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A REMARK CONCERNING UNIQUENESS
OF THE WOLD DECOMPOSITION
OF FINITE-DIMENSIONAL STATIONARY PROCESSES

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Summary. The uniqueness of the Wold decomposition of a finite-dimensional stationary process without assumption of time-containedness is proved. As a corollary the correspondence between the Wold decomposition of full rank stationary process and the Lebesgue decomposition of its spectral measure is easily obtained.

Key words: Stationary process, Wold decomposition, spectral measure.

AMS Classification: 60 G 10, secondary 47 B 15

1. INTRODUCTION AND PRELIMINARIES

An orthogonal decomposition of a stationary process into the regular and singular parts was established for the first time by H. Wold [7]. A more abstract form which points out the operator-theoretical nature of the fact can be found in [1] (cf. also [5]). It may seem to be little surprising that the natural assumption of the so-called time containedness of the regular part is of no importance for the uniqueness of the decomposition in the one-dimensional case. In fact, the same argument applies to stationary processes generated by a set of elements for which the regular part is n-dimensional. The proof requires elementary Hilbert space geometry only.

As a consequence of the uniqueness theorem we obtain a new and more elementary proof of the correspondence between the Wold decomposition of a full rank stationary process and the Lebesgue decomposition of its spectral measure.

Let $\mathcal{H}$ be a Hilbert space. We shall denote by $\mathcal{P}(\mathcal{L})$ the orthogonal projection of $\mathcal{H}$ onto a closed subspace $\mathcal{L}$ of $\mathcal{H}$. All projections are considered to be orthogonal.

A sequence $(f_n)_{n \in \mathbb{Z}}$ of vectors in $\mathcal{H}$ is called a (discrete time) stationary process if the scalar products $(f_n, f_m)$ depend of the difference $n - m$ only, i.e.

$$(f_{n+k}, f_{m+k}) = (f_n, f_m) \quad \text{for all} \quad n, m, k \in \mathbb{Z}. $$
Since an analogous relation holds for linear combinations of vectors $f_j$ it follows that there exists a unitary operator $U$ acting on the whole space $\mathcal{H}$ which satisfies
\[ Uf_n = f_{n+1} \quad \text{or equivalently} \quad U^n f_0 = f_n \]
for all $n \in \mathbb{Z}$, and $U$ is uniquely determined on the reducing subspace containing $\bigvee_{j \in \mathbb{Z}} f_j$, the closed linear span of all $f_j$. Conversely, given a unitary operator $U \in B(\mathcal{H})$ and an $x \in \mathcal{H}$, the sequence $(f_n = U^n x)_{n \in \mathbb{Z}}$ is a stationary process. The above consideration allows us to introduce the following definition.

1.1 Definition. A triplet $(\mathcal{H}, U, x)$, $\mathcal{H}$ a Hilbert space, $U \in B(\mathcal{H})$ a unitary operator and $x \in \mathcal{H}$, is called a stationary process.

Similarly, a double sequence $(f_j^i)_{i \in \mathbb{Z}}$, $i = 1, 2, \ldots, N$, of vectors from $\mathcal{H}$ is called a finite dimensional stationary process if the Gram matrix $(f_i^j, f_j^m)_{i, j = 1}^N$ depends on the difference $n - m$ only. Obviously, we can use the same reasoning as before so that the following definition describes the more general situation.

1.2 Definition. Let $U \in B(\mathcal{H})$ be a unitary operator and $\mathcal{X}$ a subset of $\mathcal{H}$. Then $(\mathcal{H}, U, \mathcal{X})$ is called a stationary process.

Consider now a stationary process $(\mathcal{H}, U, \mathcal{X})$, $\mathcal{X} \subset \mathcal{H}$. Denote by $E_{\mathcal{X}}(H_{\mathcal{X}})$ the smallest invariant (reducing, respectively) subspace of $U^*$ containing $\mathcal{X}$, i.e.
\[ E_{\mathcal{X}} = \bigvee_{k \leq 0} U^k \mathcal{X}, \quad H_{\mathcal{X}} = \bigvee_{k = -\infty}^{\infty} U^k \mathcal{X}. \]

The restriction $U^* \mid E_{\mathcal{X}}$ is an isometry so that the Wold decomposition applies. In other words, the space $E_{\mathcal{X}}$ can be decomposed into a direct sum of two subspaces reducing with respect to $U^* \mid E_{\mathcal{X}}$,
\[ E_{\mathcal{X}} = \bigoplus_{k \leq 0} U^k E_{\mathcal{X}} \oplus ((E_{\mathcal{X}} \ominus U^* E_{\mathcal{X}}) \oplus U^*(E_{\mathcal{X}} \ominus U^* E_{\mathcal{X}}) \oplus \ldots), \]
so that the restriction of $U^*$ to the first subspace is a unitary operator and the restriction to the second is a unilateral shift of multiplicity $\dim \text{span} \mathcal{X}$ (see [5], p. 4).

Let $\mathcal{R}_\mathcal{X} = \cap U^k E_{\mathcal{X}}$, and denote by $\mathcal{F}_\mathcal{X}$ the wandering subspace, $\mathcal{F}_\mathcal{X} = E_{\mathcal{X}} \ominus \bigoplus_{k \leq 0} U^k E_{\mathcal{X}}$. We shall also use the notation $M_+(\mathcal{F}_\mathcal{X}) = \bigoplus_{k \leq 0} U^k \mathcal{F}_\mathcal{X}$ and $M(\mathcal{F}_\mathcal{X}) = \bigoplus_{k = -\infty}^{\infty} U^k \mathcal{F}_\mathcal{X}$.

Moreover, this decomposition is unique in the following sense: if $E_{\mathcal{X}} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and $U^* \mid \mathcal{H}_1$ is unitary and $U^* \mid \mathcal{H}_2$ is a unilateral shift then $\mathcal{H}_1 = \mathcal{R}_\mathcal{X}$ and $\mathcal{H}_2 = M_+(\mathcal{F}_\mathcal{X})$.

Clearly,
\[ E_{\mathcal{X}} = M_+(\mathcal{F}_\mathcal{X}) \oplus \mathcal{R}_\mathcal{X}, \quad H_{\mathcal{X}} = M(\mathcal{F}_\mathcal{X}) \oplus \mathcal{R}_\mathcal{X}. \]
1.3 Definition. A stationary process \((\mathcal{H}, U, \mathcal{X})\) is called singular if \(E_{\mathcal{X}} = H_{\mathcal{X}}\). It is called regular if \(Q_{\mathcal{X}} = \{0\}\).

If we denote \(Q = 1 - P(\mathcal{H})\) then \(QU = UQ\) and \(Q(\mathcal{X}) \subseteq M_+(\mathcal{F}_\mathcal{X}) \subseteq E_{\mathcal{X}}\). Since \(QU = UQ\) the subspaces \(H_{Q\mathcal{X}}\) and \(H_{(1-Q)\mathcal{X}}\) are orthogonal and \(x = Q\mathcal{X} + (1-Q)\mathcal{X}\) for each \(x \in \mathcal{X}\). Further, the process \((\mathcal{H}, U, Q\mathcal{X})\) is regular and \((\mathcal{H}, U, (1-Q)\mathcal{X})\) is singular. Indeed,

\[
E_{(1-Q)\mathcal{X}} = \bigvee_{n \leq 0} U^n P(\mathcal{H}) \mathcal{X} = \text{clos} \left( P(\mathcal{H}) \bigvee_{n \leq 0} U^n \mathcal{X} \right) = \text{clos} \left( P(\mathcal{H}) E_{\mathcal{X}} \right) = \mathcal{R}_{\mathcal{X}} = \text{clos} \left( P(\mathcal{H}) H_{\mathcal{X}} \right) = \text{clos} \left( P(\mathcal{H}) \bigvee_{n \in \mathbb{Z}} U^n \mathcal{X} \right) = \bigvee_{n \in \mathbb{Z}} P(\mathcal{H}) \mathcal{X} = H_{(1-Q)\mathcal{X}}.
\]

Since \(Q\mathcal{X} \subseteq M_+(\mathcal{F}_\mathcal{X})\) we also have \(E_{Q\mathcal{X}} \subseteq M_+(\mathcal{F}_\mathcal{X})\) and

\[
\mathcal{R}_{Q\mathcal{X}} = \bigcap_{k \leq 0} U^k E_{Q\mathcal{X}} \subseteq \bigcap_{k \leq 0} U^k M_+(\mathcal{F}_\mathcal{X}) = \{0\}.
\]

On the other hand, it follows from the uniqueness of the Wold decomposition that if \(P\) is any projection such that it commutes with \(U\), maps \(\mathcal{X}\) into \(E_{\mathcal{X}}, (\mathcal{H}, U, P\mathcal{X})\) is regular and \((\mathcal{H}, U, (1-P)\mathcal{X})\) singular, then \(P | H_{\mathcal{X}} = Q | H_{\mathcal{X}}\). We can now sum up these facts in the following definition.

1.4 Definition. Let \((\mathcal{H}, U, \mathcal{X})\) be a stationary process. The only pair of stationary processes \((\mathcal{H}, U, Q\mathcal{X})\) and \((\mathcal{H}, U, (1-Q)\mathcal{X})\) is called the Wold decomposition of \((\mathcal{H}, U, \mathcal{X})\), if

1° \(Q\) is a projection such that \(QU = UQ\) and \(Q\mathcal{X} \subseteq E_{\mathcal{X}}, (\mathcal{H}, U, P\mathcal{X})\) is regular and \((\mathcal{H}, U, (1-P)\mathcal{X})\) singular.

2° \((\mathcal{H}, U, Q\mathcal{X})\) is regular with \(\dim \mathcal{F}_{Q\mathcal{X}} = \dim \mathcal{F}_{\mathcal{X}}\) and \((\mathcal{H}, U, (1-Q)\mathcal{X})\) is singular.

2. THE UNIQUENESS OF DECOMPOSITION

We shall use a slightly modified version of the Wold decomposition based on the fact that a bilateral shift of finite multiplicity cannot contain a bilateral shift of higher multiplicity (see [5], Proposition 2.1). Precisely, if \(W\) is a unitary operator and \(\mathcal{L}_1, \mathcal{L}_2\) two wandering subspaces of \(W\) such that \(M(\mathcal{L}_1) \subset M(\mathcal{L}_2)\) and \(\dim \mathcal{L}_1 = \dim \mathcal{L}_2 < \infty\) then \(M(\mathcal{L}_1) = M(\mathcal{L}_2)\).

The inclusion \(Q\mathcal{X} \subseteq E_{\mathcal{X}}\) in condition 1° of 1.4 implies \(E_{Q\mathcal{X}} \subseteq E_{\mathcal{X}}\) and has a natural meaning: "the past" of the regular part in the Wold decomposition depends on "the past" of the initial process only. Nevertheless, it may be replaced by a weaker one.

2.1 Proposition. Let \((\mathcal{H}, U, \mathcal{X})\) be a stationary process. Then there exists an orthogonal projection \(Q\) such that

1° \(QU = UQ\), \(Q\mathcal{X} \subseteq H_{\mathcal{X}}\),

2° \((\mathcal{H}, U, Q\mathcal{X})\) is regular with \(\dim \mathcal{F}_{Q\mathcal{X}} = \dim \mathcal{F}_{\mathcal{X}}\) and \((\mathcal{H}, U, (1-Q)\mathcal{X})\) is singular.
Conversely, if \( \dim \mathcal{F}_x < \infty \) and \( Q \) satisfies \( 1^\circ \) and \( 2^\circ \) then \( \mathcal{H}, U, Q_x, (\mathcal{H}, U, (1 - Q) \mathcal{F}) \) is the Wold decomposition of \( \mathcal{H}, U, \mathcal{F} \).

**Proof.** It is easy to see that \( Q = 1 - P(\mathcal{F}_x) \) also satisfies \( \dim \mathcal{F}_{Q_x} = \dim \mathcal{F}_x \).

To prove the second part of the assertion let us consider a projection \( Q \) satisfying \( 1^\circ \) and \( 2^\circ \). Condition \( 1^\circ \) implies \( H_{Q_x} \subset H_x \). Using the singularity of \( \mathcal{H}, U, (1 - Q) \mathcal{F} \) we have

\[
E_x = E_Q \oplus E_{(1 - Q)} = E_{Q_x} \oplus H_{(1 - Q)x}
\]

and, for \( n \in \mathbb{Z} \),

\[
U^n E_x = U^n E_{Q_x} \oplus U^n H_{(1 - Q)x} = U^n E_{Q_x} \oplus H_{(1 - Q)x}.
\]

Condition \( 1^\circ \) and regularity of \( \mathcal{H}, U, Q_x \) imply

\[
\mathcal{F}_x = \cap U^n E_x \subset \bigcap U^n E_{Q_x} \oplus H_{(1 - Q)x} = H_{(1 - Q)x} = H_x \oplus H_{Q_x},
\]

hence \( M(\mathcal{F}_{Q_x}) = H_{Q_x} \subset H_x \ominus \mathcal{F}_x = M(\mathcal{F}_x) \). Both \( \mathcal{F}_{Q_x} \) and \( \mathcal{F}_x \) are wandering subspaces of \( U \mid H_x \) and, by \( 2^\circ \), \( \dim \mathcal{F}_{Q_x} = \dim \mathcal{F}_x \). If \( \dim \mathcal{F}_x < \infty \) then \( M(\mathcal{F}_{Q_x}) = M(\mathcal{F}_x) \) by Prop. 2.1 of [5].

Clearly, \( Q \mid H_x \) is an orthogonal projection and \( QH_x = H_{Q_x} = M(\mathcal{F}_{Q_x}) \). On the other hand, \( M(\mathcal{F}_x) = (1 - P(\mathcal{F}_x)) H_x \), thus \( Q \mid H_x = (1 - P(\mathcal{F}_x)) \mid H_x \).

The proof is complete.

The following example shows that if \( \dim \mathcal{F}_x = \infty \), conditions \( 1^\circ \) and \( 2^\circ \) do not imply the uniqueness of the decomposition.

**2.2 Example.** Consider the following double sequence of orthonormal vectors in a Hilbert space \( \mathcal{H} \),

\[
\cdots e_{0,-2} e_{0,-1} e_{00} e_{01} e_{02} \cdots \\
\cdots e_{1,-1} e_{10} e_{11} \cdots \\
\cdots e_{20} \cdots \\
\cdots
\]

and define a unitary operator \( U \in B(\mathcal{H}) \) satisfying

\[
U e_{ij} = e_{i,j-1} \quad \text{for} \quad i \geq 0, \quad j \in \mathbb{Z}.
\]

If \( \mathcal{X} = \{e_k : k \geq 0\} \) then \( \mathcal{H}, U, \mathcal{F} \) is clearly a regular stationary process and \( \dim \mathcal{F}_x = \infty \). Let us define

\[
m = \sum_{k=0}^{\infty} 2^{-k} e_{kk}, \quad \mathcal{M} = H_m.
\]

The projection \( Q = 1 - P(\mathcal{M}) \) clearly satisfies condition \( 1^\circ \) and we shall show that it also satisfies condition \( 2^\circ \) of Proposition 2.1. By easy computation we have, for \( k \geq 0 \),

\[
P(\mathcal{M}) e_{k0} = P(\mathcal{M}) U^k e_{kk} = U^k P(\mathcal{M}) e_{kk} = 2^{-k} U^k P(\mathcal{M}) e_{00} = 2^{-k} e_{kk}.
\]
because 
\[ 2^k e_{kk} - e_{00} \perp \mathcal{M}. \]

Since
\[
H_{P(\mathcal{M})^*} = \bigvee_{k \geq 0} U^* P(\mathcal{M}) e_{00} = \bigvee_{k \geq 0} U^* U^k P(\mathcal{M}) e_{00} = \bigvee_{k \geq 0} (P(\mathcal{M}) \vee U^{*n-k} e_{00}) = \bigvee_{k \geq 0} U^* U^k P(\mathcal{M}) e_{00} = \bigvee_{k \geq 0} U^* P(\mathcal{M}) e_{00} = E_{P(\mathcal{M})^*},
\]
the process \((\mathcal{H}, U, P(\mathcal{M})^*)\) is singular.

Now, we shall show that \((\mathcal{H}, U, (1 - P(\mathcal{M}))^*)\) is regular. If we denote by \(\mathcal{X} = U^* P(\mathcal{M}) e_{00} = U^* U^* P(J^t) e_{00} = \bigvee_{k \geq 0} U^* U^k P(\mathcal{M}) e_{00} = \bigvee_{k \geq 0} U^* P(\mathcal{M}) e_{00} = E_{P(\mathcal{M})^*}\),
\[
H_\mathcal{X} = \bigoplus_{k \in \mathbb{Z}} U^k \mathcal{X}.
\]
To compute \(R_{P(\mathcal{M})^*}\) we shall use the inclusion
\[
U^n E_{P(\mathcal{M})^*} = U^n \bigvee_{k \geq 0} (P(\mathcal{M})^*)^n E_{\mathcal{X}} = \bigvee_{k \geq 0} (P(\mathcal{M})^*)^n E_{\mathcal{X}} \subset U^n \mathcal{X} \vee \mathcal{M}
\]
and the decomposition
\[
U^n \mathcal{X} \vee \mathcal{M} = \bigoplus_{k \geq 0} (U^n E_{\mathcal{X}} \vee \mathcal{M}) \cap U^k \mathcal{X}.
\]
For any \(n \geq 0\), we have also
\[
(U^n \mathcal{X} \vee \mathcal{M}) \cap \mathcal{X} = m \vee \bigvee_{j \geq n} e_{jj}
\]
so that
\[
U^n \mathcal{X} \vee \mathcal{M} = \bigoplus_{k \geq 0} (U^n E_{\mathcal{X}} \vee \mathcal{M}) \cap U^k \mathcal{X} = \bigoplus_{k \geq 0} U^k ((U^n E_{\mathcal{X}} \vee \mathcal{M}) \cap \mathcal{X}) = \bigoplus_{k < n} U^k (m \vee \bigvee_{j \geq n-k} e_{jj}) \bigoplus_{k \geq n} U^k \mathcal{X}.
\]
Denoting
\[
A_{nk} = \begin{cases} U^k (m \vee \bigvee_{j \geq n-k} e_{jj}), & k < n, \\ U^k \mathcal{X}, & k \geq n, \end{cases}
\]
for any \(n \geq 0\), we clearly have \(A_{n+1,k} \subset A_{nk}\) and \(\bigcap_{n \geq 0} A_{nk} = \mathcal{M} \cap U^k \mathcal{X}\). The equality
\[
U^n \mathcal{X} \vee \mathcal{M} = \bigoplus_{k \geq 0} A_{nk}
\]
now implies
\[
R_{P(\mathcal{M})^*} = \bigcap_{n \geq 0} U^n E_{P(\mathcal{M})^*} \subset \bigcap_{n \geq 0} (U^n \mathcal{X} \vee \mathcal{M}) = \bigcap_{n \geq 0} \bigoplus_{k \geq 0} A_{nk} \subset \mathcal{M}.
\]
On the other hand, \(R_{P(\mathcal{M})^*} \subset \mathcal{M}^\perp\) so that \(R_{P(\mathcal{M})^*} = \{0\}\) and the regularity of \((\mathcal{H}, U, P(\mathcal{M})^*)\) is proved.
3. STATIONARY PROCESSES WITH THE SPECTRAL MEASURE ABSOLUTELY CONTINUOUS WITH RESPECT TO THE LEBESGUE MEASURE

Let us now consider the Hilbert space $L^2 = L^2(T)$ with the norm $\|f\|^2 = \int_T |f|^2 \, dm$ where $T$ is the unit circle and $m$ the normalized Lebesgue measure on $T$. As usual, denote by $S$ the unitary operator of multiplication by $e^{it}$ on $L^2$. Given a natural number $n$, we shall denote by $L^2(n)$ the Hilbert space of all $n$-tuples $f = (f_1, \ldots, f_n)$ with $f_i \in L^2$ $(i = 1, 2, \ldots, n)$ equipped with the scalar product $(f, g) = \sum_{i=1}^{n} (f_i, g_i)$.

Let $S_n \in B(L^2(n))$ be the bilateral shift operator, $S_n f = (Sf_1, \ldots, Sf_n)$, $f \in L^2(n)$. Obviously $L^2(n) = M(\mathcal{F}, n)$ where $\mathcal{M} = \{e_j: e_{jk} = \delta_{jk}, j, k = 1, 2, \ldots, n\}$.

3.1 Definition. Let $(\mathcal{H}, U, \mathcal{X})$ be a stationary process. Denote by $E$ the spectral measure of $U$. The set of Borel measures

$$\mu_{x,y} = \{(E(\cdot), x, y): x, y \in \mathcal{X}\}$$

will be called the spectral measure of $(\mathcal{H}, U, \mathcal{X})$. We shall say that $\mu_x \ll m (\mu_x \perp m)$ iff $\mu_{x,y} \ll m (\mu_{x,y} \perp m)$, respectively for all $x, y \in \mathcal{X}$.

If $\mathcal{X}$ consists of a single element $x$ then the spectral measure of $(\mathcal{H}, U, \mathcal{X})$ is nonnegative, $\mu_x = |E(\cdot) x|^2$.

If $\mathcal{X}$ is finite, $\mathcal{X} = \{x_1, \ldots, x_n\}$, then the spectral measure of $(\mathcal{H}, U, \mathcal{X})$ can be considered as a matrix $\mu_{x} = (\mu_{ij})^n$ with nonnegative diagonal entries.

3.2 Lemma. Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces, $U_1 \in B(\mathcal{H}_1), U_2 \in B(\mathcal{H}_2)$ unitary operators and $\mathcal{X} \subset \mathcal{H}$. If $\Phi \in B(\mathcal{H}_1, \mathcal{H}_2)$ is an isometry such that $\Phi U_1 = U_2 \Phi$ then

1° $E_{\Phi\mathcal{X}} = \Phi E_{\mathcal{X}}$ and $\mathcal{F}_{\Phi\mathcal{X}} = \Phi \mathcal{F}_{\mathcal{X}}$,

2° $H_{\Phi\mathcal{X}} = \Phi H_{\mathcal{X}}$,

3° $\mathcal{R}_{\Phi\mathcal{X}} = \Phi \mathcal{R}_{\mathcal{X}}$.

Proof.

$$E_{\Phi\mathcal{X}} = \bigvee_{k \leq 0} U_{1}^{k} \Phi \mathcal{X} = \bigvee_{k \leq 0} \Phi U_{1}^{k} \mathcal{X} = \Phi \bigvee_{k \leq 0} U_{1}^{k} \mathcal{X} = \Phi E_{\mathcal{X}}$$

and

$$\mathcal{F}_{\Phi\mathcal{X}} = E_{\Phi\mathcal{X}} \ominus U_{2} E_{\Phi\mathcal{X}} = \Phi E_{\mathcal{X}} \ominus U_{2} \Phi E_{\mathcal{X}} = \Phi E_{\mathcal{X}} \ominus \Phi U_{1} E_{\mathcal{X}} = \Phi (E_{\mathcal{X}} \ominus U_{1} E_{\mathcal{X}}) = \Phi \mathcal{F}_{\mathcal{X}}.$$ 

Similarly,

$$H_{\Phi\mathcal{X}} = \Phi H_{\mathcal{X}}.$$ 

Further,

$$\mathcal{R}_{\Phi\mathcal{X}} = \bigcap_{k \leq 0} U_{2}^{k} E_{\Phi\mathcal{X}} = \bigcap_{k \leq 0} \Phi U_{2}^{k} E_{\mathcal{X}} = \bigcap_{k \leq 0} \Phi U_{2}^{k} E_{\mathcal{X}} = \Phi \mathcal{R}_{\mathcal{X}}.$$ 

3.3 Proposition. Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ be a subset of $\mathcal{H}$ such that the stationary process $(\mathcal{H}, U, \mathcal{X})$ satisfies $\dim \mathcal{F}_{\mathcal{X}} = n$ and $\mu_x \ll m$. Then $(\mathcal{H}, U, \mathcal{X})$ is regular.
Proof. Since \( p_t \leq m \) there exist functions \( f_{ij} \in L^1(T) \) such that \( f_{ij} = \frac{d \mu_{ij}}{dm} \) \((i, j = 1, \ldots, n)\). Given \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \) we have

\[
\sum_{i,j} \lambda_i \lambda_j^* \mu_{ij}(\cdot) = |E(\cdot) \sum_i \lambda_i x_i|^2 \geq 0
\]

so that \( \sum \lambda_i \lambda_j^* \mu_{ij}(\cdot) \) is a nonnegative Borel measure on \( T \) which is absolutely continuous with respect to \( m \). Consequently, its density \( \sum \lambda_i \lambda_j^* f_{ij} \) is nonnegative a.e. This implies that there exists a Borel subset \( \sigma_0 \) of \( T \) such that \( m(\sigma_0) = 1 \), all functions \( f_{ij} \) are defined on \( \sigma_0 \) and \( \sum \lambda_i \lambda_j^* f_{ij}(t) \geq 0 \) for \( t \in \sigma_0 \), \( \lambda_1, \ldots, \lambda_n \in \mathbb{C} \).

In other words, matrices \( (f_{ij}(t)) \) are positive semidefinite so that there exist functions \( \varphi_{ij} \) defined on \( \sigma_0 \) such that

\[
(f_{ij}(t)) = (\varphi_{ij}(t)) (\varphi_{ij}(t))^* \quad \text{for} \quad t \in \sigma_0.
\]

Since

\[
\sum_{k=1}^n |\varphi_{ik}(t)|^2 = f_{ii}(t)
\]

for \( t \in \sigma_0 \) we have \( \varphi_{ij} \in L^2(T) \).

Let us now set

\[
\Phi X_j = \varphi_j = (\varphi_{j1}, \ldots, \varphi_{jn}) \in L^2(n).
\]

The relations \( (x_i, x_j) = (\varphi_i, \varphi_j)(i, j = 1, \ldots, n) \) make it possible to define an isometry \( \Phi \) on \( H_x \) with values in \( L^2(n) \) which satisfies

\[
\Phi x_j = \Phi x_j \quad \text{and} \quad \Phi U = S_n \Phi.
\]

According to Lemma 3.2 the process \((L^2(n), S_n, \Phi X)\) satisfies \( \dim \mathcal{F}_x = n \). Now, using Proposition 2.1 of [5] we deduce that \((L^2(n), S_n, \Phi X)\) is regular and, consequently, \((X, U, X)\) is regular as well. The proof is complete.

If \( n = 1 \) then there are only two possibilities: either \((X, U, X)\) is singular or \( \dim \mathcal{F}_x = 1 \). So we have

3.4 Corollary. Let \((X, U, X)\) be a stationary process satisfying \( \mu_x \leq m \). Then it is either regular or singular.

4. THE LEBESGUE DECOMPOSITION OF THE SPECTRAL MEASURE

Let \((X, U, X)\) be a stationary process with the spectral measure \( \mu_X \). If \( P \) is a projection which commutes with \( U \) then \( P \) also commutes with \( E(\cdot) \) and, for \( x, y \in X \),

\[
\mu_{x,y} = (E(\cdot) x, y) = (E(\cdot) Px, Py) + (E(\cdot) (1 - P) x, (1 - P) y) = \mu_{Px,Py} + \mu_{(1-P)x,(1-P)y},
\]

or shortly,

\[
\mu_x = \mu_{Px} + \mu_{(1-P)x}.
\]

Clearly \( \mu_{Px} \leq \mu_x \) and \( \mu_{(1-P)x} \leq \mu_x \).
The spectral measure of a regular process \((\mathcal{H}, U, \mathcal{B})\) is absolutely continuous with respect to \(m\). Indeed, the unitary operator \(U \mid H_\mathcal{B}\) is a bilateral shift so that its spectral measure is equivalent to \(m ([5])\). It follows that the spectral measure of a non-singular process \((\mathcal{H}', U, \mathcal{B}')\) cannot be orthogonal to \(m\). In other words, if \(\mu_\mathcal{B} \perp m\) then \((\mathcal{H}, U, \mathcal{B})\) is singular.

On the other hand, if \(U\) is a bilateral shift and \(\mathcal{B} \subset \mathcal{H}\) such that \(H_\mathcal{B}\) is reducing to \(U\) then \((\mathcal{H}', U, \mathcal{B}')\) is singular and \(\mu_\mathcal{B} \leq m\). In view of these considerations it is not unnatural to ask what is the connection between the above decomposition and the Lebesgue decomposition of measures \(\mu_\mathcal{B}\) \((x, y \in \mathcal{B})\) into absolutely continuous and orthogonal parts with respect to \(m\).

Let \(\mathcal{B}\) be a subset of \(\mathcal{H}\), \(y \in H_\mathcal{B}\), and let us consider the nonnegative Borel measure \(\mu_y = |E(\cdot) y|^2\). Let the Lebesgue decomposition of \(\mu_y\) have the form

\[
\mu_y = \mu^a + \mu^i, \quad \mu^a \leq m, \quad \mu^i \perp m.
\]

If \(\mu_y\) is concentrated on \(B\) then \(y = E(B) y + E(B^c) y\) and the measure \(\mu_{E(B)y} = |E(\cdot) E(B) y|^2 = |E(B \cap \cdot) y|^2\) is absolutely continuous while \(\mu_{E(B^c)y}\) is orthogonal to \(m\) so that \(\mu^a = \mu_{E(B)y}\) and \(\mu^i = \mu_{E(B^c)y}\). Since subspaces reducing \(U\) are invariant to \(E(\cdot)\), elements \(E(B) y\) and \(E(B^c) y\) are in \(H_\mathcal{B}\) as well.

Now let us define subspaces

\[
\mathcal{H}^a = \{y \in H_\mathcal{B}; \mu_y \leq m\},
\]

\[
\mathcal{H}^s = \{y \in H_\mathcal{B}; \mu_y \perp m\}.
\]

Both subspaces are closed, mutually orthogonal and \(H_\mathcal{B} = \mathcal{H}^a \oplus \mathcal{H}^s\). The relation \(\mu_{Uy} = \mu_y\) implies that they are also reducing to \(U\).

4.1 Proposition. Let \(\mathcal{B} = \{x_1, \ldots, x_n\}\) be a finite subset of \(\mathcal{H}\) and let \((\mathcal{H}, U, \mathcal{B})\) be a stationary process with \(\dim \mathcal{B} = n\). If \((\mathcal{H}, U, Q\mathcal{B}), (\mathcal{H}, U, (1 - Q)\mathcal{B})\) is the Wold decomposition of \((\mathcal{H}, U, \mathcal{B})\) then

\[
\mu_\mathcal{B} = \mu_{Q\mathcal{B}} + \mu_{(1 - Q)\mathcal{B}}
\]

is the Lebesgue decomposition of the spectral measure of \((\mathcal{H}, U, \mathcal{B})\) into absolutely continuous and orthogonal parts with respect to \(m\), i.e.

\[
\mu_{x_i, x_j} = \mu_{Qx_i, x_j} + \mu_{(1 - Q)x_i, x_j}
\]

is the Lebesgue decomposition of \(\mu_{x_i, x_j}\), \(i, j = 1, 2, \ldots, n\).

Proof. According to what has been said above both \(\mathcal{H}^a\) and \(\mathcal{H}^s\) are reducing subspaces to \(U\), \(\mathcal{H}^a \perp \mathcal{H}^s\) and \(x = P(\mathcal{H}^a) x + P(\mathcal{H}^s) x\) for \(x \in \mathcal{B}\).

Obviously, \(H_{P(\mathcal{H}^a)x} \subset \mathcal{H}^a\), \(H_{P(\mathcal{H}^s)x} \subset \mathcal{H}^s\) and thus \(H_{P(\mathcal{H}^a)x} \perp H_{P(\mathcal{H}^s)x}\). Since \(\mu_{P(\mathcal{H}^a)x} \perp m\) the process \((\mathcal{H}, U, P(\mathcal{H}^a) \mathcal{B})\) is singular.

We shall show that \((\mathcal{H}, U, P(\mathcal{H}^a) \mathcal{B})\) is regular. Regularity of \((\mathcal{H}, U, Q\mathcal{B})\) implies \(H_{Qx} \subset \mathcal{H}^a\), and consequently, \(\mathcal{B} \subset M(\mathcal{B}) = H_{Qx} \subset \mathcal{H}^a\). Thus we have

\[
\mathcal{B} \subset P(\mathcal{H}^a) E_x \oplus U^*E_x = P(\mathcal{H}^a) E_x \oplus P(\mathcal{H}^a) U^*E_x \subset
\]

\[
\subset \text{clos} (P(\mathcal{H}^a) E_x) \oplus \text{clos} (P(\mathcal{H}^a) U^*E_x) = E_{P(\mathcal{H}^a)x} \oplus U^*E_{P(\mathcal{H}^a)x} = \mathcal{B}_x.
\]
It follows that $\dim \mathcal{F}_{P(\mathcal{H})^m} \geq \dim \mathcal{F} = n$. Moreover, $\mu_{P(\mathcal{H})^m} \ll m$ and, according to Proposition 3.3, $(\mathcal{H}, U, P(\mathcal{H})^m \mathcal{X})$ is regular. The decomposition $(\mathcal{H}, U, P(\mathcal{H})^m \mathcal{X})$ and $(\mathcal{H}, U, P(\mathcal{H})^n \mathcal{X})$ satisfies condition 1° and 2° of 2.1 so that $P(\mathcal{H})^m \mathcal{X} = Q \mathcal{X}$ and $P(\mathcal{H})^n \mathcal{X} = (1 - Q) \mathcal{X}$. The proof is complete.

References


Souhrn

POZNÁMKA O JEDNOZNAČNOSTI WOLDova ROZKLADU KONEČNÉROZMĚRNÝCH STACIONÁRNÍCH PROCESŮ

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V práci je dokázána jednoznačnost Woldova rozkladu konečněrozměrného stacionárního procesu bez předpokladu časové podřízenosti. Důsledkem je jednoduchý důkaz korespondence mezi Woldovým rozkladem stacionárního procesu plné hodnosti a Lebesgueovým rozkladem odpovídající spektrální měry.

Резюме

ЗАМЕЧАНИЕ ОБ ЕДИНИСТВЕННОСТИ РАЗЛОЖЕНИЯ ВОЛЬДА КОНЕЧНОМЕРНЫХ СТАЦИОНАРНЫХ ПРОЦЕССОВ

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Доказывается единственность разложения Вольда конечномерного стационарного процесса без предположения подчиненности исходному процессу. Как следствие получается элементарное доказательство соответствия разложения Вольда стационарного процесса максимального ранга и разложения Лебега его спектральной меры.

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