{diag}(2, 2, 1),$$

so that the following relations are satisfied, (of course, they do not hold for arbitrary F and F' in M_+^3):

$$(4) \quad \frac{F + F'}{2} \in M_+^3, \quad \operatorname{adj} \frac{F + F'}{2} = \frac{\operatorname{adj} F + \operatorname{adj} F'}{2}, \quad \det \frac{F + F'}{2} = \frac{\det F + \det F'}{2}.$$

If W were polyconvex, equations (1) and (4) would lead to

$$(5) \quad W\left(\frac{F + F'}{2}\right) \leq \frac{1}{2}(W(F) + W(F')).$$

For the sake of brevity, let us write

$$F^T F = \varepsilon^2 I, \quad F'^T F' = \varepsilon^2 J, \quad \left(\frac{F + F'}{2}\right)^T \left(\frac{F + F'}{2}\right) = \varepsilon^2 K$$

with $J = \operatorname{diag}(1, 1, 9)$, $K = \operatorname{diag}(1, 1, 4)$.

Then using expression (2), where the first term is homogeneous of degree 1 and the remaining terms are homogeneous of degree 2 with respect to $F^T F$, we derive from

inequality (5)

$$a_1 \operatorname{tr} K \varepsilon^2 + (a_2 \operatorname{tr} K^2 + b_1 \operatorname{tr} \operatorname{adj} K) \varepsilon^4 \leq \\ \frac{1}{2}(a_1(\operatorname{tr} I + \operatorname{tr} J) \varepsilon^2 + (a_2(\operatorname{tr} I + \operatorname{tr} J^2) + b_1(\operatorname{tr} I + \operatorname{tr} \operatorname{adj} J)) \varepsilon^4)$$

and this inequality amounts to

$$a_1 + (25a_2 + 2b_1) \varepsilon^2 \geq 0$$

which (recall that a_1 is negative) cannot be true for ε small enough. \square

References

- [1] *J. Ball*: Convexity conditions and existence theorems in nonlinear elasticity. Arch. Rat. Mech. Anal. 63 (1977), p. 337–403.
- [2] *P. G. Ciarlet*: Lectures on three-dimensional elasticity. Tata Institute Lecture Notes, Springer-Verlag, 1983.
- [3] *P. G. Ciarlet*: Topics in mathematical elasticity, vol. I. North-Holland, Amsterdam, 1985.
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Souhrn

NEPOLYKONVEXITA FUNKCE VNITŘNÍ ENERGIE SAINT VENANTOVA-KIRCHHOFFOVA MATERIÁLU

ANNIE RAOULT

Je podán protipříklad dokazující, že funkce vnitřní energie Saint Venantova-Kirchhoffova materiálu není polykonvexní.

Резюме

НЕ-ПОЛИВЫПУКЛОСТЬ ВНУТРЕННЕЙ ФУНКЦИИ МАТЕРИАЛА СЕН ВЭНАН-КИРХГОФА

ANNIE RAOULT

Дается пример материала Сен Вэнан-Кирхгофа, функция внутренней энергии которого не является поливыпуклой.

Author's address: Prof. Annie Raoult, Laboratoire d'Analyse Numérique, Université P. et M. Curie, 4 Place Jussieu, 75230 Paris cedex 05, France.