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ON A DISCRETE MODIFIED $M/GI/c/\infty$ QUEUE

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Summary. The busy period distribution of a discrete modified queue $M/GI/c/\infty$, with finitely or infinitely many servers, and with different distribution functions of customer service times is derived.

Keywords: Discrete modified $M/GI/c/\infty$ queue; busy period; idle period.
AMS classification: primary 60 K 25; secondary 60 K 30.

1. INTRODUCTION

During the last few years the use of discrete queueing systems with finitely or infinitely many servers has been increasing. The discrete systems are used as mathematical models of, for example, mass servicing machines, electronic machines, transport problems, communication channels [4], automated filmless blob-length measurements in track chambers in high-energy physics [1], particle counters [2], etc.

For a modified queue we suppose that the service times and interarrival times of all customers served during any busy period are independent random variables with not necessarily identical distribution functions. The modified $M/GI/1$ queue has been investigated by Yeo [9] and Welch [8], the modified $GI/M/1$ and $GI/GI/1$ queues by Pakes [5, 6], and $GI/M/1$ by Shanthikumar [7]. The modified $GI/GI/\infty$ has been studied, using the particle counter language, in [2].

The joint distribution of the busy and idle periods, for the $GI/M/1$ queue, has been derived by Kalashnikov [3]. For the discrete modified counter with prolonging dead time, the joint distribution of the dead and the iddle period has been obtained in [2].

An important class of discrete queueing systems appears in a process of automatized measurement systems when the interartival times, T_k, have the geometric distribution

$$P(T_k = nh) = (1 - p)\, p^{n-1}\,, \quad n \geqq 1\,, \quad k \geqq 1\,, \tag{1.1}$$

where $0 < p < 1$, and $h > 0$ is a discretization step.

For example, the measurement in track chambers in high-energy physics leads to this model [1]. Along the particle trajectory we may observe a chain of streamers which are described as circles having centres on a trajectory. The number of streamer centres is a homogeneous Poisson process, and our task is to determine the blob and gap lengths. The actual measurement is performed using the scanning apparatus, so that the experimental data on the bloblength measurements have discrete values. Interpreting the blob and the gap as the busy and idle periods we obtain a discrete queueing system $M/GI/\infty$ with (1.1). Some discrete queueing models with finitely many servers and with (1.1) may appear in communication channels [4].

In the present note we derive the busy period probability law for a discrete modified $M/GI/c/\infty$ queue for any $1 \leqq c \leqq \infty$. First we concentrate on the discrete modified $M/GI/c$ queue with finitely many servers. Then we shall continue, in more detail, with the cases of single-server and two-server queues, and with the queue having infinitely many servers. We note that the formulae presented are computationally convenient for practical use, and the computational process can be simply programmed for computer, too. Some remarks on computing, and on particular cases of queues, are given in Part 5.

2. DISCRETE MODIFIED $M/GI/c/\infty$ QUEUE

Suppose that a queue is idle before the moment $t = 0$ and let the customers arrive at discrete instants $0 \leqq \tau_1 < \tau_2 < \ldots < \infty$, which are multiples of a step $h > 0$, into a queueing system with c $(1 \leqq c \leqq \infty)$ available servers, and with a waiting room having infinitely many places (if $1 \leqq c < \infty$). Let χ_k, $k \geqq 1$, be the service time of the k-th customer, and let $T_k = \tau_{k+1} - \tau_k$, $k \geqq 1$, be the interarrival time between the arrivals of the $k + 1$-st and the k-th customers. The *busy period*, B^c, is the time interval during which at least one server is busy. The *idle period*, I^c, is the time interval during which no customer is served. The sum, $C^c = B^c + I^c$, of the busy period and the subsequent idle period is called a *cycle*.

For the *discrete modified queue* we suppose that

$$P(\tau_1 = nh) = (1 - p)\, p^n\,, \quad n \geqq 0\,, \tag{2.1}$$

$$P(T_k = nh) = (1 - p)\, p^{n-1}\,, \quad n \geqq 1\,, \quad k \geqq 1\,, \tag{2.2}$$

and the first busy period is produced by the sequence of service times $\{\chi_k\}_{k=n}^{\infty}$. $\{\chi_k\}_{k=1}^{\infty}$ is assumed to be a sequence of independent positive random variables, independent of the input process $\{T_k\}_{k=1}^{\infty}$ and of τ_1, and with the distribution laws

$$P(\chi_k = nh) = h_k(n)\,, \quad n \geqq 1\,, \quad k \geqq 1\,, \tag{2.3}$$

where $\sum_{n=1}^{\infty} h_k(n) = 1$, $k \geqq 1$. Moreover, we suppose that any successive busy period is resumed with the initial conditions, independently of the previous periods, so that the sequence of busy periods (as well as idle periods and cycles) are i.i.d. random variables. This discrete modified queue will be denoted by $\mathcal{G}^c = (p; h_1, h_2, \ldots)$.

For a given queue $\mathcal{G}^c = (p; h_1, h_2, \ldots)$ it is convenient to consider a sequence of discrete modified queues, $\{\mathcal{G}_k^c\}_{k=1}^{\infty}$, where $\mathcal{G}_k^c = (p; h_k, h_{k+1}, \ldots)$, $k \geqq 1$. Define, for any $\mathcal{G}_k^c$, the corresponding busy periods, B_k^c, idle periods, I_k^c, and cycles C_k^c, respectively. For simplicity we put $h = 1$.

Due to the known properties of the geometric input process, the idle periods have the geometric distribution law with the same parameter p, that is, for $P_k^{Ic}(n) = P(I_k^c = n)$ we have

$$P_k^{Ic}(n) = (1 - p)\, p^{n-1}\,, \quad n \geqq 1\,, \quad k \geqq 1\,. \tag{2.4}$$

Moreover, the busy and idle periods are independent random variables.

Denote by A an event that the busy period begins from $t = 0$. Due to (2.1), we have $P(A) = 1 - p$. We define $p_k(j) = P(\chi_1^k = j, A)$ for $k, j \geqq 1$. Therefore

$$p_k(j) = h_k(j)\,(1 - p)\,, \tag{2.5}$$

and for $j = 0$ we put

$$p_k(0) = p\,, \quad k \geqq 1\,. \tag{2.6}$$

We denote the conditional probability in question, $P(B_k^c = n \mid A)$, by $P_k^c(n)$, and the joint distribution, $P(B_k^c = n, A)$, by $PP_k^c(n)$. Clearly

$$P_k^c(n) = PP_k^c(n)/(1 - p)\,, \quad n \geqq 1\,, \quad k \geqq 1\,. \tag{2.7}$$

Let $W_k^c(n, j) = P(B_k^c = n,\ \chi_1^k = j, A)$, $n \geqq 1$, $1 \leqq j \leqq n$, $k \geqq 1$. Then

$$PP_k^c(n) = \sum_{j=1}^{n} W_k^c(n, j)\,, \quad n \geqq 1\,, \quad k \geqq 1\,. \tag{2.8}$$

Now let $n \geqq 1$ and $1 \leqq c < \infty$ be given (the queue with infinitely many servers will be treated in Part 4). For any i, $1 \leqq i \leqq n \wedge c$, where $x \wedge y = \min(x, y)$, and for any $1 \leqq j_1 \leqq n$, $0 \leqq j_s \leqq n - s + 1 (2 \leqq s \leqq i)$, we define $A_k^c(n; j_1, \ldots, j_i)$ as the conditional probability of $B_k^c = n$ under the condition that, for any $1 \leqq s \leqq i$, at the time $t = s$ either a customer arrives and his service time is j_s (if $j_s \geqq 1$) or no customer arrives (if $j_s = 0$). Hence

$$W_k^c(n, j) = p_k(j)\, A_k^c(n; j)\,, \quad 1 \leqq j \leqq n\,, \quad n \geqq 1\,, \quad k \geqq 1\,, \tag{2.9}$$

and if $1 \leqq i \leqq n \wedge c$, then

$$A_k^c(n; j_1, \ldots, j_i) = \sum_{j_{i+1}=0}^{n-i} p_{k+i}(j_{i+1})\, A_k^c(n; j_1, \ldots, j_i, j_{i+1})\,. \tag{2.10}$$

Using the properties of the conditional probability and the independence of the dead time of the idle period we can prove the following relationships for $A_k^c(n; j_1, \ldots, j_i)$:

(2.11)
$$\begin{cases} A_k^c(1;1) = p\,, \\ PP_k^c(1) = W_k^c(1,1) = p_k(1)\,p\,. \end{cases}$$

Let $n \geqq 2$ and suppose that we know all $A_k^c(m; j_1, \ldots, j_v)$ for any $1 \leqq v \leqq m \wedge c$, $1 \leqq m < n$, $k \geqq 1$. Then the process of evaluating $A_k^c(n; j_1, \ldots, j_i)$, $1 \leqq i \leqq n \wedge c$, will be algorithmically) divided into five steps.

(I) Existence of "gaps": There is an integer u, $2 \leqq u \leqq i$, with $j_u = 0$, such that $\max(j_1, j_2 + 1, \ldots, j_{u-1} + u - 2) < u$. Then

(2.12)
$$A_k^c(n; j_1, \ldots, j_i) = 0\,.$$

In the following let there be no "gaps".

(II) Existence of "busy periods stuck together": There is an integer u, $2 \leqq u \leqq i$, such that $\max(j_1, j_2 + 1, \ldots, j_{u-1} + u - 2) = u - 1$. Then

(2.13)
$$A_k^c(n; j_1, \ldots, j_i) = A_{k+u-1-(u)_0^*}^c(n - u + 1; j_u, \ldots, j_i)\,,$$

where (u_0^*) denotes the number of zeros in $\{j_1, \ldots, j_u\}$.

Now, let there be no "busy periods stuck together".

(III) Existence of "zeros": There are two possible cases. First, we suppose that there is an integer u, $2 \leqq u \leqq i$, with $(j_1 \geqq 2)\, j_2 \geqq 1, \ldots, j_{u-1} \geqq 1$, and $j_u = 0$. Then

(2.14)
$$A_k^c(n; j_1, \ldots, j_{u-1}, 0, j_{u+1}, \ldots, j_i) =$$
$$A_{k+(u)_1^*}^c(n; j_1 - 1, \ldots, j_{u-1} - 1, j_{u+1}, \ldots, j_i),,$$

where (u_1^*) denotes the number of "ones" in $\{j_1, \ldots, j_u\}$.

Second, let $(j_1 \geqq 2)\, j_2, \ldots, j_{i-1} \geqq 1$, $j_i = 0$. Then

(2.15)
$$A_k^c(n; j_1, \ldots, j_{i-1}, 0) = A_{k+(i)_1^*}^c(n - 1; j_1 - 1, \ldots, j_{i-1} - 1)\,.$$

In the following let there be no "zeros".

(IV) Let $1 \leqq i \leqq n \leqq c$. Here we may assume without loss of generality that

(*)
$$j_1 \geqq j_2 + 1 \geqq \ldots \geqq j_i + i - 1\,.$$

Indeed, if there is an integer u, $1 \leqq u < c \wedge n$, such that $u - 1 + j_u < u + j_{u+1}$, then

(2.16)
$$A_k^c(n; j_1, \ldots, j_i) = A_{k+\delta(j_u)}^c(n; j_1, \ldots, j_{u-1}, j_{u+1} + 1, j_u - 1, j_{u+2}, \ldots, j_i)\,,$$

where $\delta(x) = 0$ if $x \neq 1$, $\delta(x) = 1$ otherwise. Applying finitely many times this argument we get (*).

Therefore, let (*) hold. Assume that the ordered i-tuple $(j_1, j_2 + 1, \ldots, j_i + i - 1)$ there are s "stairs", that is, there exist s indices $t_1, \ldots, t_s \in \{1, \ldots, i\}$ such that $t_s = 1$ and

$$j_{t_s} = \ldots = j_{t_{s-1}-1} + t_{s-1} - 2 > j_{t_{s-1}} + t_{s-1} - 1 = \ldots = j_{t_{s-2}-1} + t_{s-2} - 2 >.$$
$$\ldots > j_{t_2} + t_2 - 1 = \ldots = j_{t_1-1} + t_1 - 2 > j_{t_1} + t_1 - 1 = \ldots = j_i + i - 1\,.$$

It is convenient to put $t_{s+1} = n$. Define recursively $\tau^0 = 0$, $\tau^v = \tau^{v-1} + t_{v+1} - t_v$, for $1 \leqq v \leqq s$. Then

$$(2.17) \qquad A_k^c(n; j_1, \ldots, j_i) = pA^c_{k+\delta(j_i)}(n-1; j_1 - 1, \ldots, j_i - 1) +$$

$$A^c_{k+\delta(i_i)+1}(n-1; j_1 - 1, \ldots, j_i - 1) \sum_{u=1}^{j_i - 1} p_{k+i}(u) +$$

$$+ \sum_{v=1}^{s} \sum_{u=1}^{t_{v+1}-t_v} p_{k+i}(\tau^{v-1} + u)\, A^c_{k+\delta(j_i)+1}(n-1; j_1 - 1, \ldots, j_{t_v - 1} - 1,$$

$$j_{t_v} - 1 + u, j_{t_v+1} - 1, \ldots, j_i - 1).$$

(V) In the following we shall deal with the case $1 \leqq i \leqq c < n$. For our aim it is sufficient to consider only the case when $i = c$. Indeed, if $1 \leqq i < c$, then using (2.10) $(c - i)$-times we get the case $i = c$.

Denote, for $A_k^c(n; j_1, \ldots, j_c)$,

$$j = \min \{j_1, j_2 + 1, \ldots, j_c + c - 1\},$$

$$s = \min \{t : j_t + t - 1 = j\}.$$

First, we suppose that $j > c$. Hence $j_1, \ldots, j_c \geqq 2$, and the $c + 1$-st customer finds all servers busy, so that he may be served only if the service of the s-th customer is finished. Therefore

$$(2.18) \qquad A_k^c(n; j_1, \ldots, j_c) = pA_k^c(n-1; j_1 - 1, \ldots, j_c - 1) +$$

$$+ \sum_{u=1}^{n-j} p_{k+c}(u)\, A^c_{k+1}(n-1; j_1 - 1, \ldots, j_{s-1} - 1, j_s - 1 + u, j_{s+1} - 1, \ldots, j_c - 1).$$

Second, let $j \leqq c$. The previous four steps guarantee $j_1 \geqq 2, j_2, \ldots, j_c \geqq 1$. Hence, the $c + 1$-st customer finds at least one server idle, consequently, he may be served immediately after his arrival. Therefore

$$(2.19) \qquad A_k^c(n; j_1, \ldots, j_c) = pA^c_{k+s^*}(n-1; j_1 - 1, \ldots, j_{s-1} - 1,$$

$$c - s + 1, j_{s+1} - 1, \ldots, j_c - 1) + \sum_{u=1}^{n-c} p_{k+c}(u)\, A^c_{k+s^*+1} \,.$$

$$.\,(n-1; j_1 - 1, \ldots, j_{s-1} - 1, c - s + u, j_{s+1} - 1, \ldots, j_c - 1),$$

where s^* denotes the number of "ones" in the set $\{j_1, \ldots, j_c\} - \{j_1, \ldots, j_s\}$.

All five steps prove the following theorem.

Theorem 1. *The busy period probability law of the discrete modified queue* $\mathscr{G}_k^c = (p; h_k, h_{k+1}, \ldots)$, $k \geqq 1$, *for any* $1 \leqq c < \infty$, *is given by formula* (2.7), *where* $PP_k^c(n)$ *is algorithmically calculated from* (2.8) through (2.19).

Corollary 1.1. *The probability law of the cycle* $P_k^{C_c}(i) = P(C_k^c = i)$ *of the discrete modified queue* $\mathscr{G}_k^c = (p; h_k, h_{k+1}, \ldots)$, $k \geqq 1$, $1 \leqq c < \infty$, *is given by*

$$P_k^{C_c}(i) = \sum_{\substack{n+m=i \\ n,m \geqq 1}} P_k^c(n)\, P_k^{I_c}(m)\,, \tag{2.20}$$

where $P_k^{I_c}(m)$ *is calculated by* (2.4).

Some practical remarks on the actual computation of $A_k^c(n; j_1, \ldots, j_i)$, $1 \leqq i \leqq \leqq n \wedge c$, will be given in Part 5.

3. DISCRETE MODIFIED SINGLE SERVER AND TWO-SERVER QUEUES

Here we concentrate on the discrete modified single server $\mathscr{G}_k^1 = (p; h_k, h_{k+1}, \ldots)$, $k \geqq 1$. In this case it is clear that it is necessary to evaluate only $A_k^1(n; j)$, for any $1 \leqq j \leqq n$ and any $k \geqq 1$. Consequently, formulae (2.13) and (2.18) have simpler forms. Summarizing this we have, for $PP_k^1(n)$ and $A_k^1(n; j)$, $k \geqq 1$, the following recursive relationships:

$$\begin{cases} A_k^1(1; 1) = p\,, \\ PP_k^1(1) \;\;= p_k(1)\, p\,. \end{cases} \tag{3.1}$$

If $n \geqq 2$, then

$$A_k^1(n; 1) = PP_{k+1}^1(n-1)\,. \tag{3.2}$$

If $2 \leqq j \leqq n-1$, then

$$A_k^1(n; j) = pA_k^1(n-1; j-1) + \sum_{i=1}^{n-j} p_k(i)\, A_{k+1}^1(n-1; j+i-1)\,, \tag{3.3}$$

$$A_k^1(n; n) = pA_k^1(n-1; n-1) = p^n\,. \tag{3.4}$$

Hence, for any $n \geqq 1$, we have

$$PP_k^1(n) = \sum_{j=1}^{n} p_k(j)\, A_k^1(n; j)\,, \tag{3.5}$$

$$P_k^1(n) \;= PP_k^1(n)/(1-p)\,. \tag{3.6}$$

Theorem 2. *The busy period probability law of the discrete modified single server queue* $\mathscr{G}_k^1 = (p; h_k, h_{k+1}, \ldots)$, $k \geqq 1$, *is given by* (3.6), *where* $PP_k^1(n)$ *and* $A_k^1(n; j)$ *are calculated from* (3.1) *through* (3.5).

For the discrete modified queue with two servers, $\mathscr{G}_k^2 = (p; h_k, h_{k+1}, \ldots)$, $k \geqq 1$, the general formulae from Part 2 reduce to the following form:

$$\begin{cases} A_k^2(1; 1) = p\,, \\ PP_k^2(1) \;\;= p_k(1)\, p; \end{cases} \tag{3.7}$$

(3.8)
$$\begin{cases} A_k^2(2;1) = PP_{k+1}^2(1)\,, \\ A_k^1(2;2) = p^2 + PP_{k+1}^2(1)\,, \\ PP_k^2(2) \ = p_k(1)\,A_k^2(2;1) + p_k(2)\,A_k^2(2;2)\,; \end{cases}$$

(3.9)
$$\begin{cases} A_k^2(2;1,0) = 0\,, \quad A_k^2(2;1,1) = p\,, \\ A_k^2(2;2,0) = p\,, \quad A_k^2(2;2,1) = p\,. \end{cases}$$

Now let $n \geqq 2$. Then

(3.10)
$$\begin{cases} A_k^2(n;1,0) = 0\,, \\ A_k^2(n;1,i) \ = A_k^2(n-1;i)\,, \quad 1 \leqq i \leqq n-1\,. \end{cases}$$

If $2 \leqq j \leqq n$, then

(3.11)
$$A_k^2(n;j,0) = A_k^2(n-1;j-1)\,.$$

For $1 \leqq i \leqq n-1$, there are two possible cases.

First, let $i + 1 \leqq j$, then

(3.12)
$$A_k^2(n;j,i) = pA_{k+\delta(i)}^2(n-1;j-1,i-1) + \\ + \sum_{u=1}^{n-i-1} p_{k+2}(u)\,A_{k+1}^2(n-1;j-1,i-1+u)\,,$$

where $\delta(x) = 1$ if $x = 1$, $\delta(x) = 0$ otherwise.

Second, let $i + 1 > j$, then

(3.13)
$$A_k^2(n;j,i) = pA_{k+\delta(j-1)}^2(n-1;i,j-2) + \\ + \sum_{u=1}^{n-j} p_{k+2}(u)\,A_{k+1}^2(n-1;i,j-2+u)\,.$$

Hence, for any $n \geqq 2$, we have

(3.14)
$$PP_k^2(n) = \sum_{j=1}^{n} \sum_{i=0}^{n-1} p_k(j)\,p_{k+1}(i)\,A_k^2(n;j,i)\,,$$

and finally

(3.15)
$$P_k^2(n) = PP_k^2(n)/(1-p)\,,$$

which proves the following theorem.

Theorem 3. *The busy period probability law of the discrete modified two-server queue* $\mathscr{G}_k^2 = (p; h_k, h_{k+1}, \ldots)$, $k \geqq 1$, *is given by* (3.15) *where we use formulae* (3.7) *through* (3.14).

4. DISCRETE MODIFIED $M/GI/\infty$ QUEUE

The method developed in Part 2 for the discrete modified queue $M/GI/c/\infty$, where $1 \leqq c < \infty$, may be used for the discrete modified queue with infinitely many servers. Obviously here we do not need the waiting room because any customer finds at least one idle server. In the following we shall see that in order to determine $P_k^\infty(n) = P(B_k^\infty = n \mid A)$ it is necessary to evaluate only $A_k^\infty(n; j)$, for any $1 \leqq j \leqq n$. Therefore step (IV) and formula (2.17) have simpler form, and we obtain the following formulae, for $\mathscr{G}_k^\infty = (p; h_k, h_{k+1}, \ldots)$, $k \geqq 1$:

(4.1) $$P_k^\infty(n) = PP_k^\infty(n)/(1-p), \quad n \geqq 1,$$

(4.2) $$PP_k^\infty(n) = \sum_{j=1}^{n} W_k^\infty(n, j), \quad n \geqq 1;$$

(4.3) $$W_k^\infty(n, j) = p_k(j) A_k^\infty(n; j), \quad 1 \leqq j \leqq n, \quad n \geqq 1;$$

(4.4) $$\begin{cases} A_k^\infty(1; 1) = p, \\ PP_k^\infty(1) = W_k^\infty(1, 1) = p_k(1)\, p. \end{cases}$$

Let $n \geqq 2$. Then

(4.5) $$A_k^\infty(n; 1) = PP_k^\infty(n-1),$$

and, for any $2 \leqq j \leqq n$, we have

(4.6) $$A_k^\infty(n; j) = pA_k^\infty(n-1; j-1) + A_{k+1}^\infty(n-1; j-1) \sum_{i=1}^{j-1} p_{k+1}(i) + \sum_{i=j}^{n-1} W_{k+1}^\infty(n-1, i)$$

(here, as usual, the sum over the empty set is defined as 0). This proves the following theorem.

Theorem 4. *The busy period probability law of the discrete modified queue* $\mathscr{G}_k^\infty = (p; h_k, h_{k+1}, \ldots)$, $k \geqq 1$, *is given by* (4.1), *where* $PP_k^\infty(n)$ *is evaluated from* (4.2) *through* (4.6).

We note that the cycle probability law of the discrete modified queque $\mathscr{G}_k^\infty = (p; h_k, h_{k+1}, \ldots)$, $k \geqq 1$, is given by (2.20), where we put $c = \infty$.

Theorem 4 generalizes the analogous result from [2] concerning the discrete modified queque $\mathscr{G}^\infty = (p; h_1, h_2, \ldots)$, where $h_1 = h_2 = \ldots$.

5. CONCLUSION

We see that the actual computation of the busy period probability law is relatively simple in the case when we have either only few servers (for example, $c = 1, 2$) or infinitely many servers. In the other cases the result of Theorem 1 may be simply programmed for a computer. Here we note only that the following relationships hold.

If $1 \leqq i \leqq n \leqq c < \infty$, then

(5.1) $$A_k^c(n; j_1, \ldots, j_i) = A_k^{c+1}(n; j_1, \ldots, j_i),$$

(5.2) $$A_k^c(n; j) = A_k^{c+1}(n; j) = A_k^\infty(n; j),$$

(5.3) $$P_k^c(n) = P_k^{c+1}(n) = P_k^\infty(n),$$

and this enables us to simplify the computation for a queue with a large number, of available servers.

If $h_k(j)$ $(k \geqq 1)$ are non-zero only for few integers j, then the calculation is simple, too. Indeed, it suffices to evaluate, for example, for $W_k^c(n, j)$, only $W_k^c(n, j)$ with $h_k(j) > 0$. Analogously we proceed with the other quantities $A_k^c(n; j_1, \ldots, j_i)$.

We say that a discrete modified queue $\mathscr{G}^c = (p; h_1, h_2, \ldots)$, $1 \leqq c \leqq \infty$, is of *order* m, if $h_m = h_{m+1} = \ldots$. If $m = 1$, then we obtain the usual (non-modified) queue, and in this case all the above formulae do not depend on the subscripts k.

If $m > 1$, then the computation of the busy period probability laws for the queues $\mathscr{G}_k^c = (p; h_k, h_{k+1}, \ldots)$, $1 \leqq k \leqq m$, $1 \leqq c \leqq \infty$, may be organized so that first of all we calculate all necessary expressions for $k = m$, then we continue for $k = m - 1, \ldots, k = 1$.

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Súhrn

O DISKRÉTNOM MODIFIKOVANOM SYSTÉME HROMADNEJ OBSLUHY TYPU $M/GI/c/\infty$

ANATOLIJ DVUREČENSKIJ

Pre diskrétny modifikovaný systém hromadnej obsluhy typu $M/GI/c/\infty$ s konečným alebo nekonečným počtom obslúh, ale s rôznymu funkciami rozdelenia obslúh zákazníkov, odvádza sa rozdelenie dĺžky periódy obsadenosti.

Резюме

О ДИСКРЕТНОЙ МОДИФИЦИРОВАННОЙ $M/GI/c/\infty$ ОЧЕРЕДИ

ANATOLIJ DVUREČENSKIJ

Для дискретной модифицированной системы $M/GI/c/\infty$ с конечным или бесконечным числом обслуживающих каналов и с разными функциями разделений времен обслуживания выводится распределение длины периода занятости.

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