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ON A SUPERCONVERGENT FINITE ELEMENT SCHEME FOR ELLIPTIC SYSTEMS

I. DIRICHLET BOUNDARY CONDITION

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Summary. Second order elliptic systems with Dirichlet boundary conditions are solved by means of affine finite elements on regular uniform triangulations. A simple averaging scheme is proposed, which implies a superconvergence of the gradient. For domains with enough smooth boundary, a global estimate $O(h^{3/2})$ is proved in the $L^2$-norm. For a class of polygonal domains the global estimate $O(h^2)$ can be proven.

Keywords: finite elements, superconvergence, post-processing, averaged gradient, elliptic systems

AMS Subject classification: 65 N 30, 73 C 99.

1. INTRODUCTION

In this article we deal with a system of linear second order elliptic equations with Dirichlet boundary conditions in a bounded plane domain. A simple averaging scheme guaranteeing a superconvergence of the derivatives of the Galerkin solution based on linear finite elements is presented. The article can be considered as a continuation of [9], where a local superconvergence for the Poisson equation has been analyzed. To the authors' knowledge, no superconvergence analysis has been published for systems of elliptic equations until now (except for [21] and a short note in [4]).

The elliptic systems considered here include Lamé's equations of linear anisotropic elasticity, the model of Cosserat continuum [7], and the standard Poisson equation with non-homogeneous boundary conditions.

In Section 2, we show that $u_h - Pu$ vanishes faster in the $H^1$-norm than $u - u_h$ or $u - Pu$, where $u_h$ is the Galerkin approximation of the solution $u$, and $Pu$ is the linear interpolation of $u$ over uniform triangular meshes. An analogous phenomenon has been observed [2, 4, 5, 11, 12, 14, 15, 16, 22] for the Poisson equation when
employing linear triangular elements over uniform (quasiuniform or piecewise uniform) triangulations. In Section 3, an avaraged gradient (based on averaging at nodes) is introduced for linear elements, and its approximation properties in the $L^2$-norm are derived. In Section 4, we combine the results of Sections 2 and 3 to obtain a global superconvergence estimate for the derivatives of $u$. Let us mention that the result of Section 2 can be used also for other averaging techniques at centroids or midpoints of sides, which were introduced in [4, 5, 11, 12]. Other important papers on post-processing with a superconvergence of the gradient include [3, 13, 20, 23] etc., see also the survey paper [10].

2. SOME LEMMAS FOR ELLIPTIC SECOND ORDER SYSTEMS AND NUMERICAL INTEGRATION

We shall consider a bounded domain $\Omega \subset \mathbb{R}^2$ with a Lipschitz boundary $\partial \Omega$. For the Euclidean norm in $\mathbb{R}^d$ we adopt the notation $\| \cdot \|$. Let us denote by $H^k(\Omega) = W^{k,2}(\Omega)$, $k = 0, 1, \ldots$, the standard Sobolev spaces with the norm $\| \cdot \|_{k,\Omega}$, and the seminorm $| \cdot |_{k,\Omega}$ of all the derivatives of $k$-th order. We also set

$$(f, v)_0,\Omega = \int_\Omega f \cdot v \, dx, \quad f, v \in (L^2(\Omega))^M,$$

where $M \geq 1$ is a given integer, and we write for brevity

$$W = (H^1(\Omega))^M.$$

Assume that the following functions are given:

$$\bar{u} \in W, \, f \in (L^2(\Omega))^M,$$

a matrix $K(x)$ of the type $\mathbb{R} \times \mathbb{R}$ with entries $K_{ij} \in P_s(\Omega)$ (i.e. polynomials of at most $s$-th order) for all $i, j \in \{1, \ldots, \times\}$ and some integer $s \geq 0$, $K$ is symmetric and positive definite uniformly with respect to $x \in \Omega$, coefficients $n_{im}, n_{itm} \in \mathbb{R}$, $1 \leq i \leq \times, 1 \leq m \leq M, t = 1, 2$. We consider the system of operators

$$N_i(v) = \sum_{m=1}^M \left( \sum_{t=1}^2 n_{itm} \frac{\partial v_m}{\partial x_t} + n_{itm} v_m \right), \quad v \in W, \quad 1 \leq i \leq \times,$$

and the bilinear form

$$a(u, v) = \int_\Omega \sum_{i,j=1}^\times K_{ij} N_i(u) N_j(v) \, dx.$$

Assume that the system $\{N_i(v)\}_{i=1}^\times$ is coercive on the space $W$, i.e. a constant $c > 0$ exists such that

$$\sum_{i=1}^\times \|N_i(v)\|^2_0,\Omega + \|v\|^2_0,\Omega \geq c \|v\|^2_1,\Omega \quad \forall v \in W.$$
We define the space of test functions
\[ V = \{ v \in W \mid yv = 0 \text{ on } \partial \Omega \} \]
(where \( y \) denotes the trace operator) and the problem to find \( u \in \bar{u} + V \) such that
\[ a(u, v) = (f, v)_{0, \Omega} \quad \forall v \in V . \]

Let us derive the classical formulation of the boundary value problem (2.1). To this end we write
\[ N_j(v) = \sum_{m=1}^{M} N_{jm}(v_m) , \]
where
\[ N_{jm}(v_m) = \sum_{t=1}^{2} n_{jmt} \frac{\partial v_m}{\partial x_t} + n_{jm}v_m . \]

It is easy to deduce that (2.1) leads to the system
\[ \sum_{i,j=1}^{\kappa} N_{ij}(N_i(u)) = f_m , \quad m = 1, \ldots, M , \]
in the domain \( \Omega \), where
\[ N_{jm}(w) = -\sum_{t=1}^{2} n_{jmt} \frac{\partial w}{\partial x_t} + n_{jm}w \]
is the operator formally adjoint to \( N_{jm} \).

Remark 2.1. The following theorem holds (see [18], Th. 3.2). Let us put
\[ \bar{N}_{im} = \sum_{t=1}^{2} n_{imt} \hat{e}_t , \quad 1 \leq i \leq \kappa , \quad 1 \leq m \leq M . \]
The system \( \{ N_i(v) \}_{i=1}^{\kappa} \) is coercive on \( W \) if and only if the rank of the matrix \( (\bar{N}_{im} \hat{\xi}) \)
equals \( M \) for all non-zero vectors \( \hat{\xi} \) from the complex two-dimensional space.

Example. Let us consider the two-dimensional theory of an elastic non-homogeneous anisotropic body. We define \( \kappa = 3, \ M = 2, \)
\[ N_1(v) = \varepsilon_{11}(v) = \frac{\partial v_1}{\partial x_1}, \quad N_2(v) = \varepsilon_{22}(v) = \frac{\partial v_2}{\partial x_2}, \]
\[ N_3(v) = \sqrt{(2)} \varepsilon_{12}(v) = \left( \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) \sqrt{2} , \]
\[ K = \begin{bmatrix} c_{1111} & \text{sym.} & c_{2222} \\ c_{2211} & c_{2222} & \sqrt{2}c_{1212} \\ c_{1211} & \sqrt{2}c_{1222} & 2c_{1212} \end{bmatrix} , \]
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\[ \sigma_{ij} = \sum_{m,k=1}^{2} c_{ijmk} \varepsilon_{mk}, \quad i,j = 1, 2, \]

holds for the stress components. Assume that

\[ c_{mklj} = c_{jmlk} = c_{ijmk} = c_{mikj} = 0, \quad s \geq 0, \]

and a positive constant \( c_0 \) exists such that

\[ (2.2) \quad \sum_{i,j,m,k=1}^{2} c_{ijmlk}(x) \varepsilon_{ijmlk} \geq c_0 \sum_{i,j=1}^{2} \varepsilon_{ij}^2 \]

holds for all symmetric matrices \((\varepsilon_{ij})\) and all \( x \in \bar{\Omega} \).

It is readily seen that

\[ (2.3) \quad \sum_{i,j=1}^{3} K_{ij} N_i(u) N_j(v) = \sum_{i,j,m,k=1}^{2} c_{ijmlk} \varepsilon_{ijmlk} \varepsilon_{mlk}, \]

\[ \sum_{i=1}^{3} N_i^2(v) = \sum_{i,j=1}^{2} \varepsilon_{ij}^2(v). \]

By virtue of (2.2), (2.3), the matrix \( K \) is uniformly positive definite. Using the theorem of Remark 2.1, one can prove that the system \( \{ N_i(v) \}_{i=1}^{3} \) is coercive on \( W \).

Moreover, we define the subspace

\[ \mathcal{P} = \{ v \in V \mid \sum_{i=1}^{3} \| N_i(v) \|_{0,\Omega} = 0 \}. \]

Let \( \mathcal{P} = \{ 0 \} \) (i.e. \( \mathcal{P} \) reduces to the zero element). Then the following inequality (of Korn's type) holds for all \( v \in V \):

\[ (2.4) \quad a(v, v) \geq c_0 \sum_{i=1}^{3} \| N_i(v) \|_{0,\Omega}^2 \geq c_1 \| v \|_{0,\Omega}^2. \]

(For the proof see e.g. [19], Lemma 11.3.2).

Henceforth we restrict ourselves to a certain subclass of domains with a Lipschitz boundary.

We say that \( \Omega \) belongs to the class \( \mathcal{C}^3(d) \) if:

(i) the boundary \( \partial \Omega \) is 3-times continuously differentiable;

(ii) a positive constant \( d \) exists such that all circles, with radius \( d \), which are tangential to \( \partial \Omega \), have no other common points with \( \partial \Omega \);

(iii) \( \Omega \) is bounded.

Henceforth \( h \) will denote a positive small parameter, tending to zero and all constants \( C, C_t \) are positive, independent of \( h \).

There exist polygonal approximations \( \Omega_h \) of \( \Omega \in \mathcal{C}^3(d) \) such that \( \Omega_h \subset \bar{\Omega} \),

\[ (H1) \quad \max_{x \in \partial \Omega_h} \text{dist} (x, \partial \Omega) \leq C h^2 \]
and the sides of $\partial \Omega_h$ are not longer than $h$. The sides of $\partial \Omega_h$ are chords or tangents of convex or of concave arcs, respectively. The points of inflexion of $\partial \Omega$ coincide with a vertex of $\partial \Omega_h$.

We consider triangulations $\mathcal{T}_h$ of $\Omega_h$, consisting of

(a) a regular part $\mathcal{T}^*_h$ generated by uniform parallelograms and carving a domain $\Omega^*_h$ (so that the sides of $\partial \Omega^*_h$ are parallel with at most 3 different directions independent of $h$) — see Fig. 2.1, and such that

$$\text{diam } T = h \quad \forall T \in \mathcal{T}^*_h$$

($T$ denotes any (closed) triangle of the triangulation). The boundary $\partial \Omega^*_h$ does not contain any tangent to $\partial \Omega$.

(b) an irregular part $\mathcal{T}_h^i$ carving the set $\Omega_h - \Omega^*_h$ and such that

$$\max_{T \in \mathcal{T}_h^i} \text{diam } T \leq h.$$  

Assume that $\mathcal{T}_h^*$ is “maximal” in the following sense:

\[(H2)\quad \max_{x \in \partial \Omega_h^*} \text{dist } (x, \partial \Omega_h^*) \leq 2h.\]

Finally, assume that the family $\mathcal{M} = \{\mathcal{T}_h\}, h \to 0$, $\mathcal{T}_h = \mathcal{T}_h^* \cup \mathcal{T}_h^i$, is strongly regular in the standard sense, i.e. constants $C_1, C_2$ exist such that:

(i) all angles of all triangles in $\{\mathcal{T}_h\}$ are greater or equal to $C_1$,

(ii) the ratio of the lengths of any two sides in any $\mathcal{T}_h \in \mathcal{M}$ is not less than $C_2$.

A constant regular $2 \times 2$ matrix exists such that

$$\mathcal{T}_h = F(\mathcal{T}_h^0), \quad \mathcal{T}_h^* = F(\mathcal{T}_h^{i0}), \quad \mathcal{T}_h^i = F(\mathcal{T}_h^{i0}),$$

where the mapping $F: \mathbb{R}^2 \to \mathbb{R}^2$ is determined by the matrix $F$,

$$\mathbf{x} = \mathcal{F} \mathbf{X} + \mathbf{x}_h^0, \quad \mathbf{x} = (x_1, x_2), \quad \mathbf{X} = (X_1, X_2),$$

Fig. 2.1.
and $\mathcal{T}^{*0}$ is generated by a uniform square mesh of step-size $h$. Here $x_h \in \mathbb{R}^2$ may depend on $h$ whereas $\mathcal{T}$ is independent of $h$. We shall write

$$\Omega^0 = F^{-1}(\Omega), \quad \Omega_h^0 = F^{-1}(\Omega_h), \quad \Omega^{*0} = F^{-1}(\Omega^{*0}).$$

Denote by $T_c, T_t \subset \Omega - \Omega_h$ the segments adjacent to the chords and tangents, respectively. We define the space

$$W_h = \{ v \in H^1(\Omega) \mid v|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h, \}
\quad v|_{T_c} \in P_1(T_c), v|_{T_t} \in P_1(T_t) \quad \forall T_c, T_t \subset \Omega - \Omega_h \}
$$

and the interpolation mapping $P : H^1(\Omega) \cap C(\overline{\Omega}) \to W_h$ by the relations:

$$Pu(x) = u(x) \quad \text{at all nodes} \quad x \in \Omega_h,$n
$$Pu(x) = u(y) \quad \text{at all nodes of} \quad \partial \Omega_h,$n

where $y$ is the point of $\partial \Omega$ nearest to $x$, $\partial Pu/\partial t = 0$ in $\Omega - \Omega_h$, where the direction $t$ is parallel with $xy$ in parts adjacent to tangent segments and perpendicular to chords in parts adjacent to the latter. Let $Pu = (Pu_1, \ldots, Pu_M)$ for $u \in W \cap (C(\overline{\Omega}))^M$.

Finally let us define

$$V_h = \{ v \in V \mid v|_T \in (P_1(T))^M \quad \forall T \in \mathcal{T}_h, \quad v = 0 \text{ in } \Omega - \Omega_h \}.$$

Note that $V_h = P(V \cap (C(\overline{\Omega}))^M)$.

To analyze the approximate solution (cf. (2.49)), we shall need the three following lemmas.

**Lemma 2.1.** Let $\Omega$ belong to the class $C^3(d)$, $u \in (H^3(\Omega))^M$, $v \in V_h$. Then

$$|a(Pu - u, v)| \leq Ch^{3/2} \|u\|_{3,\Omega} \|v\|_{1,\Omega}$$

holds for sufficiently small $h$.

**Proof.** Given any function $\phi(x)$ in $\Omega$, we define

$$\phi^0(\mathbf{X}) = \phi(\mathcal{T}\mathbf{X} + x_h^0)$$

Then we may write for all $j = 1, \ldots, x,$

$$N_j(v) = \sum_{m=1}^M \left( \sum_{t=1}^2 n_{jmt} \sum_{i=1}^2 \frac{\partial v^0_m}{\partial X_i} \mathcal{T}^{-1} + n_{jm} v^0_m \right) = N^0(v^0) = \sum_{m=1}^M \left( b^{(1)}_{jm} \frac{\partial v^0_m}{\partial X_1} + b^{(2)}_{jm} \frac{\partial v^0_m}{\partial X_2} + n_{jm} v^0_m \right),$$

where $b^{(r)}_{jm} = \text{const.}, \quad r = 1, 2, \quad m = 1, \ldots, M$. Defining

$$e = u - Pu,$n

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we have

\[ a(e, v) = \int_{\Omega^0} \sum_{i,j=1}^\infty K_{ij}^0 N_i^0(e^0) N_j^0(v^0) \left| \det F \right| dX = \]

\[ = \sum_{m=1}^M \int_{\Omega^0} \sum_{i,j=1}^\infty K_{ij}^0 N_i^0(e^0) b_{jm}^{(1)} \left| \det \mathcal{F} \right| \frac{\partial v_m^0}{\partial X_1} dX + \]

\[ + \sum_{m=1}^M \int_{\Omega^0} \sum_{i,j=1}^\infty K_{ij}^0 N_i^0(e^0) b_{jm}^{(2)} \left| \det \mathcal{F} \right| \frac{\partial v_m^0}{\partial X_2} dX + \]

\[ + \sum_{m=1}^M \int_{\Omega^0} \sum_{i,j=1}^\infty K_{ij}^0 N_i^0(e^0) n_{jm} v_m^0 \left| \det \mathcal{F} \right| dX = a_1 + a_2 + a_3 . \]

Let us consider the term \( a_1 \). Since \( v_m \) is linear in any triangle \( T \in \mathcal{F}_h \), \( \partial v_m / \partial X_1 \in \in P_0(T^0) \). We may write

\[ a_1 = \sum_{m=1}^M (S_m + S_m^*) , \]

where

\[ S_m = \sum_{i=1}^{K_A} \frac{\partial v_m}{\partial X_1} \right|_{A_t^0} \sum_{i,j=1}^\infty b_{jm}^{(1)} \left| \det \mathcal{F} \right| \int_{A_t^0} K_{ij}^0 N_i^0(e^0) dX , \]

\[ S_m^* = \sum_{i=1}^{K_B} \frac{\partial v_m}{\partial X_1} \right|_{B_t^0} \sum_{i,j=1}^\infty b_{jm}^{(1)} \left| \det \mathcal{F} \right| \int_{B_t^0} K_{ij}^0 N_i^0(e^0) dX , \]

\( A_t^0 \) are parallelograms with a diagonal parallel with the \( X_1 \)-axis (see Fig. 2.2),

\[ A_t^0 = T_k^0 \cup T_k'^0 , \]

\( T_k^0, T_k'^0 \in \mathcal{F}_h^* \) and \( B_k^0 \in \mathcal{F}_h \) are the remaining triangles.
Note that the continuity of \( v_m^0 \) implies that \( \partial v_m^0/\partial x_1 \) is constant in every parallelogram \( A_t^0 \). Let \( G_k^o \) denote the centroid of \( T_k^0 \), and \( L_0^0 \) the centroid of \( A_t^0 \).

We decompose \( S_m \) as follows:

\[
S_m = S_{m1} + S_{m2} + S_{m3},
\]

where

\[
S_{m1} = \sum_{t=1}^{K_A} \frac{\partial v_m^0}{\partial x_1} \left|_{T_k^0} \right| \sum_{i,j=1}^{n} b_{jm}^{(1)} | \text{det} |_{\mathcal{S}} \left[ \frac{h^2}{2} K_{ij}(L_t^0) \left( N_i^0(e^0) \right|_{G_k^0} + N_i^0(e^0) \right|_{G_{k'}^0} \right],
\]

\[
S_{m2} = \sum_{k=1}^{K_A} \frac{\partial v_m^0}{\partial x_1} \left|_{T_k^0} \right| \sum_{i,j=1}^{n} b_{jm}^{(1)} | \text{det} |_{\mathcal{S}} \left[ \int_{T_k^0} \left( K_{ij}(X) - K_{ij}(L_t^0) \right) N_i^0(e^0) \, dx \right],
\]

\[
S_{m3} = \sum_{k=1}^{K_A} \frac{\partial v_m^0}{\partial x_1} \left|_{T_k^0} \right| \sum_{i,j=1}^{n} b_{jm}^{(1)} | \text{det} |_{\mathcal{S}} \left[ \int_{T_k^0} K_{ij}(L_t^0) \left( N_i^0(e^0) - N_i^0(e^0) \right|_{G_k^0} \right),
\]

We shall estimate the individual terms. Let us put

\[
F_t(u^0) = N_i^0(e^0) \left|_{G_k^0} + N_i^0(e^0) \right|_{G_{k'}^0}.
\]

We show that

\[
|F_t(u^0)| \leq C h \| u^0 \|_{3,4} \leq C_1 h \| u \|_{3,4}
\]

for any \( t = 1, \ldots, K_A \). To this end we use the mapping

\[
\hat{x} = (X - X_0^i)/h
\]

(see Fig. 2.2) and define

\[
\tilde{\phi}(\hat{x}) = \phi^0(h \hat{x} + X_0^i)
\]

for any function \( \phi^0 \). By (2.12) any parallelogram \( A_t^0 \) is mapped onto the reference "unit" parallelogram \( \alpha = \tau \cup \tau' \), where \( \tau \) and \( \tau' \) denotes the upper and lower triangle respectively.

Let us drop the subscripts \( t \) for the time being. It follows easily that

\[
N_i^0(e^0) = \sum_{m=1}^{M} \left( b_{im}^{(1)} \frac{\partial \hat{e}_m}{\partial \hat{x}_1} + b_{im}^{(2)} \frac{\partial \hat{e}_m}{\partial \hat{x}_2} \right) \left[ h + n_m \hat{e}_m \right],
\]

where

\[
\hat{e}_m = \hat{u}_m - (Pu)^a.
\]

Therefore, we may write

\[
F_t(u^0) = \frac{1}{h} \sum_{m=1}^{M} \hat{f}_m(\hat{u}_m) \sum_{m=1}^{M} n_m \delta_m(\hat{u}_m),
\]

\[
\hat{f}_m(\hat{u}_m) = b_{im}^{(1)} \left( \frac{\partial \hat{e}_m}{\partial \hat{x}_1}(\gamma) + \frac{\partial \hat{e}_m}{\partial \hat{x}_2}(\gamma') \right) + b_{im}^{(2)} \left( \frac{\partial \hat{e}_m}{\partial \hat{x}_1}(\gamma) + \frac{\partial \hat{e}_m}{\partial \hat{x}_2}(\gamma') \right),
\]

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where $\gamma$ and $\gamma'$ are the centroids of $\tau$ and $\tau'$, respectively, and
\[
\hat{g}_m(\hat{u}_m) = e_m(\gamma) - e_m(\gamma').
\]
By the direct calculation we can verify that

\[(2.15) \quad f_{im}(p) = 0 \quad \forall p \in P_2(u), \quad \forall i, m.
\]
In fact, let us choose e.g.
\[p(\hat{x}_1, \hat{x}_2) = \hat{x}_1^2.
\]
Then
\[
\hat{e}_m = \hat{x}_1^2 - \hat{x}_1,
\]
\[
f_{im}(p) = b_{lm}^{(1)}(2\gamma - 1 + 2\gamma' - 1) = 0, \quad \text{a.s.o.}
\]
Since
\[
|f_{im}(\hat{u}_m)| \leq C\|\hat{u}_m\|_{3,x},
\]
and (2.15) holds, we obtain

\[(2.16) \quad |f_{im}(\hat{u}_m)| \leq C\|\hat{u}_m\|_{3,x}
\]
using the Bramble-Hilbert lemma (see [6], Theorem 4.1.3).

Obviously, we have
\[
\hat{g}_m(p) = 0 \quad \forall p \in P_1(u),
\]
\[
|\hat{g}_m(\hat{u}_m)| \leq C\|\hat{u}_m\|_{2,x},
\]
so that the Bramble-Hilbert lemma yields

\[(2.17) \quad |\hat{g}_m(\hat{u}_m)| \leq C\|\hat{u}_m\|_{2,x}.
\]

For any $w^0 \in H^n(T_k^0), \ n = 0, 1, 2, 3$, we have (see [6], pp. 118 and 122)

\[(2.18) \quad \|\hat{w}\|_{n,x} \leq C h^{n-1} \|w^0\|_{n,x}, \quad \forall T_k^0 \in \mathcal{T}_h^0.
\]
Substituting (2.16), (2.17) and (2.18) into (2.10), we arrive at

\[(2.19) \quad |F_p^1(u^0)| \leq h^{-1} \sum_{m=1}^M C_1 h^2 \|u_m^0\|_{3,A,\rho} + \sum_{m=1}^M C_2 h \|u_m^0\|_{2,A,\rho} \leq C_3 h \|u^0\|_{3,A,\rho}
\]
and (2.11) follows immediately.

Since $v_m$ is linear in $T_k$, we have
\[
|v_m^0|_{1,T_k^0} = 2^{-1/2} h \|\nabla v_m^0|_{T_k^0}^0|,
\]
so that

\[(2.20) \quad \left| \frac{\partial v_m^0}{\partial X_1} \right| \leq C h^{-1} |v_m^0|_{1,T_k^0} \leq C_1 h^{-1} |v_m^0|_{1,T_k}
\]
holds in any triangle $T_k^0$. 

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Using (2.11), (2.20) and the estimate
\[ |K^0_{ij}(L^0_i)| \leq \|K^0_{ij}\|_{C(\Omega)} \leq C \quad \forall i, j, \]
we obtain
\[ (2.21) \quad |S_{m1}| \leq \sum_{t=1}^{k_4} C h^2 \|u\|_{A_t} \|v_m\|_{A_t} \leq C h^2 \|u\|_{\Omega} \|v_m\|_{\Gamma_1}. \]

Let us pass to the term \( S_{m2} \). Considering an arbitrary triangle \( T^0_k \) and dropping the subscript \( k \), we may write
\[ |K^0_{ij}(X) - K^0_{ij}(L^0_i)| \leq \|K^0_{ij}\|_{C(\Omega)} \cdot h \leq C h, \]
\[ (2.22) \quad \left| \int_{T^0_k} (K^0_{ij}(X) - K^0_{ij}(L^0_i)) N^0_t(e^0) \, dX \right| \leq Ch (\text{mes } T^0)^{1/2} \|N^0_t(e^0)\|_{0,T^0}. \]

The well-known error estimate implies that
\[ (2.23) \quad \|N^0_t(e^0)\|_{0,T^0} \leq C \sum_{m=1}^{M} u_m^0 - (Pu_m)^0 \|_{1,T^0} \leq C_1 \sum_{m=1}^{M} h |u_m^0|_{2,T^0} \leq C h |u|_{2,T}. \]

Combining (2.22), (2.23) and (2.20), we obtain
\[ (2.24) \quad |S_{m2}| \leq \sum_{k} C h^2 |v_m|_{1,T_k} |u|_{2,T_k} \leq C h^2 |u|_{2,T} |v_m|_{1,T}. \]

To estimate \( S_{m3} \), we set
\[ w^0 = N^0_t(e^0) \]
and
\[ E_k(w^0) = \int_{T_k^0} (w^0 - w^0(G^0_k)) \, dX. \]

Since \( e^0_m = u^0_m - Pu_m^0 \in H^3(T^0_k) \), we have \( N^0_t(e^0) = w^0 \in H^2(T^0_k) \). Using the mapping (2.12), we come to
\[ |E_k(w^0)| = \frac{1}{2} h^2 |\hat{E}(\hat{\phi})|, \quad \hat{E}(\hat{\phi}) = \int_{\tau} (\hat{\phi} - \hat{\phi}(\gamma)) \, dx. \]

It holds, however, that
\[ \hat{E}(\hat{\phi}) = 0 \quad \forall \hat{\phi} \in P_1(\tau), \]
\[ |\hat{E}(\hat{\phi})| \leq \mathcal{C} \|\phi\|_{2,\tau}. \]

Thus the Bramble-Hilbert lemma yields
\[ |\hat{E}(\hat{\phi})| \leq \mathcal{C} |\phi|_{2,\tau}. \]

By virtue of (2.18), we obtain
\[ (2.25) \quad |E_k(w^0)| \leq C_1 h^2 |\hat{\phi}|_{2,\tau} \leq C_1 h^3 |w^0|_{2,T^0_k}. \]

It is readily seen that
(2.26) \[ |w^0|_{2,T^0} = |N^0_0(e^0)|_{2,T^0} \leq C \sum_{m=1}^M (|e^0_m|_{3,T^0} + |e^0_m|_{2,T^0}) \leq \]
\[ \leq C \sum_{m=1}^M \|u^0_m\|_{3,T^0} \leq C_1 \|u\|_{3,T^0}. \]

Inserting (2.26) into (2.25), using (2.20) and the boundedness of \( K^0_{ij}(L^0) \), we get

(2.27) \[ |S_{m3}| \leq CH^2 \sum_{k=1}^{K^A} \|u\|_{3,T^k} \|v^m_m\|_{1,T^k} \leq Ch^2 \|u\|_{3,\Omega} \|v^m_m\|_{1,\Omega}. \]

It remains to estimate the term \( S^w_m \) in (2.6). Note that the estimate (2.20) holds for the triangles \( B^0_k \in \mathcal{T}_h^0 \) by virtue of the condition (ii) of the strong regularity of \( \{\mathcal{T}_h\} \). For triangles \( B_k \) adjacent to tangent segments we apply the following argument.

If we denote by \( \Pi u \) the linear interpolate of \( u \) in \( B_k \), then for any node \( x \in \partial \Omega \cap \Omega \)
\[ |(\Pi u_m(x) - Pu_m(x)| = |u_m(x) - u_m(y)| \leq \|\nabla u_m\|_{C(\Omega)} \leq C_1 h^2 \|u_1\|_{3,\Omega}, m = 1, \ldots, M, \]

as follows from (H 1) and the Sobolev embedding theorem. Consequently,

\[ \|((\Pi u_m)^0 - (Pu_m)^0)\|_{1,B^0_k} \leq Ch^2 \|u^0_m\|_{3,\Omega}, \]
\[ \|v^0_m - (Pu_m)^0\|_{1,B^0_k} \leq \|v^0_m - ((\Pi u_m)^0)\|_{1,B^0_k} + \|((\Pi u_m)^0 - (Pu_m)^0)\|_{1,B^0_k} \leq \]
\[ \leq C(h \|u^0_m\|_{2,B_k} + h^2 \|u^0_m\|_{3,\Omega}). \]

Then (2.23) holds for any \( B_k^0 \) with an additional term \( C_3 h^2 \|u\|_{3,\Omega} \) on the right-hand side, where \( C_3 \) is positive only if \( B_k \) is adjacent to a tangent segment. Thus we obtain

(2.28) \[ \int_{B^0_k} K_{ij}^0 N^0_0(e^0) \, dX \leq C(\text{mes } B^0_k)^{1/2} \|N^0_0(e^0)\|_{0,B^0_k} \leq \]
\[ \leq C_1 (h^2 \|u\|_{2,B_k} + C_3 h^3 \|u\|_{3,\Omega}). \]

Combining (2.20) and (2.28), we arrive at the estimate

(2.29) \[ |S^w_m| \leq \sum_{k=1}^{K^A} Ch^{-1} \|v_m^m\|_{1,B_k} h^2 \|u\|_{2,B_k} + C_1 \sum_{k=1}^{K^A} C_3 h^{-1} \|v_m^m\|_{1,B_k} h^3 \|u\|_{3,\Omega} \leq \]
\[ \leq C(h \|u\|_{2,\Omega_B} |v_m^m|_{1,\Omega_B} + C_4 h^2 \|u\|_{3,\Omega} h^{-1/2} |v_m^m|_{1,\Omega_B}), \]

where \( \Omega_B \) is the union of all triangles (both from \( \mathcal{T}_h^* \) and from \( \mathcal{T}_h^i \)), which do not belong to any parallelogram \( A_t \).

On the basis of the hypotheses (H 1) and (H 2) we conclude that

(3.30) \[ \max_{x \in \partial \Omega} \text{dist} (x, \partial \Omega) \leq 4h \]
holds for sufficiently small \( h \).

Next we employ the following result of V. P. Iljin ([8], Theorem 4.4 and Remark 12.4). Let \( Q^\varepsilon \) be a boundary strip of the width \( \varepsilon \), lying in \( \Omega \in \mathcal{C}^3(d), 0 < \varepsilon < d \). Then

(2.31) \[ \|w\|_{0,Q^\varepsilon} \leq C \varepsilon^{1/2} \|w\|_{1,\Omega} \]
holds for any \( w \in H^1(\Omega) \), where \( C \) depends on \( d \) and the domain \( \Omega \) only.
Using (2.30) and (2.31) with \( \varepsilon = 4h \), we get

\[
(2.32) \quad \| u_m \|_{2, \Omega_B} \leq C h^{1/2} \| u_n \|_{3, \Omega}.
\]

Inserting this into (2.29), we come to the estimate

\[
(2.33) \quad | S_m^e | \leq C h^{3/2} \| u \|_{3, \Omega} \| v_m \|_{1, \Omega}.
\]

Gathering the estimates (2.21), (2.24), (2.27) and (2.33), we obtain from (2.6) and (2.9)

\[
(2.34) \quad | a_1 | \leq C h^{3/2} \| u \|_{3, \Omega} \| v \|_{1, \Omega}.
\]

The term \( a_2 \) in the decomposition of \( a(e, v) \) can be estimated by an analogous way. The parallelograms \( A^0_i \) are substituted by those with a diagonal parallel with the \( X_2 \)-axis. Altogether, we get the same upper bound for \( |a_2| \) as that in (2.34).

To estimate the last term \( a_3 \), we decompose it as follows

\[
(2.35) \quad a_3 = \sum_{m=1}^{M} (S_m + S_m^e),
\]

where

\[
S_m = \sum_{t=1}^{K_A} \sum_{i,j=1}^{\infty} n_{jm} | \det \mathcal{F} | \int_{A_t^0} N^0_{ij}(e^0) \, dX,
\]

\[
S_m^e = \sum_{k=1}^{K_B} \sum_{i,j=1}^{n} n_{jm} | \det \mathcal{F} | \int_{B_k^0} N^0_{ij}(e^0) \, dX,
\]

with the same decomposition of \( \bar{G}_h \) as in (2.7) and (2.8). Next we write

\[
(2.36) \quad S_m = S_{m1} + S_{m2} + S_{m3},
\]

\[
S_{m1} = \sum_{i=1}^{K_A} \sum_{i,j=1}^{\infty} n_{jm} | \det \mathcal{F} | K_{ij}^0(L_i) v_m^0(L_i) \frac{h^2}{2} (N^0_{i}(e^0) |_{G_k^0} + N^0_{i}(e^0) |_{G_k^0}),
\]

\[
S_{m2} = \sum_{k=1}^{K_A} \sum_{i,j=1}^{\infty} n_{jm} | \det \mathcal{F} | \int_{T_k^0} (K_{ij}^0(X) v_m^0(X) - K_{ij}^0(L_i) v_m^0(L_i)) N^0_{i}(e^0) \, dX,
\]

\[
S_{m3} = \sum_{k=1}^{K_A} \sum_{i,j=1}^{\infty} n_{jm} | \det \mathcal{F} | \int_{T_k^0} K_{ij}^0(L_i) v_m^0(L_i) (N^0_{i}(e^0) - N^0_{i}(e^m)) |_{G_k^0} \, dX.
\]

Since we have (see [6], p. 142)

\[
(2.37) \quad \| v_m^0 \|_{C(T_k^0)} \leq C h^{-1} \| v_m^0 \|_{0, T_k^0},
\]

the estimate

\[
(2.38) \quad | S_{m1} | \leq C h^2 \| u \|_{3, \Omega} \| v_m \|_{0, \Omega}
\]

follows by arguments similar to those of (2.11)—(2.21).
We can write

\[ \int_{T_h^0} \left| K_{ij}^0(X) v_m^0(X) - (K_{ij}^0 v_m^0) (L) \right|^2 \, dX \leq 2 \int_{T_h^0} \left| K_{ij}^0(X) - K_{ij}^0(L) \right|^2 (v_m^0)^2 \, dX + 2(K_{ij}^0(L))^2 \int_{T_h^0} \left| v_m^0(X) - v_m^0(L) \right|^2 \, dX \leq C h^2 \left( \| v_m^0 \|^2_{0, T_h^0} + \| v_m^0 \|^2_{1, T_h^0} \right). \]

Combining this estimate with (2.23), we arrive at

\[ |S_{m2}| \leq C \sum_{k=1}^{K^A} h^2 \| v_m \|_{1, T_k} |u|_{2, T_k} \leq C h^2 \| v_m \|_{1, \Omega} |u|_{2, \Omega}. \]

Using the estimates (2.37), (2.25) and (2.26), we obtain

\[ |S_{m3}| \leq C \sum_{k=1}^{K^A} h^{-1} \| v_m \|_{0, T_k} h^3 \| u \|_{3, T_k} \leq C_1 h^2 \| u \|_{3, \Omega} \| v_m \|_{0, \Omega}. \]

On the basis of (2.23) and the derivation of (2.28) we have

\[ \left| \int_{B_h^0} K_{ij}^0 v_m^0 N_i^0(\psi^0) \, dX \right| \leq C \| v_m \|_{0, B_h^0} \| N_i^0(\psi^0) \|_{0, B_h^0} \leq \| v_m \|_{0, B_h^0} (C_1 h |u|_{2, B_h^0} + C_3 h^2 \| u \|_{3, \Omega}). \]

Consequently,

\[ |S_{m*}| \leq C h \| v_m \|_{0, \Omega} |u|_{2, \Omega} + C_3 h^2 \| u \|_{3, \Omega} C h^{-1/2} \| v_m \|_{0, \Omega} \]

follows easily. Using (2.30) and (2.31) twice, we are led to the estimate

\[ |S_{m*}| \leq C h^2 \| u \|_{3, \Omega} \| v_m \|_{1, \Omega}. \]

From (2.35), (2.36), and the estimates (2.38)–(2.41), we get

\[ |a_3| \leq C h^2 \| u \|_{3, \Omega} \| v \|_{1, \Omega}. \]

Combining the bounds for \( a_1, a_2, a_3 \), we arrive at the estimate (2.5).

\[ \textbf{Remark 2.2.} \] Assume that the domain \( \Omega \) has a polygonal boundary, which consists of line segments parallel with one of three different directions and the ratio of the lengths of any two parallel sides is rational. Then we can put

\[ \mathcal{T}_h = \mathcal{T}_h^*, \ \Omega = \Omega_h = \Omega_h^*, \]

and a stronger estimate

\[ |a(u - Pu, v)| \leq C h^2 \| u \|_{3, \Omega} \| v \|_{1, \Omega}, \]

holds for sufficiently small \( h \).

In fact, the term \( S_{m*} \) in (2.6) vanishes, since \( v_m^0 = 0 \) on \( \partial \Omega \) implies that

\[ \frac{\partial v_m^0}{\partial X_1} = 0 \] on any triangle \( B_k^0 \).
Consequently, collecting the estimates (2.21), (2.24) and (2.27), (and their analogues\(^1\)) for the terms \(a_2, a_3\), we arrive at the estimate (2.42).

**Lemma 2.2.** Let \(\Omega\) belong to the class \(C^3(d)\), \(f \in H^2(\Omega)\) and \(v \in V_h\) for \(M = 1\). Define the approximation
\[
(f, v)_{0, \Omega}^h
\]
by the centroid rule on the triangulation \(\mathcal{T}_h\). Then
\[
|(f, v)_{0, \Omega} - (f, v)_{0, \Omega}^h| \leq Ch^2 \|f\|_{2, \Omega} \|v\|_{1, \Omega}.
\]

**Proof.** Following the idea of [11], we define the local error on a single triangle
\[
E_k(w) = \int_{T_k} w \, dx - w(G_k) \, \text{mes} \, T_k, \quad T_k \in \mathcal{T}_h,
\]
and write
\[
|(f, v)_{0, \Omega} - (f, v)_{0, \Omega}^h| \leq \sum_{k=1}^{K} \{|E_k(fv(G_k))| + |E_k(f(v - v(G_k)))|\}.
\]
Applying the affine mapping
\[
x = B_T \xi + b_T,
\]
which transforms the reference unit triangle \(\tau\) onto \(T_k\), we easily deduce
\[
|E_k(w)| \leq Ch^2 |\hat{E}(\hat{w})|,
\]
where
\[
\hat{E}(\hat{w}) = \int_{\tau} (\hat{w} - \hat{w}(\gamma)) \, d\xi, \quad \hat{w}(\xi) = w(B_T \xi + b_T).
\]
Since
\[
|\hat{E}(\hat{w})| \leq C \|\hat{w}\|_{2, \tau}, \quad \hat{E}(\hat{p}) = 0 \quad \forall \hat{p} \in P_1(\tau),
\]
the Bramble-Hilbert lemma yields
\[
|\hat{E}(\hat{w})| \leq C |\hat{w}|_{2, \tau}.
\]
Using the estimate (cf. (2.18))
\[
|\hat{w}|_{2, \tau} \leq Ch |w|_{2, T_k},
\]
we obtain
\[
|E_k(w)| \leq Ch^3 |w|_{2, T_k}.
\]
Moreover, we have (by virtue of the strong regularity of \(\{\mathcal{T}_h\}\))
\[
|v(G_k)| \leq \|v\|_{C(T_k)} \leq Ch^{-1} \|v\|_{0, T_k} \quad \forall T_k \in \mathcal{T}_h,
\]
\(^1\) Here suitable "regularized" domains \(\bar{Q} = \Omega, \tilde{Q} \in C^3(d)\), have to be used when employing Iljin's inequality (2.31).
so that

\[ (2.46) \quad |E_k(f v(G))| = |E_k(f) v(G_k)| \leq C h^2 \|f\|_{2, \tau_k} \|v\|_{1, \tau_k} \]

follows from (2.44) and (2.45).

Using again (2.43), we obtain

\[ (2.47) \quad |E_k(f(v - v(G_k)))| \leq C h^2 |\mathcal{E}(\delta - \delta(y)))| = C h^2 \left| \int_\tau f(\delta - \delta(y)) \, d\xi \right|. \]

If we put

\[ \mathcal{F}(\delta) = \int_\tau \delta(\delta - \delta(y)) \, d\xi. \]

then

\[ \mathcal{F}(p) = 0 \quad \forall p \in P_0(\tau), \]

since \( \delta \) is linear, and

\[ |\mathcal{F}(\delta)| \leq C \|\delta - \delta(y)\|_{0, \tau} \|\delta\|_{1, \tau}. \]

From the Bramble-Hilbert lemma

\[ |\mathcal{F}(\delta)| \leq C \|\delta - \delta(y)\|_{0, \tau} \|\delta\|_{1, \tau} \]

follows.

On the other hand, we have

\[ |\delta(\delta) - \delta(\gamma)| \leq |V v \cdot (\delta - \gamma)| \leq \|V \delta\|, \quad \delta \in \tau. \]

Consequently, we get

\[ \|\delta - \delta(\gamma)\|_{0, \tau} \leq |\delta|_{1, \tau}, \]

\[ |\mathcal{F}(\delta)| \leq C |\delta|_{1, \tau} \|\delta\|_{1, \tau} \leq C_1 \|f\|_{1, \tau_k} \|v\|_{1, \tau_k}. \]

Substituting into (2.47), we arrive at

\[ (2.48) \quad |E_k(f(v - v(G_k)))| \leq C h^2 \|f\|_{1, \tau_k} \|v\|_{1, \tau_k}. \]

Combining (2.48) with (2.46), we obtain

\[ |(f, v)_{0, \Omega} - (f, v)^*_{0, \Omega}| \leq C \sum_{k=1}^K h^2 \|f\|_{2, \tau_k} \|v\|_{1, \tau_k} \leq C h^2 \|f\|_{2, \Omega} \|v\|_{1, \Omega}. \]

To introduce an approximation of the problem (2.1), let us assume that \( \bar{u} \in W \cap (C(\Omega))^M \). Then the discrete problem can be defined as follows:

Find \( u_h \in \mathcal{P} \bar{u} + V_h \) such that

\[ (2.49) \quad \alpha(u_h, v_h) = (f, v_h)^*_{0, \Omega} \quad \forall v_h \in V_h, \]

where the right-hand side is defined as a sum over \( m = 1, \ldots, M \), of the approximations defined in Lemma 2.2.
Theorem 2.1. Let $\Omega$ belong to the class $\mathcal{C}^3(d)$. Let $u \in (H^3(\Omega))^M$ and $u_h$ be the solution of (2.1) and (2.49), respectively, where $f \in (H^2(\Omega))^M$. Then
\[ \|u_h - Pu\|_{1,\Omega} \leq Ch^{3/2} \left( \|u\|_{3,\Omega} + \|f\|_{2,\Omega} \right) \]
holds for sufficiently small $h$.

Proof. As $u \in \overline{\mathcal{U}} + V$, $u_h \in P\overline{\mathcal{U}} + V_h$, and the mapping $P$ is additive, it is readily seen that
\[ \mathbf{v}_h \equiv Pu - u_h \in V_h. \]
Thus employing the inequality of Korn's type (2.4) and the definitions (2.1), (2.49), we come to
\begin{align*}
(2.50) \quad \|Pu - u_h\|_{1,\Omega}^2 & \leq C(a(Pu - u_h, \mathbf{v}_h) \leq C|a(Pu - u, \mathbf{v}_h)| + \\
& + |a(u, \mathbf{v}_h) - a(u_h, \mathbf{v}_h)| = C(|a(Pu - u, \mathbf{v}_h)| + |(f, \mathbf{v}_h)|_{0,\Omega} - |(f, \mathbf{v}_h)|_{0,\Omega h}|).
\end{align*}

Estimating the terms in the right-hand side with the help of Lemmas 2.1 and 2.2, we obtain
\[ \|Pu - u_h\|_{1,\Omega} \leq Ch^{3/2} \left( \|u\|_{3,\Omega} + \|f\|_{2,\Omega} \right). \]

Remark 2.3. Let the domain $\Omega$ have a polygonal boundary which consists of line segments parallel with one of three different directions and the ratio of the lengths of any two parallel sides is rational. Assume that the solution $u$ of (2.1) belongs to $(H^3(\Omega))^M$ and the right-hand side $f \in (H^2(\Omega))^M$.

Then we put $\mathcal{T}_h = \mathcal{T}_h^*, \Omega = \Omega_h^*$, and
\[ (2.51) \quad \|u_h - Pu\|_{1,\Omega} \leq Ch^{3/2} \left( \|u\|_{3,\Omega} + \|f\|_{2,\Omega} \right) \]
holds for sufficiently small $h$.

This result follows from Remark 2.2, Lemma 2.2 and (2.50).

3. AVERAGED GRADIENT AND ITS APPROXIMATION PROPERTIES

The results of this section will be used later to obtain better approximation of the first derivatives of the solution $u$ than the derivatives of $u_h$ (cf. (2.49)). For the time being, however, let $\mathbf{v}_h = (v_{1h}, \ldots, v_{Mh}) \in (W_h)^M$ be arbitrary. The gradient of its $m$-th component $v_{mh}$ (the subscript $m$ will be dropped in the whole section) is a piecewise constant vector function. This enables us to define an averaged gradient $G_h(v_h)$ on the domain $\Omega_h^*$ as the linear interpolation of $\mathcal{T}_h^*$ of the nodal values
\begin{equation}
(3.1) \quad (G_h(v_h))(x) = \sum_{j=1}^{m_n} w_j^* \text{grad} v_h(x). \end{equation}

Here $n = n(x) \in \{1, \ldots, 6\}$ is the number of triangles of $\mathcal{T}_h^*$ which contain the node.
\( x, m_n \) is the number of the triangles \( T_j^x \) sketched in Fig. 3.n, the weights \( w_j^x, j \in \{1, \ldots, m_n\} \), are real numbers the values of which are marked also in Fig. 3.n, in the corresponding triangles \( T_j^x \).

\[
\text{Fig. 3.1.}
\]

\[
\text{Fig. 3.2.}
\]

\[
\text{Fig. 3.3.}
\]

\[
\text{Fig. 3.4.}
\]

\[
\text{Fig. 3.5.}
\]

\[
\text{Fig. 3.6.}
\]

A crucial point of our analysis will be the proof of the estimate

\[
\| \text{grad} \, v - G_h(Pv) \|_{0,0,h^*} = O(h^2)
\]

for any sufficiently smooth scalar function \( v \) (note that we have only \( \| \text{grad} \, v - - \text{grad} \, Pv \|_{0,0,h^*} = O(h) \)).

At first let us consider the difference \( \text{grad} \, v - G_h(Pv) \) on a single triangle \( T \in \mathcal{S}_h^* \). From (3.1) we see that the values of \( G_h(v_h) \) on \( T \) depend only upon the values of \( \text{grad} \, v_h \) on \( D(T) \), where

\[
D(T) = \bigcup_{\substack{T' \cap T = \emptyset \\\ T' \in \mathcal{S}_h^*}} T'.
\]

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Furthermore, we set $A = D(\tau)$ for the reference triangle $\tau$ with the vertices $(0,0)$, $(1,0), (0,1)$ — see Fig. 3.7 below. It is clear that for any $T \in \mathcal{T}_h$ there exists a regular $2 \times 2$ matrix $B_T$ and a vector $b_T \in \mathbb{R}^2$ such that

$$F_T(\tau) = T,$$

where

$$F_T(\hat{x}) = B_T \hat{x} + b_T.$$

As the system $\mathcal{M} = \{\mathcal{T}_h\}$ is regular (see [6] for Zlámal's condition), we get

$$\|B_T\| \leq Ch.$$

Moreover, the strong regularity of $\mathcal{M}$ yields

$$\|B_T^{-1}\| \leq Ch^{-1}.$$

For $D(T)$ given by (3.2), we put

$$\delta_T = F_T^{-1}(D(T)).$$

Obviously $\delta_T \subseteq A$, and there is only a finite number of (reference) domains $\delta_T$, when $T$ passes through all triangulations $\mathcal{T}_h$.

Let $\hat{\delta}$ be a continuous and piecewise linear function on $\delta_T$. Let the averaged gradient $\hat{G}_T(\hat{\delta})$ on the reference triangle $\tau$ be a linear function which is defined at the vertices $\hat{x}$ of $\tau$ likewise (3.1), i.e.

$$\hat{G}_T(\hat{\delta})(\hat{x}) = \sum_{j=1}^{m_n} w_j^x \text{grad } \hat{\delta}|_{T_j},$$

where $T_j = F_T^{-1}(T_j)$ and $x = F_T(\hat{x})$. There is again only a finite number of different formulae for $\hat{G}_T(\hat{\delta})$. For simplicity we drop the subscript $T$ for the time being.

**Lemma 3.1.** There exists a constant $\tilde{C} > 0$ such that

$$\|\text{grad } \hat{\delta} - \hat{G}(\hat{\delta})\|_{0,\tau} \leq \tilde{C} |\hat{\delta}|_{3,\delta} \quad \forall \hat{\delta} \in H^3(\delta),$$

where $\hat{G}(\hat{\delta})$ is the linear interpolation of $\hat{\delta}$ on $\delta$.

**Proof.** Let $\hat{\delta} \in H^3(\delta)$ be arbitrary and let $y$ be that vertex of $\tau$, where the convex function $\hat{x} \rightarrow \|\hat{G}(\hat{\delta})(\hat{x})\|^2$ attains its maximum over $\tau$. Then for a suitable $k \in \{1, \ldots, 6\}$ we may write

$$\|\hat{G}(\hat{\delta})\|_{0,\tau} \leq (\text{mes } \tau)^{1/2} \|\hat{G}(\hat{\delta})(y)\| \leq (\text{mes } \tau)^{1/2} \sum_{j=1}^{m_n} |w_j^y| \|\text{grad } \hat{\delta}|_{T_jy} \| \leq 3(\text{mes } \tau)^{1/2} \|\text{grad } \hat{\delta}|_{T_jy} \| \leq 3 \|\text{grad } \hat{\delta}\|_{0,\delta},$$

because $\sum_{j=1}^{m_n} |w_j^y| \leq 3$ and $T_j^y \subseteq \delta$. 

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For a fixed $\hat{g} \in L^2(\tau)$ we define a linear functional $\Phi$ by

$$\Phi(\hat{v}) = (\text{grad } \hat{v} - \hat{G}(\hat{P}\hat{v}), \hat{g})_{0,\tau}, \quad \hat{v} \in H^3(\delta).$$

We apply (3.8) to see that $\Phi$ is bounded on $H^3(\delta)$, i.e.

$$|\Phi(\hat{v})| \leq \|\text{grad } \hat{v} - \hat{G}(\hat{P}\hat{v})\|_{0,\tau} \|\hat{g}\|_{0,\tau} \leq (\|\hat{v}\|_{1,\tau} + 3\|\text{grad } \hat{P}\hat{v}\|_{0,\delta}) \|\hat{g}\|_{0,\tau} \leq C\|\hat{v}\|_{3,\delta} \|\hat{g}\|_{0,\tau}.$$  

Next, we show that

$$(3.10) \quad \hat{G}(\hat{P}\hat{v}) = \text{grad } \hat{v} \quad \text{on } \tau$$

for all quadratics $\hat{v} \in P_2(\delta)$. Since both functions in the left- and right-hand side of (3.10) are then linear, it suffices to prove (3.10) for all vertices of $\tau$. Let us confine ourselves to the vertex $(0, 0)$ and to the first components of the gradients (the proof for the second components and other vertices is essentially the same). Let us write

$$\hat{v}(x_1, x_2) = c_0 + c_1x_1 + c_2x_2 + c_3x_1^2 + c_4x_1x_2 + c_5x_2^2,$$

and let us number the triangles from the union $\mathcal{A} = D(\tau)$ as marked in Fig. 3.7.

Then the averaging, which corresponds to the situation of Fig. 3.1, yields

$$((\hat{G}(\hat{P}\hat{v}))(0, 0))_1 = \frac{1}{2}\partial_1\hat{P}\hat{v}|_{T_1} + \frac{1}{2}(\partial_1\hat{P}\hat{v})|_{T_{10}} - \partial_1\hat{P}\hat{v}|_{T_9} - \partial_1\hat{P}\hat{v}|_{T_{11}}) =$$

$$= \frac{1}{2}(c_1 + c_3) + \frac{1}{2}(c_1 + c_3 + c_4 - (c_1 + 3c_3) - (c_1 + c_3 + c_4)) =$$

$$= c_1 = (\partial_1\hat{v})(0, 0).$$

Fig. 3.7.
The point $x$ at the Fig. 3.2 is common to two triangles. Mapping e.g. the left one to $T$, we obtain

$$((\hat{G} (\hat{P} \hat{v}))(0,0))_t = \frac{1}{2}(\hat{\partial}_1 \hat{P} \hat{v}|_{T_1} + \hat{\partial}_1 \hat{P} \hat{v}|_{T_2} - \frac{1}{2}(\partial_1 \hat{P} \hat{v}|_{T_1} + \partial_1 \hat{P} \hat{v}|_{T_2}) =$$

$$= \frac{1}{2}(c_1 + c_3 + c_1 + c_3) - \frac{1}{2}(c_1 + 3c_3 + c_1 + 3c_3) = c_1 .$$

The other cases can be proved by an analogous way (see also [9], p. 108, for the situation of Fig. 3.6).

Thus (3.10) implies that the functional $\Phi$ vanishes for all $\hat{\theta}$ quadratic. From the Bramble-Hilbert lemma and (3.9), we come to

$$|\Phi(\hat{\theta})| \leq \hat{C} |\hat{\theta}|_{3,\delta} \|\hat{\theta}\|_{0,T},$$

which proves the lemma.

**Lemma 3.2.** There exists a constant $C > 0$ such that for any $T \in \mathcal{T}_h \ast$ ($\mathcal{T}_h \ast \subset \mathcal{T}_h \in \mathfrak{M}$) it holds that

$$\|\text{grad } v - G_h(Pv)\|_{0,T} \leq Ch^2 |v|_{3,D} \quad \forall v \in H^3(D),$$

where $D \equiv D(T)$ is defined by (3.2).

**Proof.** Let $T \in \mathcal{T}_h \ast$ be given. In accordance with (3.3) we get

$$\text{(3.11)} \quad \text{grad } \hat{v}(\hat{x}) = (B_T^{-1})^T \text{grad } \hat{\theta}(\hat{x}),$$

provided $v \in H^3(D)$ and $\hat{\theta} \in H^3(\delta)$ are coupled by the relation

$$\hat{\theta}(\hat{x}) = v(x) \quad (x = F_T(\hat{x}) \in D).$$

Obviously, the formula (3.11) is valid for continuous and piecewise linear functions $v$ and $\hat{\theta}$, too. Therefore, using (3.1), (3.6) and the fact that $F_T(\delta) = D$, we easily find that a formula similar to (3.11) holds also for the averaged gradients, i.e.

$$(G_h(v))(x) = (B_T^{-1})^T (\hat{G}(\hat{\theta}))(\hat{x}) \quad (x = F_T(\hat{x}) \in T),$$

where $v$ and $\hat{\theta}$ are continuous and piecewise linear on $D$ and $\delta$, respectively. Hence,

$$(G_h(v))(x) = (B_T^{-1})^T (\hat{G}(\hat{\theta}))(\hat{x}),$$

for all $x \in T$, $v \in H^3(D)$ and the corresponding $\hat{x} \in \tau$, $\hat{\theta} \in H^3(\delta)$. Combining this result with (3.11) and (3.7), and using the well-known relation $(Pv) \hat{=} \hat{P} \hat{\theta}$, we arrive at

$$\text{(3.12)} \quad \|\text{grad } v - G_h(Pv)\|_{0,T}^2 \leq$$

$$\leq \|B_T^{-1}\|^2 \|\text{grad } \hat{\theta} - \hat{G}(\hat{P} \hat{\theta})\|_{0,T}^2 |\det B_T| \leq \hat{C} \|B_T^{-1}\|^2 |\det B_T| |\hat{\theta}|_{3,\delta}^2 .$$

Since (see [6], p. 118)

$$|\hat{\theta}|_{3,\delta} \leq C \|B_T\|^3 |\det B_T|^{-1/2} |v|_{3,D} ,$$

from the estimates (3.12), (3.4) and (3.5) the assertion of the lemma follows. \[\blacksquare\]
The following estimate will be applied in the next section to each component of the solution $u = (u_1, \ldots, u_M)$.

**Theorem 3.1.** There exists a constant $C > 0$ such that

$$
\|\text{grad } v - G_h(Pv)\|_{0, \Omega_h^*} \leq Ch^2\|v\|_{3, \Omega_h^*} \quad \forall v \in H^3(\Omega).
$$

**Proof.** From Lemma 3.2 we have

$$
\|\text{grad } v - G_h(Pv)\|_{0, \Omega_h^*} = \sum_{T \in \mathcal{T}_h^*} \|\text{grad } v - G_h(Pv)\|_{0,T}^2 \leq C^2h^4 \sum_{T \in \mathcal{T}_h^*} |v|_{3,T}^2 = 13C^2h^4|v|_{3, \Omega_h^*}^2 \quad \forall v \in H^3(\Omega),
$$

since any $T \in \mathcal{T}_h^*$ is contained in at most 13 sets $D(T_i), i = 1, \ldots, k$ ($k \leq 13$), where $T_2, T_3, \ldots, T_k \in \mathcal{T}_h^*$ are the “neighbouring” triangles to $T = T_1$ (cf. Fig. 3.7).

4. THE MAIN RESULT ON THE SUPERCONVERGENCE

Now we are able to prove the main theorem of the paper. We collect the above results to obtain a superconvergence of the derivatives for our elliptic problem (2.1) with the solution $u = (u_1, \ldots, u_M)$. Let us recall that $u_h = (u_{1h}, \ldots, u_{Mh})$ denotes the solution of the discrete problem (2.49). We adopt the following notations. By $\mathcal{G}_h(u_h)$ we denote $2 \times M$ matrix, the $j$-th column of which equals to $G_h(u_jh)$; and let $\partial u_j/\partial x$ be $2 \times M$ matrix of the first partial derivatives of $u$.

**Theorem 4.1.** Let the assumptions of Theorem 2.1 be satisfied. Then

$$
\begin{align*}
\left\| \frac{\partial u}{\partial x} - \mathcal{G}_h(u_h) \right\|_{0, \Omega_h^*} &\leq C h^{3/2}(\|u\|_{3, \Omega} + \|f\|_{2, \Omega})
\end{align*}
$$

holds for sufficiently small $h$.

**Proof.** Obviously,

$$
\begin{align*}
\left\| \frac{\partial u}{\partial x} - \mathcal{G}_h(u_h) \right\|_{0, \Omega_h^*} &\leq \sum_{j=1}^M \left\| \text{grad } u_j - G_h(u_jh) \right\|_{0, \Omega_h^*} \\
&\leq \sum_{j=1}^M \left( \left\| \text{grad } u_j - G_h(Pu_j) \right\|_{0, \Omega_h^*} + \left\| G_h(Pu_j - u_jh) \right\|_{0, \Omega_h^*} \right),
\end{align*}
$$

and from Theorem 3.1 we may easily bound the first terms

$$
\begin{align*}
\left\| \text{grad } u_j - G_h(Pu_j) \right\|_{0, \Omega_h^*} &\leq Ch^2|u_j|_{3, \Omega_h^*}, \quad j = 1, \ldots, M.
\end{align*}
$$

To bound the second terms we use some ideas of Sections 2 and 3. As $Pu_j - u_jh$ is a continuous piecewise linear function, we find likewise (3.8) that

$$
\begin{align*}
\left\| G_h(Pu_j - u_jh) \right\|_{0, \mathcal{T}_h^*} &\leq 3\left\| \text{grad } (Pu_j - u_jh) \right\|_{0, \mathcal{D}(T)},
\end{align*}
$$

whenever $T \in \mathcal{T}_h^*$ and $j \in \{1, \ldots, M\}$. Any $T \in \mathcal{T}_h^*$ is contained in at most 13 sets
(4.3) \[ \| G_h (P_{uj} - u_{jh}) \|_{0, \Omega_h^*} = 3 \sqrt{(13)} \| \text{grad} (P_{uj} - u_{jh}) \|_{0, \Omega_h^*} \leq 3 \sqrt{(13)} \| Pu - u_n \|_{1, \Omega}, \]

which can be further estimated by Theorem 2.1. The theorem now follows from (4.1), (4.2), and (4.3).

**Remark 4.1.** In case of a polygonal domain discussed in Remark 2.3, the Theorem 4.1 holds even with the power \( h^2 \) instead of \( h^{3/2} \) and we obtain the global super-convergence in the \( \| \cdot \|_{0, \Omega} \)-norm. The regularity assumption \( u \in (H^3(\Omega))^M \), however is unreasonable for general non-smooth boundaries and \( f \in (H^2(\Omega))^M \). For boundaries of the class \( C^{(\infty)} \), we have \( u \in (H^3(\Omega))^M \) if \( \tilde{u} \in (H^3(\Omega))^M \) and \( f \in (H^1(\Omega))^M \), as follows from [1] and [17], Lemma 3.2, Chapt. 5.

**Corollary 4.1.** Extending the definition of \( \mathcal{G}_h(u_h) \) in the simple manner

\[ \mathcal{G}_h(u_h) = \frac{\partial u_n}{\partial x} \text{ in } \Omega_h^1 = \Omega_h - \Omega_h^*, \]

one can deduce even the global superconvergence estimate

\[ \left\| \frac{\partial u}{\partial x} - \mathcal{G}_h(u_h) \right\|_{0, \Omega_h} \leq C h^{3/2} (\| u \|_{3, \Omega} + \| f \|_{2, \Omega}), \]

under the same assumptions as in Theorem 4.1.

**Proof.** By the above Theorem 4.1, it suffices to estimate the term

\[ \left\| \frac{\partial u}{\partial x} - \mathcal{G}_h(u_h) \right\|_{0, \Omega_h} \leq \sum_{j=1}^{M} \| \text{grad} u_j - \text{grad} u_{jh} \|_{0, \Omega_h} \leq \sum_{j=1}^{M} (\| u_j - P_{uj} \|_{1, \Omega_h} + \| u_{jh} - P_{uj} \|_{1, \Omega}). \]

The bound for the last term follows immediately from Theorem 2.1. Thus it remains to deal with the term \( \| u_j - P_{uj} \|_{1, \Omega_h} \). Denoting by \( \pi \) the standard interpolation operator, we easily find like in the proof of Lemma 2.1 (cf. the derivation of (2.28)) that

(4.4) \[ \| u_j - P_{uj} \|_{1, \Omega_h} \leq \| u_j - \pi u_j \|_{1, \Omega_h} + \| \pi u_j - P_{uj} \|_{1, \Omega_h} \leq C(h \| u_j \|_{2, \Omega} + h^{3/2} \| u_j \|_{3, \Omega}). \]

On the basis of the hypotheses (H 1) and (H 2) we conclude that

\[ \max_{x \in \partial \Omega_h} \text{dist}(x, \partial \Omega_h^*) \leq C h. \]

Consequently the use of the Iljxin inequality (2.31) and (4.4) yields

\[ \| u_j - P_{uj} \|_{1, \Omega_h} \leq C h^{3/2} \| u_j \|_{3, \Omega}. \]
References

O JEDNOM SUPERKONVERGENTNÍM SCHÉMATU
V METODĚ KONEČNÝCH PRVKŮ PRO ELIPTICKÉ SYSTÉMY
I. DIRICHLETOVY OKRAJOVÉ PODMÍNKY

IVAN Hlaváček, Michal Křížek

V článku se uvažují systémy eliptických rovnic druhého řádu (zahrnující Lamého rovnice pružnosti) s nehomogenními Dirichletovými okrajovými podmínkami na omezené rovinné oblasti. Předkládá se jednoduché průměřující schéma, které zaručuje superkonvergenci derivací řešení při použití standardních lineárních trojúhelníkových prvků. Je dokázán globální odhad chyby v $L^2$-normě řádu $O(h^{3/2})$ pro oblasti s hladkou hranicí. Pro jistou třídu polygonálních oblastí se odvozuje globální odhad chyby řádu $O(h^2)$.

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