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Aplikace matematiky, Vol. 31 (1986), No. 6, 480--485

Persistent URL: http://dml.cz/dmlcz/104225

31 (1986) APLIKACE MATEMATIKY No. 6, 480—485

ON INTERPOLATION IN PERIODIC AUTOREGRESSIVE PROCESSES

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(Received October 15, 1985)

Summary. The periodic autoregressive processes are useful in statistical analysis of seasonal time series. Some procedures (e.g. extrapolation) are quite analogous to those in the classical autoregressive models. The problem of interpolation needs, however, some special methods. They are demonstrated in the paper on the case of the process of the second order with the period of length 2.

Keywords: covariance function, interpolation, multivariate AR(1) process, periodic autoregressive process, projection.

AMS subject classificatiin: 62M10.

1. INTRODUCTION

The periodic autoregressive process $\{X_t\}$ of the nth order with a period p is given by the relation

$$X_{n+(j-1)p+k} = \sum_{i=1}^{n} b_{ki} X_{n+(j-1)p+k-i} + \eta_{n+(j-1)p+k} \quad (k = 1, \dots, p), \tag{1.1}$$

where b_{ki} are given constants and η_t are uncorrelated random variables with vanishing mean such that $\operatorname{Var} \eta_{n+(j-1)p+k} = \sigma_k^2$. We shall assume that $0 < \sigma_k^2 < \infty$ $(k = 1, \dots, p)$. If the variables $X_1, \dots, X_n$ are given, then the relation (1.1) is considered for $p = 1, 2, \dots$. But under certain conditions, which will be briefly discussed later on, it is possible to consider the process $\{X_t\}_{t=-\infty}^{\infty}$ similarly as in the stationary autoregressive models.

The history of the periodic autoregressive processes is described in [2]. We only remind that important results about the structure of the periodic autoregressive processes and about asymptotic properties of some estimators were derived by Pagano in [6]. Statistical analysis of the periodic autoregressive processes based on the Bayes approach is described in [2] and [3].

In the special case when $n = 2$, $p = 2$ the model (1.1) can be written in the form

$$\begin{aligned} X_{2t+1} &= b_{11} X_{2t} + b_{12} X_{2t-1} + \eta_{2t+1}, \\ X_{2t+2} &= b_{21} X_{2t+1} + b_{22} X_{2t} + \eta_{2t+2}. \end{aligned} \tag{1.2}$$

It follows from the general results given in [6] that any periodic autoregressive model can be transformed into classical multidimensional autoregressive model. In the special case (1.2) we introduce random vectors

$$\mathbf{Z}_t = (X_{2t-1}, X_{2t})' , \quad \mathbf{Y}_t = (\eta_{2t-1}, \eta_{2t})'$$

and matrices

$$\mathbf{B}_0 = \begin{pmatrix} 1, & 0 \\ -b_{21}, & 1 \end{pmatrix}, \quad \mathbf{B}_1 = \begin{pmatrix} -b_{12}, & -b_{11} \\ 0, & -b_{22} \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} \sigma_1, & 0 \\ 0, & \sigma_2 \end{pmatrix}.$$

Then (1.2) can be expressed as

$$\mathbf{B}_0\mathbf{Z}_{t+1} + \mathbf{B}_1\mathbf{Z}_t = \mathbf{Y}_{t+1} \tag{1.3}$$

(cf. [2], p. 381). It is well known that the two-dimensional autoregressive process $\{\mathbf{Z}_t\}_{t=-\infty}^{\infty}$ is stationary if and only if the roots λ_1, λ_2 of the equation

$$|\mathbf{B}_0\lambda + \mathbf{B}_1| = 0 \tag{1.4}$$

satisfy $|\lambda_1| < 1$, $|\lambda_2| < 1$. It can be easily checked that (1.4) is equivalent to

$$\lambda^2 - (b_{11}b_{21} + b_{12} + b_{22})\,\lambda + b_{12}b_{22} = 0\,.$$

In our case $\operatorname{Var} \mathbf{Y}_{t+1} = \mathbf{D}^2$. Since we are going to use some formulas derived under the assumption that the white noise in the multidimensional autoregressive model has the unit variance matrix, we must transform (1.3) to such case. It is quite easy, because (1.3) is equivalent to

$$\mathbf{A}_0\mathbf{Z}_{t+1} + \mathbf{A}_1\mathbf{Z}_t = \mathbf{D}^{-1}\mathbf{Y}_{t+1}\,, \tag{1.5}$$

where

$$\mathbf{A}_0 = \mathbf{D}^{-1}\mathbf{B}_0\,, \quad \mathbf{A}_1 = \mathbf{D}^{-1}\mathbf{B}_1\,. \tag{1.6}$$

Now, $\operatorname{Var} \mathbf{D}^{-1}\mathbf{Y}_{t+1} = \mathbf{I}$. Clearly, the equation $|\mathbf{A}_0\lambda + \mathbf{A}_1| = 0$ has the same roots λ_1, λ_2 as the equation (1.4).

Till the end of this paper we shall assume that $\{\mathbf{Z}_t\}$ is stationary.

2. COVARIANCE FUNCTION OF STATIONARY MULTIVARIATE AR(1) PROCESS

Although most of the results of this section are known, we introduce them for sake of completness. The covariance function $\mathbf{R}(t)$ of the process $\{\mathbf{Z}_t\}$ is defined by $\mathbf{R}(t) = \mathsf{E}\mathbf{Z}_{s+t}\mathbf{Z}'_s$. We shall denote the elements of $\mathbf{R}(t)$ by $R_{ij}(t)$, i.e., $\mathbf{R}(t) = (R_{ij}(t))_{i,j=1}^2$. Since the elements $R_{ij}(0)$ of the matrix $\mathbf{R}(0)$ occur very often, we use the notation $R_{ij} = R_{ij}(0)$ for $i, j = 1, 2$.

Theorem 2.1. *Let* $\mathbf{U} = -\mathbf{B}_0^{-1}\mathbf{B}_1$. *Then*

$$\mathbf{B}_0\mathbf{R}(0)\,\mathbf{B}'_0 - \mathbf{B}_1\mathbf{R}(0)\,\mathbf{B}'_1 = \mathbf{D}^2\,, \tag{2.1}$$

$$\mathbf{R}_k = \mathbf{U}\mathbf{R}(k-1) \quad \text{for} \quad k \geqq 1\,. \tag{2.2}$$

Proof. We multiply (1.3) from the right by $\mathbf{Z}'_{t+1-k}$ and then we take expectation. Since $\mathsf{E}\mathbf{Y}_{t+1}\mathbf{Z}'_{t+1-k} = \mathbf{0}$ for $k \geqq 1$, we get (2.2) immediately.

Multiplying (1.3) from the right by $\mathbf{Z}'_{t+1}$ and taking expectation, we obtain

$$\mathbf{B}_0\mathbf{R}(0) + \mathbf{B}_1\mathbf{R}(-1) = \mathsf{E}\mathbf{Y}_{t+1}\mathbf{Z}'_{t+1} . \tag{2.3}$$

Multiplying (1.3) from the right by $\mathbf{Y}'_{t+1}$ and taking expectation we have

$$\mathsf{E}\mathbf{B}_0\mathbf{Z}_{t+1}\mathbf{Y}'_{t+1} = \mathbf{D}^2 .$$

From here

$$\mathsf{E}\mathbf{Y}_{t+1}\mathbf{Z}'_{t+1} = (\mathbf{B}_0^{-1}\mathbf{D}^2)' .$$

Because

$$\mathbf{R}(-1) = [\mathbf{R}(1)]' = -[\mathbf{B}_0^{-1}\mathbf{B}_1\mathbf{R}(0)]'$$

and $\mathbf{R}(0)$ is symmetric, (2.3) is equivalent to (2.1). □

From (2.2) we have, of course,

$$\mathbf{R}(k) = \mathbf{U}^k\mathbf{R}(0) , \quad k \geqq 1 .$$

The only problem is to get the solution $\mathbf{R}(0)$ from (2.1). This can be done by one of the following two methods. The first method is based on a system of linear equations which we get grom (2.1) when we compute the elements of the matrices on the both sides. Because $\mathbf{R}(0)$ and $\mathbf{D}$ are symmetric, we get three equations for three unknown variables R_{11}, R_{12}, R_{22}. The second method is a modern one and it uses some properties of the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ of the matrices $\mathbf{A}$ and $\mathbf{B}$. If $\mathbf{A} = (a_{ij})$, $\mathbf{B} = (b_{ij})$ then $\mathbf{A} \otimes \mathbf{B} = (a_{ij}\mathbf{B})$. We shall use the following fundamental properties of the Kronecker product:

$$(\mathbf{A} \otimes \mathbf{B})^{-1} = \mathbf{A}^{-1} \otimes \mathbf{B}^{-1} \quad \text{whenever } \mathbf{A}^{-1} \text{ and } \mathbf{B}^{-1} \text{ exist ;} \tag{2.4}$$

$$\mathbf{A}_1\mathbf{A}_2 \otimes \mathbf{B}_1\mathbf{B}_2 = (\mathbf{A}_1 \otimes \mathbf{B}_1)(\mathbf{A}_2 \otimes \mathbf{B}_2) \quad \text{whenever the products exist .} \tag{2.5}$$

Further it is known that if $\mathbf{A}$ and $\mathbf{B}$ are $m \times m$ and $n \times n$ matrices with eigenvalues $\xi_1, \ldots, \xi_m$ and $\delta_1, \ldots, \delta_n$, respectively, then the eigenvalues of $\mathbf{A} \otimes \mathbf{B}$ are $\xi_i\delta_j$ $(i = 1, \ldots, m; j = 1, \ldots, n)$.

Let $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$, where $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are columns of $\mathbf{A}$. Then the symbol $\operatorname{vec} \mathbf{A}$ means

$$\operatorname{vec} \mathbf{A} = (\mathbf{a}'_1, \ldots, \mathbf{a}'_n)' .$$

If $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are matrices such that the product $\mathbf{ABC}$ exists, then it can be proved (see Neudecker [5]), that

$$\operatorname{vec} \mathbf{ABC} = (\mathbf{C}' \otimes \mathbf{A}) \operatorname{vec} \mathbf{B} . \tag{2.6}$$

Theorem 2.2. *The elements R_{ij} of the matrix $\mathbf{R}(0)$ are given by the formula*

$$\operatorname{vec} \mathbf{R}(0) = [(\mathbf{B}_0 \otimes \mathbf{B}_0) - (\mathbf{B}_1 \otimes \mathbf{B}_1)]^{-1} \operatorname{vec} \mathbf{D}^2 ,$$

where

$$\operatorname{vec} \mathbf{R}(0) = (R_{11}, R_{12}, R_{12}, R_{22})' , \quad \operatorname{vec} \mathbf{D}^2 = (\sigma_1^2, 0, 0, \sigma_2^2)' .$$

Proof. The assertion follows from (2.1) using (2.6). It remains only to prove that the matrix $\mathbf{B}_0 \otimes \mathbf{B}_0 - \mathbf{B}_1 \otimes \mathbf{B}_1$ is regular. Applying (2.4) and (2.5) we have

$$\mathbf{B}_0 \otimes \mathbf{B}_0 - \mathbf{B}_1 \otimes \mathbf{B}_1 = (\mathbf{B}_0 \otimes \mathbf{B}_0)\left[\mathbf{I} - (\mathbf{B}_0^{-1} \otimes \mathbf{B}_0^{-1})(\mathbf{B}_1 \otimes \mathbf{B}_1)\right] =$$
$$= (\mathbf{B}_0 \otimes \mathbf{B}_0)(\mathbf{I} - \mathbf{B}_0^{-1}\mathbf{B}_1 \otimes \mathbf{B}_0^{-1}\mathbf{B}_1) = (\mathbf{B}_0 \otimes \mathbf{B}_0)(\mathbf{I} - \mathbf{U} \otimes \mathbf{U}).$$

The eigenvalues of $\mathbf{U} \otimes \mathbf{U}$ are $\lambda_1^2, \lambda_1\lambda_2, \lambda_1\lambda_2, \lambda_2^2$, and thus

$$|\lambda\mathbf{I} - \mathbf{U} \otimes \mathbf{U}| = (\lambda - \lambda_1^2)(\lambda - \lambda_1\lambda_2)^2(\lambda - \lambda_2^2)$$

does not vanish for $\lambda = 1$. Therefore, $\mathbf{I} - \mathbf{U} \otimes \mathbf{U}$ is regular. The regularity of $\mathbf{B}_0 \otimes \mathbf{B}_0$ is obvious, since $\mathbf{B}_0$ is regular and $(\mathbf{B}_0 \otimes \mathbf{B}_0)^{-1} = \mathbf{B}_0^{-1} \otimes \mathbf{B}_0^{-1}$ exists. □

3. INTERPOLATION OF $\mathbf{Z}_s$

In this section we solve the problem of interpolation, when all variables from one period are missing. In the case of model (1.2) it means that we want to interpolate the variables X_{2s-1} and X_{2s} when all other variables X_t $(t \neq 2s - 1, t \neq 2s)$ are known. In fact, we shall see that it is sufficient to know only $X_{2s-3}, X_{2s-2}, X_{2s+1}, X_{2s+2}$.

Theorem 3.1. *The best linear interpolation $\mathbf{Z}_s^*$ of the random vector $\mathbf{Z}_s$ based on $\{\mathbf{Z}_t, t \neq s\}$ is*

$$\mathbf{Z}_s^* = \mathbf{F}\mathbf{Z}_{s-1} + \mathbf{G}\mathbf{Z}_{s+1}, \tag{3.1}$$

where

$$\mathbf{F} = -(\mathbf{B}_0'\mathbf{D}^{-2}\mathbf{B}_0 + \mathbf{B}_1'\mathbf{D}^{-2}\mathbf{B}_1)^{-1}\,\mathbf{B}_1'\mathbf{D}^{-2}\mathbf{B}_0, \tag{3.2}$$

$$\mathbf{G} = -(\mathbf{B}_0'\mathbf{D}^{-2}\mathbf{B}_0 + \mathbf{B}_1'\mathbf{D}^{-2}\mathbf{B}_1)^{-1}\,\mathbf{B}_0'\mathbf{D}^{-2}\mathbf{B}_1. \tag{3.3}$$

The residual variance matrix $\mathbf{V}$ *is*

$$\mathbf{V} = \operatorname{Var}(\mathbf{Z}_s - \mathbf{Z}_s^*) = \mathbf{R}(0) - \mathbf{F}[\mathbf{R}(1)]' - \mathbf{G}\,\mathbf{R}(1). \tag{3.4}$$

Proof. We use the form (1.5) of our model. According to formula (18) in [1] the best linear interpolation $\mathbf{Z}_s^*$ is given by

$$\mathbf{Z}_s^* = -\mathbf{H}_{ss}^{-1}[\mathbf{H}_{s,s-1}\mathbf{Z}_{s-1} + \mathbf{H}_{s,s+1}\mathbf{Z}_{s+1}],$$

where

$$\mathbf{H}_{ss} = \mathbf{A}_0'\mathbf{A}_0 + \mathbf{A}_1'\mathbf{A}_1, \quad \mathbf{H}_{s,s-1} = \mathbf{A}_1'\mathbf{A}_0, \quad \mathbf{H}_{s,s+1} = \mathbf{A}_0'\mathbf{A}_1.$$

Inserting for $\mathbf{A}_0$ and $\mathbf{A}_1$ from (1.6) we get formula (3.1). Further, using the orthogonality properties, we obtain

$$\operatorname{Var}(\mathbf{Z}_s - \mathbf{Z}_s^*) = \mathsf{E}(\mathbf{Z}_s - \mathbf{Z}_s^*)(\mathbf{Z}_s - \mathbf{Z}_s^*)' = \mathsf{E}\mathbf{Z}_s\mathbf{Z}_s' - \mathsf{E}\mathbf{Z}_s^*\mathbf{Z}_s'.$$

This yields (3.4). □

Of course, an equivalent formula for the residual variance matrix can be obtained from the relation

$$\operatorname{Var}(\mathbf{Z}_s - \mathbf{Z}_s^*) = \mathsf{E}\mathbf{Z}_s\mathbf{Z}_s' - \mathsf{E}\mathbf{Z}_s^*\mathbf{Z}_s^{*\prime} .$$

The elements of $\mathbf{V}$ will be denoted by V_{ij} $(i, j = 1, 2)$.

4. INTERPOLATION OF A SINGLE VALUE

The main point of this paper is to derive a formula for the best linear interpolation when one value of the process $\{X_t\}$ satisfying (1.2) is missing. A solution is based on the following theorem.

Theorem 4.1. *Let $\mathscr{H}$ be a Hilbert space of random variables with vanishing mean and $\mathscr{H}_1$ its Hilbert subspace. Let $\beta, \gamma_1, \ldots, \gamma_m \in \mathscr{H}$. Denote $\mathscr{H}_2$ the Hilbert subspace generated by $\gamma_1, \ldots, \gamma_m$. Put $\gamma = (\gamma_1, \ldots, \gamma_m)'$. Let $\hat{\beta}$ be the projection of β onto $\mathscr{H}_1$ and let $\hat{\gamma}$ be the vector of the projections of the components of γ onto $\mathscr{H}_1$. Let $\tilde{\gamma} = \gamma - \hat{\gamma}$. If $\mathsf{E}\tilde{\gamma}\tilde{\gamma}'$ is a regular matrix, then the projection β^* of β onto the Hilbert subspace $\mathscr{H}_1 + \mathscr{H}_2$ is*

$$\beta^* = \hat{\beta} + (\mathsf{E}\beta\tilde{\gamma}')(\mathsf{E}\tilde{\gamma}\tilde{\gamma}')^{-1}\,\tilde{\gamma} .$$

Proof. See Luenberger [4], p. 92, Theorem 3. □

Let X_{2s-1} be the missing value of the process $\{X_t\}$. All the other variables X_t $(t \neq 2s - 1)$ are supposed to be known. (However, it is sufficient to know only X_t for $t = 2s - 3,\ 2s - 2,\ 2s,\ 2s + 1,\ 2s + 2$.) The best linear interpolation of X_{2s-1} will be denoted by X_{2s-1}^*. We use Theorem 4.1, where $\beta = X_{2s-1}$, $m = 1$, $\gamma = \gamma_1 = X_{2s}$. Then

$$(\hat{\beta}, \hat{\gamma})' = \mathbf{Z}_s^* = (\hat{X}_{2s-1}, \hat{X}_{2s})' , \quad \tilde{\gamma} = X_{2s} - \hat{X}_{2s} .$$

Obviously,

$$\mathsf{E}\tilde{\gamma}\tilde{\gamma}' = \mathsf{E}(X_{2s} - \hat{X}_{2s})^2 = V_{22} .$$

Further,

$$\mathsf{E}\beta\tilde{\gamma}' = \mathsf{E}X_{2s-1}(X_{2s} - \hat{X}_{2s}) .$$

Denote f_{ij} and g_{ij} the elements of the matrices $\mathbf{F}$ and $\mathbf{G}$, respectively $(i, j = 1, 2)$. Then

$$\hat{X}_{2s} = f_{21}X_{2s-3} + f_{22}X_{2s-2} + g_{21}X_{2s+1} + g_{22}X_{2s+2}$$

and

$$\mathsf{E}\beta\tilde{\gamma}' = R_{12} - (f_{21} + g_{21})\,R_{11}(1) - f_{22}\,R_{12}(1) - g_{22}\,R_{21}(1) .$$

Therefore, the final formula for interpolation is

$$X_{2s-1}^* = \hat{X}_{2s-1} +$$
$$+ V_{22}^{-1}[R_{12} - (f_{21} + g_{21})\,R_{11}(1) - f_{22}\,R_{12}(1) - g_{22}\,R_{21}(1)]\,(X_{2s} - \hat{X}_{2s}) ,$$

where $(\hat{X}_{2s-1}, \hat{X}_{2s})' = \mathbf{Z}_s^*$ from (3.1).

The case, when only X_{2s} is missing and all variables X_t for $t \neq 2s$ are known, is quite analogous. We get

$$X_{2s}^* = \hat{X}_{2s} +$$

$$+ V_{11}^{-1}[R_{12} - (f_{12} + g_{12}) R_{22}(1) - f_{11} R_{21}(1) - g_{11} R_{12}(1)] (X_{2s-1} - \hat{X}_{2s-1}) .$$

The methods described in this paper can be generalized to $n \geqq 2$, $p \geqq 2$. It seems, however, that it is more useful to derive formulas for a given particular model rather than to try to write down cumbersome formulas for the general case.

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Souhrn

O INTERPOLACI V MODELECH PERIODICKÉ AUTOREGRESE

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Modely periodické autoregrese jsou vhodné pro statistickou analýzu sezónních časových řad. Některé postupy (např. extrapolace) jsou pro ně zcela analogické jako v případě klasických autoregresních modelů. Odvození interpolačních vzorců však vyžaduje použít speciálních netriviálních metod. Ty jsou v práci demonstrovány na modelu druhého řádu, který má délku periody rovnu dvěma.

Резюме

ОБ ИНТЕРПОЛЯЦИИ В ПЕРИОДИЧЕСКИХ ПРОЦЕССАХ АВТОРЕГРЕССИИ

JIŘÍ ANDĚL, ASUNCIÓN RUBIO

Некоторые процедуры в периодических процессах авторегрессии строятся так же как в классической модели авторегрессии — например экстраполяция. В проблеме интерполяции надо применить специальные методы. В статье это показано для процессов второго порядка с периодом 2.

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