Ivan Hlaváček
Domain optimization in axisymmetric elliptic boundary value problems by finite elements


Persistent URL: http://dml.cz/dmlcz/104304

Terms of use:
© Institute of Mathematics AS CR, 1988

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.
Domain Optimization in Axisymmetric Elliptic Boundary Value Problems by Finite Elements

Ivan Hlaváček

(Received March 4, 1987)

Summary. An axisymmetric second order elliptic problem with mixed boundary conditions is considered. A part of the boundary has to be found so as to minimize one of four types of cost functionals. The existence of an optimal boundary is proven and a convergence analysis for piecewise linear approximate solutions presented, using weighted Sobolev spaces.

Keywords: domain optimization, finite elements, elliptic problems.

AMS Subject class: 65N99, 65N30, 49A22.

Introduction

One often meets elliptic boundary value problems in three-dimensional domains $\Omega$, which are generated by the rotation of a bounded plane domain $D$ around an axis. Then the most suitable approach is to use the cylindrical coordinates. If the data of the problem are axially symmetric, the problem is reduced to the two-dimensional domain $D$ (see [2] for a detailed finite element analysis).

Let a part $\Gamma$ of the boundary $\partial D$ be optimized. We arrive at a generalization of a two-dimensional domain optimization problem, which has been analyzed thoroughly by Béhis and Glowinski in [1]. In the axisymmetric problem, however, we are forced to employ weighted Sobolev spaces, where the radial coordinate plays the role of the weight.

Section 1 contains some results valid for the whole family $\{W_{2,p}^{(1)}(D(z))\}$, $z \in U_{ad}$, of the weighted spaces under consideration. In Section 2 we formulate the state problem and prove the continuous dependence of its solution on the design variable. Section 3 contains the definitions of four domain optimization problems and the proof of existence of optimal solutions. Approximate problems are proposed in Section 4 and a convergence proof is presented for three types of the cost functionals.
1. SOME PRELIMINARY RESULTS

Let \( \Omega \subset \mathbb{R}^3 \) be a bounded domain with Lipschitz boundary \( \partial \Omega \), axially symmetric with respect to the \( x_3 \)-axis, if \( x = (x_1, x_2, x_3) \) denote the Cartesian coordinates. Let \( H^k(\Omega) \) denote the standard Sobolev space \( W^{(k),2}(\Omega) \), \( k = 1, 2, \ldots \), with the usual norm \( \| \cdot \|_{k,\Omega} \). Passing to the cylindrical coordinates \( r, z, \vartheta \), for any function \( u(x_1, x_2, x_3) \) we define

\[
\hat{u}(r, z, \vartheta) = u(r \cos \vartheta, r \sin \vartheta, z).
\]

If \( x \in \Omega \), then \((r, z) \in D, \vartheta \in [0, 2\pi]\) and by (1) a mapping \( Z: H^1(\Omega) \to ZH^1(D \times \times [0, 2\pi]) \) is defined. Introducing the norm in \( ZH^1 \)

\[
\| \hat{u} \|_1 = \left( \int_D \int_0^{2\pi} \left[ \hat{u}^2 + \left( \frac{\partial \hat{u}}{\partial r} \right)^2 + r^{-2} \left( \frac{\partial \hat{u}}{\partial \vartheta} \right)^2 + \left( \frac{\partial \hat{u}}{\partial z} \right)^2 \right] r \, dr \, dz \, d\vartheta \right)^{1/2},
\]

for \( \hat{u} = Zu \) we obtain

\[
\| \hat{u} \|_1 = \| u \|_{1,\Omega}.
\]

Let \( W_0 \) be the subspace of axisymmetric functions, i.e.

\[
W_0 = \left\{ \hat{u} \in ZH^1(D \times [0, 2\pi]) | \frac{\partial \hat{u}}{\partial \vartheta} = 0 \ a.e. \right\}.
\]

It is easy to see that \( Z \) is an isomorphism and \( W_0 \) is closed in \( ZH^1 \). Moreover, we have

\[
(2\pi)^{-1} \| \hat{u} \|_1^2 = \int_D \left[ u^2 + \left( \frac{\partial u}{\partial r} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 \right] r \, dr \, dz \quad \forall \hat{u} \in W_0.
\]

The square-root of the integral in the right-hand side represents a norm

\[
\| \hat{u} \|_{1,r,D}
\]

in the weighted Sobolev space \( W^{(1)}_{2,r}(D) \) (see e.g. [3]). Consequently, the subspace \( W_0 \) can be identified with the space \( W^{(1)}_{2,r}(D) \).

In the optimization process we shall consider a specific class of admissible domains \( D(\alpha) \), where

\[
D(\alpha) = \{(r, z) | 0 < r < \alpha(z), 0 < z < 1\}
\]

and the function \( \alpha(z) \) — the design variable — belongs to the following set of admissible functions

\[
U_{ad} = \left\{ \alpha \in C^{(0),1}([0, 1]) , \quad (i.e. \ Lipschitz \ functions) ,
\right. \\
\left. \alpha_{\min} \leq \alpha(z) \leq \alpha_{\max} , \ |d\alpha/dz| \leq C_1 , \ \int_0^1 \alpha^2(z) \, dz = C_2 \right\}
\]

214
with given positive constants $a_{\min}, a_{\max}, C_1, C_2$. Assume that $a_{\min} > a_{\max}/2$.

Let $\Gamma(z)$ denote the graph of the function $z$, 

$$\Gamma_1 = \partial D(z) \cap \{z = 0\}, \quad \Gamma_2 = \partial D(z) \cap \{z = 1\}$$

(see Fig. 1).

Next we present several properties of the spaces $W^{(1)}(\Omega(z)), \alpha \in U_{\alpha}$. For any $\alpha \in U_{\alpha}$ the space $W^{(1)}(\Omega(z))$ is a reflexive Banach space and the set $C^\infty(D(z))$ is dense in it (see e.g. [3] — chapt. 6. § 2).

Let $S_0(z) \subset \partial \Omega(z)$ be generated by the rotation of a curve segment $\Gamma_0(z) \subset \partial D(z)$ around the z-axis $\theta$, mes $S_0(z) > 0$. Then the Trace Theorem [3] says that a linear mapping $\tilde{\gamma}: H^1(\Omega(z)) \to L^2(S_0(z))$ exists such that $\tilde{\gamma}u = u$ holds for smooth functions and

$$\|\tilde{\gamma}u\|_{L^2(S_0(z))} \leq C(z)\|u\|_{1,\Omega(z)}.$$  \hspace{1cm} (4)

Passing to the cylindrical coordinates, we obtain the existence of a linear mapping

$$\gamma: W^{(1)}(\Omega(z)) \to L_{2,z}(\Gamma_0(z)), \quad \Gamma_0(z) \subset \partial D(z) \cap \theta,$$

such that $\gamma \hat{u} = \hat{u}$ for smooth functions and

$$\|\gamma \hat{u}\|_{0,z,\Gamma_0(z)} \leq C(z)\|\hat{u}\|_{1,z,\Omega(z)},$$  \hspace{1cm} (5)

where

$$L_{2,z}(\Gamma_0(z)) = \left\{v \int_{\Gamma_0(z)} v^2 r \, ds \equiv \|v\|_{0,\Gamma_0(z)}^2 < +\infty \right\}.$$  

The constant $C(z)$ in (5) is the same as that in (4), but it depends on the domain $D(z)$ and on $\Gamma_0(z)$. Hence a question arises, if a “common” constant for all admissible variables exists. An answer is contained in the following
Lemma 1. There exists a positive constant $C$ such that

$$\|\gamma u\|_{0,r,\Gamma_1(\alpha) \cup \Gamma_2(\alpha)} \leq C\|u\|_{1,r,\Gamma(a)}$$

holds for all $\alpha \in U_{ad}$ and $u \in W^{1}_{2,r}(D(\alpha))$.

Proof. Let $D_0 = (0, 1) \times (0, 1)$ be the unit square.

The mapping $(r, z) \to (\xi_1, \xi_2)$, defined through the relations

$$\xi_1 = r/\alpha(z), \quad \xi_2 = z$$

maps the domain $D(\alpha)$ onto $D_0$. It is readily seen that for the functions

$$\bar{u}(\xi_1, \xi_2) = u(\xi_1\alpha(\xi_2), \xi_2)$$

the following estimate holds

$$\int_{D_0} (|\nabla \bar{u}|^2 + \bar{u}^2) \xi_1 d\xi \leq C_0\|u\|_{1,r,\Gamma(a)}^2, \quad \forall \alpha \in U_{ad},$$

where $C_0$ depends on $\alpha_{\min}$, $\alpha_{\max}$ and $C_1$, but not on $\alpha$. We may write

$$\int_{\Gamma_1 \cup \Gamma(\alpha)} u^2 r \, ds = \int_{\Gamma_1} u^2 r \, dr + \int_0^1 u^2 \alpha(z)(1 + (\alpha')^2)^{1/2} \, dz \leq \alpha_{\max}^2 \int_{\Gamma_1} (\bar{u}(\xi_1))^2 \xi_1 \, d\xi_1 + (1 + C_4^2)^{1/2} \alpha_{\max} \int_0^1 \bar{u}^2(1, \xi_2) \, d\xi_2 \leq C_4 \int_{\Gamma_0} \bar{u}^2 \xi_1 \, ds, \quad C_4 = C_4(\alpha_{\max}, C_1),$$

where $\Gamma_1$ and $\Gamma_0$ denotes the corresponding image of $\Gamma_1$ and $\Gamma_0 = \Gamma_1 \cup \Gamma(\alpha)$, respectively. Combining (7), (6) and the inequality (5), applied to the square $D_0$, we obtain

$$\int_{\Gamma_0} u^2 r \, ds \leq C_4 C_2 C_0 \|u\|_{1,r,\Gamma(a)}^2 \quad \text{Q.E.D.}$$

Lemma 2. The set

$$M(D(\alpha)) = \{ u \in C^0(\overline{D(\alpha)}) | \text{supp } u \cap \Gamma_2(\alpha) = \emptyset \}$$

is dense in the subspace

$$V(D(\alpha)) = \{ v \in W^{1}_{2,r}(D(\alpha)) | \gamma v = 0 \text{ on } \Gamma_2(\alpha) \}$$

for any positive $\alpha \in C^{(0),1}([0, 1])$. 

216
Proof. Let \( \hat{u} \in V(D(\alpha)) \) and denote \( u = Z^{-1} \hat{u} \). By the above mentioned density a sequence \( \{ \hat{w}_k \} \), \( k \to \infty \), exists such that \( \hat{w}_k \in C^\infty(\text{Cl} (D(\alpha))) \) and
\[
\| \hat{w}_k - \hat{u} \|_{1,r,D(\alpha)} \to 0 .
\]
From (2), (3)
\[
\| Z^{-1} \hat{w}_k - u \|_{1,\Omega(\alpha)} \to 0
\]
follows. It is well-known (see e.g. [3]), that a sequence \( \{ u_n \} \), \( n \to \infty \), \( u_n \in C^\infty(\text{Cl} (\Omega(\alpha))) \), exists such that
\[
\text{supp } u_n \cap S_2(\alpha) = \emptyset ,
\]
\[
\| u_n - u \|_{1,\Omega(\alpha)} \to 0 ,
\]
(where \( S_2(\alpha) \) is generated by rotation of \( \Gamma_2(\alpha) \)).

Let us define \( \hat{u}_n = Zu_n \) and \( \omega_n(r, z) = (2\pi)^{-1} \int_0^{2\pi} \hat{u}_n(r, z, \theta) \, d\theta \).

We can see that
\[
\text{supp } \omega_n \cap \Gamma_2(\alpha) = \emptyset , \quad \omega_n \in C^\infty(\text{Cl} (D(\alpha))) .
\]
Moreover, we may write
\[
|\omega_n(r, z) - \hat{\omega}_k(r, z)|^2 = (2\pi)^{-2} \left[ \int_0^{2\pi} (\hat{u}_n(r, z, \theta) - \hat{\omega}_k(r, z)) \, d\theta \right]^2 \leq (2\pi)^{-1} \int_0^{2\pi} (\hat{u}_n - \hat{\omega}_k)^2 \, d\theta ,
\]
\[
\left| \frac{\partial}{\partial r} (\omega_n - \hat{\omega}_k) \right|^2 \leq (2\pi)^{-1} \int_0^{2\pi} \left( \frac{\partial \hat{u}_n}{\partial r} - \frac{\partial \hat{\omega}_k}{\partial r} \right)^2 \, d\theta
\]
and a similar estimate for \( \partial (\omega_n - \hat{\omega}_k) / \partial z \).

Consequently, we have for \( n \to \infty \), \( k \to \infty \)
\[
\| \omega_n - \hat{\omega}_k \|_{1,r,D(\alpha)} = \int_0^1 dz \int_0^{2\pi} \left[ (\omega_n - \hat{\omega}_k)^2 + |\nabla (\omega_n - \hat{\omega}_k)|^2 \right] r \, dr \leq \int_0^1 dz \int_0^{2\pi} (2\pi)^{-1} \int_0^{2\pi} \left[ (\hat{u}_n - \hat{\omega}_k)^2 + |\nabla (\hat{u}_n - \hat{\omega}_k)|^2 \right] d\theta \, dr \leq (2\pi)^{-1} \| \hat{u}_n - \hat{\omega}_k \|_1 = (2\pi)^{-1} \| u_n - Z^{-1} \hat{w}_k \|_{1,\Omega(\alpha)} \to 0 ,
\]
since \( \| u_n - Z^{-1} \hat{w}_k \|_{1,\Omega(\alpha)} \leq \| u_n - u \|_{1,D(\alpha)} + \| u - Z^{-1} \hat{w}_k \|_{1,\Omega(\alpha)} \to 0 \)
on the basis of (10), (9).
Combining (11) and (8), we derive finally
\[ \| \omega_n - \tilde{u} \|_{1,r,D(\alpha)} \leq \| \omega_n - \tilde{\omega}_k \|_{1,r,D(\alpha)} + \| \tilde{\omega}_k - \tilde{u} \|_{1,r,D(\alpha)} \to 0. \]
Since \( \omega_n \in M(D(\alpha)) \), the lemma is proved.

**Lemma 3.** There exists a positive constant \( C_3 \) such that
\[ \int_{D(\alpha)} |\nabla u|^2 r \, dr \, dz \geq C_3 \| u \|_{1,r,D(\alpha)}^2 \]
holds for all \( u \in V(D(\alpha)) \) and \( \alpha \in \mathcal{U}_a_d \).

**Proof.** Let us introduce a constant \( \delta \in (\alpha_{\text{max}}, 2z_{\text{min}}) \) and the rectangle
\[ \tilde{D} = (0, \delta) \times (0, 1). \]
We shall construct an extension \( Eu \in W^{1,2}_r(\tilde{D}) \) of the function \( u \in V(D(\alpha)) \) as follows:
\[ Eu(r, z) = u(2\alpha(z) - r, z) \quad \text{on} \quad \tilde{D} - D(\alpha) \]
and \( Eu = u \) on \( D(\alpha) \). Then we have obviously \( \gamma(Eu) = 0 \) on \( z = 1 \) and we can prove that
\[ \int_{D} |\nabla (Eu)|^2 r \, dr \, dz \leq C \int_{D(\alpha)} |\nabla u|^2 r \, dr \, dz, \]
(13') \[ \| Eu \|_{1,r,D} \leq C \| u \|_{1,r,D(\alpha)} \]
where the constants \( \tilde{C}, C \) are independent of \( \alpha \). To this end, one employs the estimate
\[ |\nabla Eu(r, z)|^2 \leq (2 + 4(\alpha')^2) |\nabla u(2\alpha(z) - r, z)|^2 \]
for \( (r, z) \in \tilde{D} - D(\alpha) \).
Then we prove by a standard way, that
\[ \int_{D} (Eu)^2 r \, dr \, dz \leq \int_{D} \left( \frac{\partial Eu}{\partial z} \right)^2 r \, dr \, dz \quad \forall u \in V(D(\alpha)). \]
(First we verify this for \( u_n \in M(\tilde{D}) \) and then pass to the limit on the basis of Lemma 2.)
Therefore we have
\[ \int_{D} (Eu)^2 r \, dr \, dz \leq \int_{D} |\nabla Eu|^2 r \, dr \, dz \]
and from (13)
\[ \int_{D(\alpha)} u^2 r \, dr \, dz \leq \int_{D} (Eu)^2 r \, dr \, dz \leq \int_{D} |\nabla Eu|^2 r \, dr \, dz \leq \tilde{C} \int_{D(\alpha)} |\nabla u|^2 r \, dr \, dz \]
follows. The assertion of the lemma is an immediate consequence of (14), with $C_3 = (1 + \bar{C})^{-1}$.

**Lemma 4.** The embedding

$$W^{(1)}_{2,r}(D) \subset L_{2,r}(D)$$

is compact.

**Proof.** Let $M$ be a bounded subset of $W^{(1)}_{2,r}(D)$. Since

$$\|Z^{-1} \hat{a}\|^2_{1,\Omega} = \|\hat{a}\|^2_{1} = 2\pi \|\hat{a}\|^2_{1,r,D},$$

the set $Z^{-1}M$ is bounded in $H^1(\Omega)$.

Rellich’s Theorem ([3]) implies that $Z^{-1}M$ is precompact in $L^2(\Omega)$.

Let $Z^{-1} \hat{a}_n \to \omega$ in $L^2(\Omega)$. We also have

$$\|Z^{-1} \hat{a}_n\|^2_{0,\Omega} = 2\pi \|\hat{a}_n\|^2_{0,r,D}$$

and therefore

$$\|Z^{-1} \hat{a}_n - Z^{-1} \hat{a}_m\|_{0,\Omega} \to 0 \Rightarrow \|\hat{a}_n - \hat{a}_m\|_{0,r,D} \to 0$$

(for $n \to \infty$, $m \to \infty$). Since the space $L_{2,r}(D)$ is complete, the sequence $\{\hat{a}_n\}$ is convergent in $L_{2,r}(\Omega)$. Consequently, the set $M$ is precompact in $L_{2,r}(D)$.

2. THE STATE PROBLEM AND THE CONTINUOUS DEPENDENCE OF ITS SOLUTION ON THE DESIGN VARIABLE

We shall consider the following boundary value problem

\begin{align*}
(15) & \quad - \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \left( A_i(x) \frac{\partial u}{\partial x_i} \right) = F \quad \text{in} \quad \Omega(x), \\
(16) & \quad \sum_{i=1}^{3} \nu_i A_i \frac{\partial u}{\partial x_i} = G \quad \text{on} \quad S_1(x) \cup S(x), \\
(17) & \quad u = 0 \quad \text{on} \quad S_2(x),
\end{align*}

where $\Omega(x)$ is generated by rotation of $D(x)$ around the $x_3$-axis, $S_i(x)$ by rotation of $\Gamma_i(x)$, $i = 1, 2$ and $S(x)$ by rotation of $\Gamma(x)$, (see Fig. 1), $\nu_i$ are components of the unit outward normal with respect to $\partial \Omega(x)$.

Let $\bar{D}$ be the cylindrical domain generated by rotation of the rectangle $D = (0, \delta) \times (0, 1)$, $\delta > \alpha_{\max}$.

Assume that the function $F$ in (15) is determined as the restriction to $\Omega(x)$ of an axisymmetric function $F \in L^2(\hat{\Omega})$,

$$G = \begin{cases} 0 & \text{on} \quad S(x), \\ G_1 & \text{on} \quad S_1(x), \end{cases}$$
where $G_1$ is determined as the restriction to $S_1(x)$ of an axisymmetric function $G_1 \in L^2(\hat{\Omega})$, $\hat{\Omega} = \partial \Omega \cap \{x \mid x_3 = 0\}$.

Assume that the coefficients $A_i$ are restrictions to $\Omega(x)$ of axisymmetric functions $A_i \in L^\infty(\hat{\Omega})$, $A_1 = A_2$ a.e. and a positive constant $a_0$ exists such that

$$A_i(x) \geq a_0 \quad \text{a.e. in } \hat{\Omega}.$$ 

Let us denote $A_1 = A_2 = a_r$, $A_3 = a_z$.

Passing to the cylindrical coordinate system, we transform the standard variational formulation of the problem (15), (16), (17) to the following state problem:

$$\text{find } y \in V(D(x)) \text{ such that}$$

$$(18) \quad a(a; y, v) = L(a; v) \quad \forall v \in V(D(x)),$$

where

$$a(a; y, v) = \int_{D(x)} \left( a_r \frac{\partial y}{\partial r} \frac{\partial v}{\partial r} + a_z \frac{\partial y}{\partial z} \frac{\partial v}{\partial z} \right) r \, dr \, dz,$$

$$L(a; v) = \int_{D(x)} f v r \, dr \, dz + \int_{\Gamma_1(x)} g v r \, ds,$$

where the function $f \in L_{2,r}(\hat{D})$ and $g \in L_{2,r}(\hat{\Gamma}_1)$, $\hat{\Gamma}_1 = \partial \hat{D} \cap \{z = 0\}$, are given. Finally, $a_r, a_z \in L^\infty(\hat{D})$,

$$a_r(r, z) \geq a_0, \quad a_z(r, z) \geq a_0 \quad \text{a.e. in } \hat{D}.$$

Remark 1. The variational formulation (18) corresponds with the following "classical" one:

$$- \frac{1}{r} \frac{\partial}{\partial r} \left( r a_r \frac{\partial y}{\partial r} \right) - \frac{\partial}{\partial z} \left( a_z \frac{\partial y}{\partial z} \right) = f \quad \text{in } D(x),$$

$$v_r a_r \frac{\partial y}{\partial r} + v_z a_z \frac{\partial y}{\partial z} = 0 \quad \text{on } \Gamma(x),$$

$$y = 0 \quad \text{on } \Gamma_2(x).$$

Lemma 5. Positive constants $C_4, C_5$ exist such that the inequalities

$$(20) \quad a(a; u, u) \geq C_4 \|u\|^2_{1,r,D(x)} \quad \forall u \in V(D(x)),$$

$$(21) \quad |a(a; u, v)| \leq C_5 \|u\|_{1,r,D(x)} \|v\|_{1,r,D(x)} \quad \forall u, v \in W^{1,2}_r(D(x))$$

hold for any $\alpha \in U_{ad}$.

Proof. Using (19) and Lemma 3, we obtain

$$a(a; u, u) \geq a_0 \int_{D(x)} |\nabla u|^2 r \, dr \, dz \geq C_3 a_0 \|u\|^2_{1,r,D(x)}.$$
By virtue of the boundedness of $a_r, a_z$, we may write

$$a(\alpha; u, v) \leq C_5 \int_{D(\alpha)} \left| \nabla u \right| \left| \nabla v \right| r \, dr \, dz \leq C_5 \| u \|_{1, r, D(\alpha)} \| v \|_{1, r, D(\alpha)},$$

where $C_5 = \max \{ \| a_r \|_{\infty, D}, \| a_z \|_{\infty, D} \}$.

**Lemma 6.** A positive constant $C_6$ exists such that

$$|L(\alpha; u)| \leq C_6 \| u \|_{1, r, D(\alpha)} \quad \forall u \in W_2^1(D(\alpha))$$

holds for any $\alpha \in U_{ad}$.

**Proof.** Using Lemma 1, we may write

$$|L(\alpha; u)| = \left( \int_{D(\alpha)} \left( f u r \, dr \, dz + \int_{J_1(\alpha)} g u r \, dr \right) \right) \leq \left( \| f \|_{0, r, D} \| u \|_{0, r, D(\alpha)} + \| g \|_{0, r, J_1(\alpha)} \| u \|_{0, r, J_1(\alpha)} \right) \leq (\| f \|_{0, r, D} + C \| g \|_{0, r, J_1}) \| u \|_{1, r, D(\alpha)}.$$

**Lemma 7.** The state problem (18) has a unique solution $y = y(\alpha)$ for any $\alpha \in U_{ad}$.

**Proof.** Follows from the Riesz Theorem, since the space $V(D(\alpha))$ can be equipped with the inner product $a(\alpha; u, v)$, on the basis of Lemma 5. Moreover, we employ Lemma 6 to show the continuity of the right-hand side in (18). Q.E.D.

**Proposition 1.** Assume that a sequence $\{ \alpha_n \}_{n=1}^{\infty}, \alpha_n \in U_{ad}$, converges to a function $\alpha$ in $C([0, 1])$. Let us construct extensions $E y(\alpha_n) \in W_2^1(D(\alpha))$ of the solutions $y(\alpha_n)$ of the state problem (18) by means of the formula (12).

Then

$$E y(\alpha_n) \mid_{D(\alpha)} \rightharpoonup y(\alpha) \quad \text{(weakly) in } W_2^1(D(\alpha)),$$

where $y(\alpha)$ is the solution of the state problem (18) on $D(\alpha)$.

**Proof.** Let us denote $y(\alpha_n) = y_n, D(\alpha_n) = D_n, D(\alpha) = D$. Using Lemmas 5 and 6, we may write

$$C_4 \| y_n \|_{1, r, D_n}^2 \leq a(\alpha_n; y_n, y_n) = L(\alpha_n; y_n) \leq C_6 \| y_n \|_{1, r, D_n}.$$

Consequently, we have

$$\| y_n \|_{1, r, D_n} \leq C_6/C_4 \quad \forall n.$$

On the basis of the relation (13')

$$\| E y_n \|_{1, r, D} \leq C \| y_n \|_{1, r, D_n} \leq C_7$$

holds. Since $W_2^1(D)$ is reflexive Banach space, there exist a subsequence $\{ y_k \} \subset \{ y_n \}$
and \( y \in W_{2,p}^{(1)}(\hat{D}) \) such that 
\[
Ey_k \rightharpoonup y \quad \text{(weakly) in } \ W_{2,p}^{(1)}(\hat{D}).
\]

Let us show that \( y|_D \) solves the state problem (18). It is easy to verify that the limit function \( x \in U_{ad} \). Let \( v \in V(D) \) be arbitrary and let us construct \( Ev \in W_{2,p}^{(1)}(\hat{D}) \). Obviously, \( Ev \in V(\hat{D}) \), so that the restriction 
\[
Ev|_{D_n} \in V(D_n) \quad \forall n,
\]

since the trace \( \gamma(Ev) \) vanishes on \( \Gamma_2(\hat{D}) \). Consequently, we may insert \( Ev \) into the definition (18) to obtain 
\[
(26) \quad a(x_k; y_k, Ev) = L(x_k; Ev). 
\]

Let us pass to the limit with \( k \to \infty \) on both sides of (26). We have 
\[
|a(x; y_k, Ev) - a(x; y, v)| \leq |a(x_k; y_k, Ev) - a(x; Ey_k, Ev)| + \\
+ |a(x; Ey_k, Ev) - a(x; y, v)| \equiv I_k + J_k,
\]

\[
I_k \leq \int_{\Delta(D_k, D)} \left| a_r \frac{\partial Ey_k}{\partial r} \frac{\partial Ev}{\partial r} + a_z \frac{\partial Ey_k}{\partial z} \frac{\partial Ev}{\partial z} \right| r \, dr \, dz \leq \\
\leq C \| \nabla Ey_k \|_{0,r,D} \| \nabla Ev \|_{0,r,\Delta(D_k, D)},
\]

where 
\[
\Delta(D_k, D) = (D_k - D) \cup (D - D_k).
\]

Since 
\[
\lim_{k \to \infty} \text{mes} \left( \Delta(D_k, D) \right) = 0
\]

and 
\[
\| \nabla Ey_k \|_{0,r,D} \leq \| Ey_k \|_{1,r,D} \leq C_7,
\]

we conclude that \( I_k \to 0 \) for \( k \to \infty \).

By virtue of Lemma 5 the functional 
\[
u \to a(x; u, v)
\]
is linear and continuous on \( W_{2,p}^{(1)}(\hat{D}) \). Consequently, 
\[
J_k \to 0
\]

follows from the weak convergence (25). Thus we arrive at the conclusion 
\[
(27) \quad \lim_{k \to \infty} a(x_k; y_k, Ev) = a(x; y, v).
\]
We may write
\[
|L(a_k; Ev) - L(a; v)| \leq \left| \int_{\Gamma_1(a_k)} gEv r ds - \int_{\Gamma_1(a)} gEv r ds \right| + \\
\left| \int_{D_k} fEv dr dz - \int_{D} fEv dr dz \right| = M_k + N_k,
\]

since
\[
(28') \quad \lim_{k \to \infty} \text{mes} \left( \triangle(\Gamma_1(a_k), \Gamma_1(a)) \right) = 0.
\]

Moreover, it holds
\[
N_k \leq \int_{\Delta(D_k, D)} |fEv| r dr dz \to 0.
\]

Altogether, we have
\[
(28) \quad \lim_{k \to \infty} L(a_k; Ev) = L(a; v).
\]

Passing to the limit in (26) and using (27), (28), we arrive at the condition (18).

It is not difficult to verify that the subspace \( V(D) \) is weakly closed in \( W^{(1)}_2(D) \).

Since every function \( Ey_n \in V(\bar{D}) \), the weak limit \( y \in V(D) \) and its restriction \( y|_D \in V(D) \). Hence \( y|_D \) is a solution of (18).

Since the solution of (18) is unique and (25) implies that
\[
E y_k|_D \to y|_D \quad \text{(weakly)} \quad \text{in} \quad W^{(1)}_2(D),
\]

the latter convergence holds for the whole sequence \( \{E y_n|_D\} \). Q.E.D.

3. DOMAIN OPTIMIZATION PROBLEMS AND THE EXISTENCE OF OPTIMAL SOLUTION

We shall consider the following four types of the cost functional:
\[
j_1(\alpha, y) = \int_{D(\alpha)} (y - y_d)^2 r dr dz ,
\]
\[
j_2(\alpha, y) = \int_{D(\alpha)} (y(z), z) - y_d^2 dz ,
\]

where \( y(\alpha(z), z) \equiv y \) denotes the trace of \( y \) on the curve \( \Gamma(\alpha) \), \( y_d \in L^2(\varnothing \bar{D}) \) and \( y \in \varnothing \in L^2([0, 1]) \) are given functions,
\[
j_3(\alpha, y) = a(\alpha; y, y),
\]

223
where $K_i \in L_{2,r}(\bar{D})$ are given functions, $i = 1, 2$.

We define the **Domain Optimization Problems**:

$$j_4(x, y) = \int_{D(a)} \left[ \left( a_r \frac{\partial y}{\partial r} - K_1 \right)^2 + \left( a_z \frac{\partial y}{\partial z} - K_2 \right)^2 \right] r \, dr \, dz ,$$

where $K_i \in L_{2,r}(\bar{D})$ are given functions, $i = 1, 2$.

We define the **Domain Optimization Problems**:

find $\alpha^0 \in U_{ad}$ such that

$$j_i(\alpha^0, y(\alpha^0)) \leq j_i(\alpha, y(\alpha)) \quad \forall \alpha \in U_{ad} , \quad i \in \{1, 2, 3, 4\} ,$$

where $y(\alpha)$ denotes the solution of the state problem (18).

To the proof of existence of an optimal $\alpha^0$ we shall need the following

**Proposition 2.** Let the assumptions of Proposition 1 be satisfied. Then

$$\lim_{n \to \infty} j_i(\alpha_n, y(\alpha_n)) = j_i(\alpha, y(\alpha)) , \quad i = 1, 2, 3 ,$$

$$\liminf_{n \to \infty} j_4(\alpha_n, y(\alpha_n)) \geq j_4(\alpha, y(\alpha)) .$$

**Proof.** Case $i = 1$. Denoting again $y(\alpha_n) = y_n$, $y(\alpha) = y$, $D(\alpha_n) = D_n$, $D(\alpha) = D$, we conclude on the basis of Proposition 1 and Lemma 4 that

$$E y_n |_D \to y \quad \text{in} \quad L_{2,r}(D) .$$

We may write

$$\int_D \left[ (E y_n - y_d)^2 - (y - y_d)^2 \right] r \, dr \, dz = \int_D (E y_n - y) (E y_n + y - 2y_d) r \, dr \, dz \leq$$

$$\leq \| E y_n - y \|_{0,r,D} \| E y_n + y - 2y_d \|_{0,r,D} .$$

By virtue of (30), however, we obtain for $n \to \infty$

$$\| E y_n - y \|_{0,r,D} \to 0 , \quad \| E y_n + y - 2y_d \|_{0,r,D} \leq C .$$

Consequently,

$$\int_D (E y_n - y_d)^2 r \, dr \, dz \to \int_D (y - y_d)^2 r \, dr \, dz .$$

It is easy to see that

$$\int_{D_n} (y_n - y_d)^2 r \, dr \, dz = \int_D (E y_n - y_d)^2 r \, dr \, dz +$$

$$+ \int_{D_n - D} (E y_n - y_d)^2 r \, dr \, dz - \int_{D - D_n} (E y_n - y_d)^2 r \, dr \, dz .$$

224
Let us estimate the last two terms as follows:

\[
\left| \int_{D_n - D} - \int_{D - D_n} \right| \leq \int_{\triangle(D_n, D)} (E'y_n - y_d)^2 \, dr \, dz \leq \\
\left( \int_{\triangle(D_n, D)} r^2 \, dr \, dz \right)^{1/2} \left( \int_{D_0} (E'y_n - y_d)^4 \, dr \, dz \right)^{1/2},
\]

where \( D_0 = D - (0, a_{\min}/2) \times (0, 1). \)

Since the restriction \( E'y_n|_{D_0} \in H^1(D_0) \) and the embedding of \( H^1(D_0) \) into \( L^4(D_0) \) is continuous, we may write

\[
\| E'y_n - y_d \|_{L^4(D_0)} \leq C \| E'y_n - y_d \|_{1, D_0} \leq \\
\leq C(2/a_{\min})^{1/2} \| E'y_n - y_d \|_{1, r, D} \leq \mathcal{C},
\]

where also (24) has been used.

Since

\[
\text{mes} (\triangle(D_n, D)) \to 0,
\]

the right-hand side of (33) tends to zero. Combining this result with (31) and (32), the assertion of the Proposition follows.

Case \( i = 2. \) Denoting \( \alpha_m(z) = \alpha(z) - 1/m, \) \( m = 2, 3, \ldots, \) we may write for \( n \) sufficiently great with respect to \( m: \)

\[
\int (\alpha_n(z)) - J_2(\alpha, y) = M_1 + M_2 + M_3,
\]

\[
M_1 = \int_0^1 (y_n(\alpha_n(z)) - y_n(z))^2 \, dz - \int_0^1 (y_n(\alpha_m(z)) - y_n(z))^2 \, dz,
\]

\[
M_2 = \int_0^1 (y_n(\alpha_m(z)) - y_n(z))^2 \, dz - \int_0^1 (y(\alpha_m(z)) - y_n(z))^2 \, dz,
\]

\[
M_3 = \int_0^1 (y(\alpha_m(z)) - y_n(z))^2 \, dz - \int_0^1 (y(\alpha(z)) - y_n(z))^2 \, dz
\]

and estimate the individual terms.

It is readily seen that

\[
|M_1| \leq \int_0^1 |y_n(\alpha_n(z)) - y_n(\alpha_m(z))| |y_n(\alpha_n(z)) - y_n(\alpha_m(z)) - 2y_n(z)| \, dz \leq \\
\leq \| y_n(\alpha_n) - y_n(\alpha_m) \|_0 \| y_n(\alpha_n) + y_n(\alpha_m) - 2y_n(z) \|_0,
\]

where \( \| \cdot \|_0 \) denotes the norm in \( L^2([0, 1]). \) Furthermore,

\[
\| y_n(\alpha_n) \, y_n(\alpha_m) \|_0^2 = \int_0^1 dz \left( \int_{\alpha_m(z)} \frac{\partial y_n}{\partial r} \, dr \right)^2 
\]

\[
225
\]
\[ \leq \int_0^1 dz (\beta_n + 1/m) \int_{\alpha_m}^{\beta_n} \left( \frac{\partial y_n}{\partial r} \right)^2 dr \leq (\beta_n + 1/m) \left\| y_n \right\|^2_{1, D_n^0} . \]

where
\[
\beta_n = \max_{z \in [0, 1]} |\alpha_m(z) - \alpha(z)| ,
\]

\[ D_n^0 = D_n - (0, \alpha_{\min}/2) \times (0, 1) , \]

(considering only \( m \) such that \( \alpha(z) - 1/m \geq \alpha_{\min}/2 \).

By virtue of (23)
\[
\left\| y_n \right\|^2_{1, D_n^0} \leq (2/\alpha_{\min}) \left\| y_n \right\|^2_{1, r, D_n} \leq C .
\]

For sufficiently great \( n, m, n > n_0(m) \), we derive
\[
\left\| y_n(\alpha_n) + y_n(\alpha_m) - 2y_n \right\|^2_0 \leq 3 \int_0^1 \left( y_n^2(\alpha_n) + y_n^2(\alpha_m) + 4y_n^2 \right) dz \leq C .
\]

In fact, Lemma 1 and (23) yield that
\[
\int_0^1 y_n^2(\alpha_n) dz \leq \int_{\Gamma(\alpha_n)} (y_n)^2 ds \leq \left\| y_n \right\|^2_{0, r, \Gamma(\alpha_n)} \cdot 2/\alpha_{\min} \leq C(2/\alpha_{\min}) \left\| y_n \right\|^2_{1, r, D_n} \leq C .
\]

Using (39), (36) and (37), we obtain
\[
\int_0^1 y_n^2(\alpha_m) dz \leq 2 \int_0^1 y_n^2(\alpha_n) dz + 2 \int_0^1 (y_n(\alpha_m) - y_n(\alpha_n))^2 dz < C .
\]

Combining (36), (37) and (38), we arrive at the estimate
\[
|M_1| \leq C(\beta_n + 1/m)^{1/2} .
\]

In a similar way we derive
\[
|M_2| \leq \left\| y_n(\alpha_m) - y(\alpha_m) \right\|_0 \left\| y_n(\alpha_m) + y(\alpha_m) - 2y \right\|_0 .
\]

It holds, however, that
\[
\left\| y_n - y \right\|^2_{0, \Gamma(\alpha_m)} = \int_{\Gamma(\alpha_m)} (y_n(\alpha_m) - y(\alpha_m))^2 ds \to 0 \quad \text{for} \quad n \to \infty .
\]

In fact, the mapping
\[
y: H^1(G_m^0) \to L^2(\partial G_m^0) ,
\]

where
\[
G_m^0 = G_m - (0, \alpha_{\min}/2) \times (0, 1)
\]

and
\[
G_m = \{(r, z) \mid 0 < r < \alpha_m(z), 0 < z < 1\} ,
\]

226
is completely continuous ([3] — chap. 2, § 6.2). The weak convergence (22) of \( \{Ey_n\} \) implies that

\[
y_n \rightarrow y \text{ (weakly) in } H^1(G^0_m),
\]

since \( H^1(G^0_m) = W^{1,2}_r(G^0_m) \) and the norms are equivalent. Consequently, \( y_n \rightarrow y \) in \( L^2(\Gamma(x_m)) \) follows, i.e. (42) holds.

Thus we obtain that

\[
\|y_n(x_m) - y(x_m)\|_0^2 \leq \int_{\Gamma(x_m)} (y_n(x_m) - y(x_m))^2 \, ds = \|y_n - y\|_{0,\Gamma(x_m)}^2 \rightarrow 0
\]

for \( n \rightarrow \infty \).

Using (43), it is easy to see that

\[
\|y_n(x_m) + y(x_m) - 2y\|_0 \leq \|y_n(x_m) - y(x_m)\|_0 + 2\|y(x_m) - y\|_0 \leq C
\]

for any fixed \( m \) and sufficiently great \( n \).

Substituting (43), (44) into (41), we are led to the conclusion that

\[
M_2 \rightarrow 0 \text{ for } n \rightarrow \infty \text{ and any fixed } m.
\]

Finally, we may write

\[
|M_3| \leq \|y(x_m) - y(x)\|_0 \|y(x_m) + y(x) - 2y\|_0,
\]

\[
\|y(x_m) - y(x)\|_0^2 = \int_0^1 (y(x_m) - y(x))^2 \, dz =
\]

\[
= \int_0^1 dz \left( \int_{x_m(z)}^{x_m(z)} \frac{\partial y}{\partial r} \, dr \right)^2 \leq \frac{1}{m} \int_0^1 dz \int_{x_m(z)}^{x_m(z)} \left( \frac{\partial y}{\partial r} \right)^2 \, dr \leq m^{-1}\|y\|_{1,\rho_0}^2 \leq Cm^{-1}\|y\|_{1,\rho,D}^2 \rightarrow 0
\]

for \( m \rightarrow \infty \) (here \( D^0 = D \setminus (0, x_{\text{min}}/2) \times (0, 1) \)).

Moreover,

\[
\|y(x_m) + y(x) - 2y\|_0 \leq \|y(x_m) - y(x)\|_0 + 2\|y(x) - y\|_0 \leq C
\]

follows easily. Substituting into (46), we obtain

\[
M_3 \rightarrow 0 \text{ for } m \rightarrow \infty.
\]

It suffices to combine (34), (40), (45) and (47) to arrive at

\[
\lim_{n \rightarrow \infty} j_2(x_n, y_n) = j_2(x, y).
\]
Case $i = 3$. By virtue of the definition (18) we have

$$j_3(x, y) = a(x; y, y) = L(x; y),$$

$$j_3(x_n, y_n) = a(x_n; y_n, y_n) = L(x_n; y_n).$$

Consequently, we may write

$$j_3(x_n, y_n) - j_3(x, y) = K_1 + K_2,$$

where

$$K_1 = L(x_n; y_n) - L(x; E_y^m),$$

$$K_2 = L(x; E_y^m - y).$$

Let us estimate

$$|K_1| \leq \left| \int_{D_n} f y_n r dr dz - \int_D f E_y^m r dr dz \right| + \left| \int_{R_1(x_n)} g y_n r dr - \int_{R_1(x)} g E_y^m r dr \right| \leq$$

$$= \int_{\triangle(D_n, D)} |f E_y^m| r dr dz + \int_{\triangle(R_1(x_n), R_1(x))} |g E_y^m| r dr = K_{11} + K_{12}.$$ 

We obviously have

$$K_{11} \leq \|f\|_{0,r,\triangle(D_n, D)} \|E_y^m\|_{0,r,D} \to 0$$

on the basis of (24), (34) and

$$K_{12} \leq \|g\|_{0,r,\triangle(R_1(x_n), R_1(x))} \|E_y^m\|_{0,r,R_1} \to 0$$

on the basis of (28'), (24) and (5) (which can be applied to $z(x) \equiv \delta$, as well), since

$$\|E_y^m\|_{0,r,R_1} \leq C \|E_y^m\|_{1,r,D} \leq \tilde{C}.$$ 

Altogether,

$$\lim_{n \to \infty} K_1 = 0.$$ 

By virtue of Lemma 6 the functional $u \to L(x; u)$ is continuous on $W^{(1)}_2(D)$. Consequently, the weak convergence (22) implies that

$$\lim_{n \to \infty} K_2 = 0.$$ 

Combining these two results we are led to the assertion to be proved.

Case $i = 4$. We have

$$j_4(x_n, y_n') \geq \int_{G_m} \left[ \left( a_r \frac{\partial y_n}{\partial r} - K_1 \right)^2 + \left( a_z \frac{\partial y_n}{\partial z} - K_2 \right)^2 \right] r dr dz$$

for any $n, m$ such that $G_m \subset D_n$. The functional on the right-hand side is weakly lower semi-continuous on $W^{(1)}_2(G_m)$ (being convex and Gâteaux-differentiable).
The restrictions $y_n^m |_{G_m}$ converge to $y^m |_G$ weakly in the space $W^{1, 1}_2(G_m)$, as follows from (22). Consequently, we have for any $m$

$$\liminf_{n \to \infty} j_4(x_n, y_n) \geq \int_{G_m} \left( a_r \frac{\partial y}{\partial r} - K_1 \right)^2 + \left( a_z \frac{\partial y}{\partial z} - K_2 \right)^2 \, r \, dr \, dz.$$ 

Passing to the limit with $m \to \infty$, we obtain the assertion

$$\liminf_{n \to \infty} j_4(x_n, y_n) \geq j_4(x, y).$$

Q.E.D.

Now we are able to prove the main result of the Section 3.

**Theorem 1.** There exists at least one solution of the Domain Optimization Problem (29)i, $i \in \{1, 2, 3, 4\}$.

**Proof.** Let $\{x_n^i, x_n \in U_{ad}\}$ be a minimizing sequence of $j_i(x, y(z))$, where $i \in \{1, 2, 3, 4\}$, i.e.

$$\lim_{n \to \infty} j_i(x_n, y(z_n)) = \inf_{x \in U_{ad}} j_i(x, y(z)).$$

By means of Arzelà-Ascoli Theorem we show that the set $U_{ad}$ is compact in $C([0, 1])$. Hence there exist a subsequence $\{x_k\}$ and $x^0 \in U_{ad}$ such that

$$x_k \to x^0 \text{ in } C([0, 1]).$$

Then Proposition 2 and (48) imply that

$$j_i(x^0, y(z^0)) \leq \liminf_{k \to \infty} j_i(x_k, y(z_k)) = \inf_{x \in U_{ad}} j_i(x, y(z)).$$

Consequently, at $x^0$ a minimum is attained. Q.E.D.

4. APPROXIMATIONS BY FINITE ELEMENT METHOD

In the present Section we propose an approximate solution of the domain optimization problem, making use of piecewise linear design variable and linear triangular finite elements for solving the state problem.

Let $N$ be a positive integer and $h = 1/N$. We denote by $\triangle_j$, $j = 1, 2, \ldots, N$, the subintervals $[(j - 1) h, j h]$ and introduce the set

$$U^h_{ad} = \{ x_k \in U_{ad}; x_k |_{\triangle_j} \in P_1(\triangle_j) \forall j \}.$$ 

where $P_1(\triangle_j)$ is the set of linear functions defined on $\triangle_j$. Let $D_h = D(x_h)$ denote the domain bounded by the graph $\Gamma_h = \Gamma(x_h)$ of the function $x_h \in U^h_{ad}$. The polygonal domain $D_h$ will be carved into triangles by the following way. We choose $x_0 \in (0, a)$
and introduce a uniform triangulation of the rectangle \( R = [0, \alpha_0] \times [0, 1] \), independent of \( \alpha_h \), if \( h \) is fixed.

In the remaining part \( D_h \) let the nodal points divide the segments \([\alpha_0, \alpha_h(jh)]\), \( j = 1, 2, \ldots, N \), into \( M \) equal segments, where
\[
M = 1 + \left[ \frac{1}{\beta - \alpha_0} N \right]
\]
and the square brackets denote the integer part of the number inside.

One can verify that the segments parallel with the \( r \)-axis are not longer than \( h \) and shorter than \( h(\alpha - \alpha_0)/(\beta - \alpha_0) \). One also deduces the following estimate for the interior angles \( \omega \) of the triangles \( T \):
\[
tg \omega \geq \frac{\alpha - \alpha_0}{\beta - \alpha_0} \left( 1 + C_1 + C_2^2 \right)^{-1}.
\]
Consequently, one obtains a regular family \( \{ T_h(\alpha_h) \} \), \( h \to 0, \alpha_h \in U_{ad} \), of triangulations, with
\[
\max_{T \in T_h(\alpha_h)} (\text{diam } T) \leq h/\sin \omega_0,
\]
\[
\omega_0 = \arctg \left[ \frac{\alpha - \alpha_0}{\beta - \alpha_0} \left( 1 + C_1 + C_2^2 \right)^{-1} \right] .
\]
The family is even strongly regular, i.e. the ratio of the maximal and minimal side in \( T_h \) is not greater than a constant, independent of \( h \) and \( \alpha_h \).

Let us consider the standard space \( V_h \) of linear finite elements
\[
V_h = \{ v_h \in C(\bar{D}_h) \cap V(D_h) \mid v_h|_T \in P_1(T) \ \forall T \in T_h(\alpha_h) \}.
\]
We define the approximate state problem:

find \( y_h \equiv y_h(\alpha_h) \in V_h \) such that
\[
a(\alpha_h; y_h, v_h) = L_h(\alpha_h; v_h) \ \forall v_h \in V_h.
\]
Here \( L_h(\alpha_h; v_h) \) denotes a suitable approximation of the functional \( L(\alpha_h; v_h) \), which satisfies the following conditions:

positive constants \( C_8, C_9 \) and \( \mathcal{B} \) exist such that
\[
| L_h(\alpha_h; v_h) - L(\alpha_h; v_h) | \leq C_8 h^\mathcal{B} \| v_h \|_{1,r,D_h},
\]
\[
| L_h(\alpha_h; v_h) | \leq C_9 \| v_h \|_{1,r,D_h},
\]
hold for any \( \alpha_h \in U_{ad} \) and any \( v_h \in V_h \).

Let us define
\[
L_h(\alpha_h; v_h) = \sum_{T \in T_h(\alpha_h)} [frv_h]_{G(T)} \text{mes } (T) + \sum_{I \in \Gamma_h(\alpha_h) \cap \Gamma_1(\alpha_h)} [grv_h]_{G(I)} \text{mes } (I),
\]
\[
\text{under the assumption } \alpha_h \in U_{ad}.
\]
where $G(T)$ denotes the centre of gravity of the triangle $T$ and $G(l)$ the midpoint of the interval $I = T \cap \Gamma_1(\alpha_h)$.

**Lemma 8.** Let $L_h(\alpha_h; v_h)$ be defined by the formula (52). Assume that $f \in H^1(\tilde{D}) \cap \cap C(\tilde{D})$, $r^2D^2 f \in L^2(\tilde{D})$ for $|x| = 2$ and $g$ is piecewise from $C^2$.

Then (50), (51) hold, with $\psi = 1$.

**Proof.** If we denote the local error by

$$E_T(w) = \int_T w \, dx - w(G) \, \text{mes} \, T, \quad T \in \mathcal{T}_h(\alpha_h)$$

and $fr \equiv F$, we may write

$$\int_{\Delta_h} f v_h \, r \, dr \, dz - \sum_T [fr v_h]_{G(T)} \, \text{mes} \, T = |\sum_T E_T(Fv_h)| \leq \sum_T (|E_T(Fv_h(G))| + |E_T(F(v_h - v_h(G)))|).$$

Applying the affine mapping

$$(r, z) = B_T \hat{x} + b_T, \quad \hat{x} = (\hat{r}, \hat{z}),$$

which transforms the reference unit triangle $\tau$ with the vertices $(0, 0), (1, 0), (0, 1)$ onto $T$, we easily deduce

$$|E_T(w)| \leq Ch^2|\hat{E}(\hat{w})|,$$

where $\hat{w}(\hat{x}) = w(B_T \hat{x} + b_T)$,

$$\hat{E}(\hat{w}) = \int_\tau (\hat{w} - \hat{w}(\gamma)) \, d\hat{x}$$

and $\gamma$ is the centroid of the triangle $\tau$. Since

$$|\hat{E}(\hat{w})| \leq C\|\hat{w}\|_{2,\tau}, \quad \hat{E}(p) = 0 \quad \forall p \in P_1(\tau),$$

the Bramble-Hilbert Lemma yields [5]

$$|\hat{E}(\hat{w})| \leq C|\hat{w}|_{2,\tau}.$$

Using the estimate ([5] – pp. 118, 122)

$$|\hat{w}|_{2,\tau} \leq Ch|w|_{2,\tau},$$

we obtain

$$|E_T(w)| \leq Ch^3|w|_{2,\tau}.$$
Moreover, by the strong regularity of the family \( \{ \mathcal{T}_h(\alpha_h) \} \), we have ([5], p. 142)

\[
(56) \quad |v_h(G)| \leq \|v_h\|_{C(T)} \leq Ch^{-1}\|v_h\|_{0,T} \quad \forall T \in \mathcal{T}_h(\alpha_h),
\]
so that

\[
(57) \quad |E_T(Fv_h(G))| = |E_T(F)| |v_h(G)| \leq Ch^2|F|_{2,T}\|v_h\|_{0,T}
\]

follows from (55), (56).

Using again (54), we derive

\[
(58) \quad |E_T(F(v_h - v_h(G)))| \leq Ch^2|\hat{F}(\hat{v}_h - \hat{v}_h(\gamma))| = Ch^2 \left| \int_T \hat{F}(\hat{v}_h - \hat{v}_h(\gamma)) \, dx \right|.
\]

If we define

\[
\mathcal{F}(\hat{F}) = \int_T \hat{F}(\hat{v}_h - \hat{v}_h(\gamma)) \, dx,
\]

then

\[
\mathcal{F}(p) = 0 \quad \forall p \in P_0(\tau),
\]

since \( \hat{v}_h \) is linear and

\[
|\mathcal{F}(\hat{F})| \leq C \|\hat{v}_h - \hat{v}_h(\gamma)\|_{0,T} \|\hat{F}\|_{1,T}.
\]

From the Bramble-Hilbert Lemma

\[
|\mathcal{F}(\hat{F})| \leq C\|\hat{v}_h - \hat{v}_h(\gamma)\|_{0,T} \|\hat{F}\|_{1,T}
\]

follows. Since

\[
|\hat{v}_h - \hat{v}_h(\gamma)| \leq |\nabla \hat{v}_h(\hat{x} - \gamma)| \leq |\nabla \hat{v}_h|,
\]

we have

\[
\|\hat{v}_h - \hat{v}_h(\gamma)\|_{0,T} \leq |\hat{v}_h|_{1,T}
\]

and

\[
|\mathcal{F}(\hat{F})| \leq C|\hat{v}_h|_{1,T} \|\hat{F}\|_{1,T} \leq C\|F\|_{1,T} \|v_h\|_{1,T}.
\]

Substituting into (58), we arrive at

\[
(59) \quad |E_T(F(v_h - v_h(G)))| \leq Ch^2\|F\|_{1,T} \|v_h\|_{1,T}.
\]

Combining (57), (59), we see that the upper bound in (53) is

\[
(60) \quad E_f \equiv Ch^2\|F\|_{2,Dh} \|v_h\|_{1,Dh}.
\]

For any \( v_h \in V_h \) the following inequality holds

\[
(61) \quad \|v_h\|_{1,Dh} \leq Ch^{-1/2}\|v_h\|_{1,r,Dh}.
\]

To prove this, we consider the subset \( \mathcal{F}_h^{(1)} \) of triangles \( T \in \mathcal{T}_h(\alpha_h) \), such that

\[
T \cap \emptyset \neq \emptyset
\]

232
(where \( \emptyset \) is the z-axis). Let us use the mapping \( r = h_1 \hat{r}, \ z = z_T + h \hat{z} \), which transforms the "unit" triangle \( \tau \) onto \( T \in \mathcal{T}_h^{(1)} \). On \( T \) we have

\[
\nu_h^2 + |\nabla \nu_h|^2 = p^2 + c^2, \quad p \in P_1(T), \quad c \in P_0(T),
\]

(62) \[
\|\nu_h\|_{1,T}^2 = \int_T (p^2 + c^2) \, dr \, dz = h_1 \int_\tau (\hat{p}^2 + c^2) \, d\hat{r} \, d\hat{z},
\]

where \( \hat{p}(\hat{r}, \hat{z}) = p(h_1 \hat{r}, z_T + h \hat{z}) \). Since in the finite-dimensional space \( P_1(\tau) \times P_0(\tau) \) all norms are equivalent, we have

\[
\int_\tau (\hat{p}^2 + c^2) \, d\hat{r} \, d\hat{z} \leq C \int_\tau (\hat{p}^2 + c^2) \, \hat{r} \, d\hat{r} \, d\hat{z}.
\]

Substituting into (62) leads us to the following estimate

(63) \[
\|\nu_h\|_{1,T}^2 \leq Ch_1 \int_\tau (\hat{p}^2 + c^2) \, \hat{r} \, d\hat{r} \, d\hat{z} \leq C h^{-1} \int_T (p^2 + c^2) \, r \, dr \, dz = Ch^{-1} \|\nu_h\|_{1,r,T}^2.
\]

If \( T \in \mathcal{T}_h(\sigma_h) \subset \mathcal{T}_h^{(1)} \), then \( h_1 \leq r \) so that \( 1 \leq Ch^{-1} \) and

(64) \[
\|\nu_h\|_{1,T}^2 = \int_T (\nu_h^2 + |\nabla \nu_h|^2) \, dr \, dz \leq C h^{-1} \|\nu_h\|_{1,r,T}^2.
\]

Adding the results (63) and (64), we arrive at (61).

Consequently, instead of (60) we may write

(65) \[
E_f \leq Ch^{3/2} \|F\|_{2,c_h} \|\nu_h\|_{1,r,c_h}.
\]

Next we have

(66) \[
\left| \int_{I_1(\sigma_h)} g v_h \, dr - \sum_I \left[ g v_h \right]_{G(I)} \mes I \right| \leq \sum_{I \in I_1} |E_I(g v_h)|,
\]

where the local error is defined by the formula

\[
E_I(w) = \int_I w \, dr - w(G) \ h_I, \quad h_I = \mes I.
\]

It is easy to see that by means of the mapping

\[
r = r_I + h_I \hat{r}
\]

which maps the unit interval \( \sigma \) onto \( I \), we obtain

\[
E_I(w) = \int_0^1 \left( \hat{w} - \hat{w}(1/2) \right) \, h_I \, d\hat{r} = h_I \hat{E}(\hat{w}),
\]

\[
\hat{w}(\hat{r}) = w(r_I + h_I \hat{r}).
\]
Since
\[ |E(\hat{w})| \leq C\|\hat{w}\|_{2,\sigma}, \quad \hat{E}(p) = 0 \quad \forall p \in P_1(\sigma), \]
the Bramble-Hilbert Lemma yields that
\[ |\hat{E}(\hat{w})| \leq C|\hat{w}|_{2,\sigma}. \]
Making use of the relation
\[ |\hat{w}|_{2,\sigma}^2 = h_I^2|w|_{2,I}^2, \]
we obtain
\[ |E_1(w)| \leq \sqrt{h}E(\hat{w}) \leq Ch^{5/2}|w|_{2,I} \quad \forall w \in H^2(I). \]
If we insert \( w = grv_h \), then
\[ |w|_{2,I}^2 = |grv_h|_{2,I}^2 \leq C\|g\|_{C^2}^2 \|v_h\|_{1,I}^2, \]
where
\[ \|g\|_{C^2} = \max_1 \|g\|_{C^2(S_i)} \]
and \( S_i \subset I_1 \) are subintervals, where \( g\big|_{S_i} \in C^2(S_i) \).

Consequently, we obtain the following upper bound for the right-hand side in (66)
\[ E_g \leq \sum_{I \in I_1} h_I^{5/2}\|g\|_{C^2}\|v_h\|_{1,I} \leq Ch^2\|g\|_{C^2}\|v_h\|_{1,\Gamma_1} \]  
Here we have also used the estimate \( Ch^{-1} \) for the number of subintervals \( I \in I_1 \).

One can show that
\[ \|v_h\|_{0,\Gamma_1} \leq Ch^{-1}\|v_h\|_{0,\Gamma_1} \]
by an argument similar to that of (61). Furthermore, we have
\[ |v_h|_{1,I}^2 = \left| \frac{dv_h}{dr} \right|^2 h_I \leq h_I |v_h|_{1,T}^2 (\text{mes } T)^{-1} \leq Ch^{-1}|v_h|_{1,T}^2 \]
since
\[ \text{mes } T \geq C h^2 \]
holds by virtue of the strong regularity of \( \{\mathcal{T}_h(z_h)\} \). Consequently, we may write
\[ \|v_h\|_{1,I_1}^2 = \|v_h\|_{0,I_1}^2 + \sum_{I \in I_1} |v_h|_{1,I}^2 \leq Ch^{-1}\|v_h\|_{0,\Gamma_1}^2 + \]
\[ + Ch^{-1}\|v_h\|_{1,D_h}^2 \leq Ch^{-2}\|v_h\|_{1,\Gamma_1}^2, \]
using again (61). Substituting into (67), we obtain
\[ E_g \leq Ch\|g\|_{C^2}\|v_h\|_{1,\Gamma_1}, \]

Finally, it is easy to combine (65) and (68) to get
\[ |L_h(x_h; v_h) - L(x_h; v_h)| \leq E_f + E_g \]
\[ \leq C(h^{3/2} \| f \|_{2,D_h} + h \| g \|_{C^2}) \| v_h \|_{1,r,D_h}. \]

Consequently, the condition (50) is satisfied with \( \mathcal{G} = 1 \).

Note that if \( f \in H^1(\bar{D}) \) and \( r^2D^2f \in L^2(\bar{D}) \) for \( |x| = 2 \), then \( fr \in H^2(\bar{D}) \).

To verify the condition (51), we employ Lemma 6 in the following estimate
\[ |L_h(x_h; v_h)| \leq |L(x_h; v_h)| + |L_h(x_h; v_h) - L(x_h; v_h)| \]
\[ \leq C_6 \| v_h \|_{1,r,D_h} + C_8 h^2 \| v_h \|_{1,r,D_h} \leq C_9 \| v_h \|_{1,r,D_h}. \]

**Lemma 9.** The approximate state problem (49) has a unique solution for any \( x_h \in U_{ad}^h \).

**Proof.** Lemmas 5 and 8 guarantee that we can apply Riesz-Theorem in the space \( V_h \) with the inner product \( (u, v) \equiv a(x_h; u, v) \).

**Proposition 3.** Let the assumptions of Lemma 8 be satisfied. Let \( \{x_h\}, h \to 0, \) be a sequence of \( x_h \in U_{ad}^h, \) converging to \( x \) in \( C([0, 1]) \). Let us construct extensions \( Ey_h \) of the solutions \( y_h(x_h) \) of the approximate state problem (49) by means of the formula (12).

Then
\[ (69) \quad Ey_h|_{D(x)} \rightharpoonup y(x) \quad \text{(weakly) in } W^{1,1}_{2,r}(D(x)), \]
where \( y(x) \) is the solution of the state problem (18) on \( D(x) \).

**Proof.** Let us define \( y^*_h \in V_h \) to be the solution of the following problem
\[ (70) \quad a(x_h; y^*_h, v_h) = L(x_h; v_h) \quad \forall v_h \in V_h. \]

Recall that \( y_h \in V_h \) satisfies the condition
\[ a(x_h; y_h, v_h) = L_h(x_h; v_h) \quad \forall v_h \in V_h. \]

Subtracting, we obtain
\[ a(x_h; y^*_h - y_h, v_h) = L(x_h; v_h) - L_h(x_h; v_h) \]
and inserting \( v_h := y^*_h - y_h \), we arrive at
\[ (71) \quad C_4 \| y^*_h - y_h \|_{1,r,D_h}^2 \leq a(x_h; y^*_h - y_h, y^*_h - y_h) = \]
\[ = L(x_h; y^*_h - y_h) - L_h(x_h; y^*_h - y_h) \leq C_8 h \| y^*_h - y_h \|_{1,r,D_h}. \]

using Lemma 5 and Lemma 8.
For the extensions we make use of (13') and (71) to obtain

\[(72) \quad \|E y_h^* - E y_h\|_{1,r,D} \leq C \|y_h^* - y_h\|_{1,r,D_h} \leq \mathcal{C} h.\]

From (70) it follows that

\[C_4 \|y_h^*\|_{1,r,D_h}^2 \leq L(\zeta_h; y_h^*) \leq C_9 \|y_h^*\|_{1,r,D_h},\]

by virtue of Lemma 5 and Lemma 8.

Consequently,

\[(73) \quad \|E y_h^*\|_{1,r,D} \leq C \|y_h^*\|_{1,r,D_h} \leq \mathcal{C}
\]

follows and there exists a subsequence (we shall denote it by the same symbol) and \(y \in W^{2,\varepsilon}_2(\hat{D})\) such that

\[(74) \quad E y_h^* \rightharpoonup y \text{ (weakly) in } W^{2,\varepsilon}_2(\hat{D}).\]

Let \(v \in V(D(z))\) be given. Let us construct the extension \(E v \in V(\hat{D})\) by the formula (12). By virtue of Lemma 2, there exists a sequence \(\{v_\kappa\}, \kappa \to 0\), such that \(v_\kappa \in C^\infty(\text{Cl}(\hat{D}))\) supp \(v_\kappa \cap \hat{P}_2 = 0\) and

\[(75) \quad \|v_\kappa - E v\|_{1,r,D} \to 0 \text{ for } \kappa \to 0.\]

Consider the Lagrange linear interpolate \(\pi_h v_\kappa\) of \(v_\kappa\) over the triangulation \(T_h(\zeta_h)\). Obviously, we have \(\pi_h v_\kappa \in V_h\). Let \(\kappa\) be fixed, for a time being. We can insert \(\pi_h v_\kappa\) into (70) to obtain

\[(76) \quad a(\zeta_h; y_h^*, \pi_h v_\kappa) = L(\zeta_h; \pi_h v_\kappa).\]

We shall pass to the limit with \(h \to 0\). Let again \(\alpha_m = \alpha(z) - 1/m\), where \(m = 2, 3, \ldots\),

\[G_m = \{(r, z) \mid 0 < r \leq \alpha_m(z), 0 < z < 1\}.
\]

Then

\[G_m \subset D_h\]

for \(h < h_0(m)\) and we may write

\[(77) \quad |a(\zeta_h; y_h^*, \pi_h v_\kappa) - a(\alpha_m; y, v_\kappa)| =
\]

\[= |a(\zeta_m; y_h^*, v_\kappa) + a(\alpha_m; y_h^*, \pi_h v_\kappa - v_\kappa) +
\]

\[+ \hat{a}(\zeta_h - \alpha_m; y_h^*, \pi_h v_\kappa) - a(\alpha_m; y, v_\kappa)| \leq
\]

\[\leq |a(\zeta_m; y_h^* - y, v_\kappa)| + |a(\zeta_m; y_h^*, \pi_h v_\kappa - v_\kappa)| +
\]

\[+ |\hat{a}(\zeta_h - \alpha_m; y_h^*, \pi_h v_\kappa)|,
\]

where

\[\hat{a}(\zeta_h - \alpha_m; \cdot, \cdot) = a(\zeta_h; \cdot, \cdot) - a(\alpha_m; \cdot, \cdot).\]
Consider a positive $\varepsilon$. From (74) we conclude that the first term on the right-hand side of (77) is not greater than $\varepsilon/6$ if $h < h(x, m)$.

To estimate the second term, we employ the well-known inequality (see e.g. [5])

\[
\|\pi_h v_x - v_x\|_{1,D_h} \leq C h \|v_x\|_{1,2,D_h} \leq Ch \|v_x\|_{2,D}.
\]

Combining (73) and (78) we obtain

\[
|a(\alpha_m; y^*, \pi_h v_x - v_x)| \leq C_5 \|y^*_h\|_{1,r,G_m} \|\pi_h v_x - v_x\|_{1,r,G_m} \leq Ch \|v_x\|_{2,D} < \varepsilon/6 \quad \text{for} \quad h < h_2.
\]

It remains to estimate the third term. To this end, we realize that

\[
\|\pi_h v_x\|_{1,T} \leq C \|v_x\|_{2,T} \quad \forall h
\]

holds for all triangles $T \in \mathcal{T}_h(\alpha_h)$.

Let $G_m^h$ be the smallest union $U$ of triangles $T \in \mathcal{T}_h(\alpha_h)$ such that $D_h \in G_m \subseteq U$. Obviously, we have

\[
\text{mes } G_m^h \leq 1/m + 2h + \|\alpha_h - \alpha\|\infty,
\]

where $\|\cdot\|\infty$ denotes the norm in $C([0, 1])$. Consequently,

\[
\|\pi_h v_x\|_{1,D_h-G_m^h} \leq \|\pi_h v_x\|_{1,G_m^h} = \sum_{T \in G_m^h} \|\pi_h v_x\|_{1,T} \leq C^2 \|v_x\|_{2,G_m^h}.
\]

Using again (73), we may write

\[
|a(\alpha_h - \alpha_m; y^*_h, \pi_h v_x)| \leq C_5 \|y^*_h\|_{1,r,D_h} \|\pi_h v_x\|_{1,r,D_h-G_m} \leq C \|v_x\|_{2,G_m^h}.
\]

Combining (77), (79), (81), we deduce the following inequality

\[
|a(\alpha_h - \alpha_m; y^*_h, \pi_h v_x) - a(\alpha; y, v_x)| \leq |a(\alpha_h; y^*_h, \pi_h v_x)| + |a(\alpha; y, v_x)| \leq |a(\alpha_h; y^*_h, \pi_h v_x) - a(\alpha_m; y, v_x)| + |\alpha - \alpha_m| + |\alpha - \alpha_m| + |\alpha - \alpha_m| \leq \varepsilon/2 + C \|v_x\|_{2,G_m^h} + C_5 \|y\|_{1,r,D} \|v_x\|_{1,r,D-G_m} \quad \text{for} \quad h < h_3(\varepsilon, m).
\]

Making use of (80), we conclude that

\[
\lim_{h \to 0} a(\alpha_h; y^*_h, \pi_h v_x) = a(\alpha; y, v_x).
\]

Next we may write

\[
|L(\alpha_h; \pi_h v_x) - L(\alpha; v_x)| \leq |L(\alpha_h; \pi_h v_x - v_x)| + |L(\alpha; v_x) - L(\alpha; v_x)| = \mathcal{L}_1 + \mathcal{L}_2,
\]

\[
|\mathcal{L}_1| \leq C_6 \|\pi_h v_x - v_x\|_{1,r,D_h} \leq Ch \|v_x\|_{2,D},
\]

\[
|\mathcal{L}_2| \leq \int_{\triangle(D_h,D)} |fv_x| r \, dr \, dz + \int_{\triangle(G_m^h, D)} |gv_x| r \, dr \, dz
\]
and since
\[ \triangle(D_h, D) \to 0, \quad \triangle(\Gamma_1(z_h), \Gamma_1) \to 0 \]
for \( h \to 0 \), we conclude that
\[ \lim_{h \to 0} L(z_h; \pi_h \nu_h) = L(z; \nu_z). \]
Passing to the limit with \( h \to 0 \) in \( (76) \) and using \( (82), (83) \), we arrive at
\[ a(z; y, \nu_z) = L(z; \nu_z). \]
Passing to the limit with \( \kappa \to 0 \) and using Lemma 5, Lemma 6 and \( (75) \), we obtain
\[ a(z; v) = L(z; v). \]
The space \( V(D) \) is weakly closed in \( W^1_2(D) \) and every function \( E_y^* \) belongs to \( V(D) \). Hence the weak limit \( y \in V(D) \) and its restriction to \( D \equiv D(z) \) belongs to \( V(D) \). Consequently, \( y|_D \) is a solution of \( (18) \). Since the solution is unique, \( y|_D = y(z) \) and \( (74) \) implies that
\[ E_y^*|_D \to y(z) \] (weakly) in \( W^1_2(D) \);
the latter convergence holds for the whole sequence \( \{E_y^*|_D\} \).
The remainder of the proof is an easy consequence of \( (72) \). Q.E.D.

For a fixed parameter \( h \), we define the Approximate Domain Optimization Problems:

find \( a^0_h \in U_{ad}^h \) such that
\[ j_i(a^0_h, y^i_h(a^0_h)) \leq j_i(a_h, y^i_h(a_h)) \quad \forall a_h \in U_{ad}^h, \]
where \( i \in \{1, 2, 3\} \) and \( y^i_h(a_h) \) is the solution of the approximate state problem \( (49) \).

**Proposition 4.** The Approximate Domain Optimization Problems have at least one solution for any \( i \in \{1, 2, 3\} \) and any \( h = 1/N, N = 2, 3, \ldots \).

**Proof.** It is readily seen that
\[ a_h \in U_{ad}^h \Leftrightarrow a \in \mathcal{A}, \]
if \( a \in \mathbb{R}^{N+1} \) denotes the vector of \( z_h(jh), j = 0, 1, \ldots, N \), and \( \mathcal{A} \) is a compact set in \( \mathbb{R}^{N+1} \). One can show that the nodal values of \( y_h(a_h) \) depend continuously on \( a \); the same assertion can be then verified for \( j_i(a_h, y_h(a_h)) \equiv J_i(a) \). Consequently, the function \( J_i(a) \) attains its minimum on the set \( \mathcal{A} \).

**Proposition 5.** Let the assumptions of Proposition 3 be satisfied. Then
\[ \lim_{h \to 0} j_i(a_h, y_h(a_h)) = j_i(z, y(z)) \]
holds for \( i \in \{1, 2, 3\} \), where \( y_h(a_h) \) and \( y(z) \) is the solution of the problem \( (49) \) and \( (18) \), respectively.

238
Proof is parallel to that of Proposition 2. We replace $a$ by $a_h$, $n$ by $y_h$, $D_n$ by $D_h$, instead of Proposition 1 and Lemma 6 we make use of Proposition 3 and Lemma 6, respectively. The boundedness of all $\|y_h\|_{1,r,D_h}$ follows from (72) and (73).

In proving the assertion for $f_3(a_h, y_h)$, we have moreover to estimate the following term by means of Lemma 8 (50)

$$|L_h(a_h; y_h) - L(a_0; y_h)| \leq C_8 h \|y_h\|_{1,r,D_h} \leq Ch.$$ 

**Theorem 2.** Let the assumptions of Lemma 8 hold. Let $\{a_h\}, h \to 0$, be a sequence of solutions of the Approximate Domain Optimization Problem (85), $i \in \{1, 2, 3\}$. Then a subsequence $\{a_h\}$ exists such that

$$a_h \to a^0 \quad \text{in} \quad C([0, 1]),$$

$$Ey_h|_{D(a^0)} \to y(a^0) \quad \text{(weakly) in} \quad W_{x,r}^{1,3}(D(a^0)),$$

where $a^0$ is a solution of the Domain Optimization Problem (29i), $Ey_h$ are the solutions $y_h(a_h)$, extended according to the formula (12), $y(a^0)$ is the solution of the state problem (18) on $D(a^0)$.

The limit of any uniformly convergent subsequence of $\{a_h\}$ represents a solution of (29i) and an analogue of (87) holds.

**Proof.** Since $U_{ad}$ is compact in $C([0, 1])$, a subsequence $\{a_h\} \subset \{a_h\}$ exists such that (86) holds and $a^0 \in U_{ad}$.

Let $a \in U_{ad}$ be given. There exists a sequence $\{\beta_h\}, \beta_h \in U_{ad}$, such that $\beta_h \to a$ in $C([0, 1])$. This follows from Lemma A1 below (Appendix).

We have

$$j_i(a_h, y_h(a_h)) \leq j_i(\beta_h, y_h(\beta_h)) \quad \forall h$$

by definition.

Passing to the limit with $h \to 0$ and using Proposition 5 on both sides, we obtain

$$j_i(a^0, y(a^0)) \leq j_i(a, y(a)).$$

Consequently, $a^0$ is a solution of the problem (29i).

The convergence (87) follows from Proposition 3. The rest of the Theorem is obvious.

**APPENDIX**

**Lemma A1.** To any $a \in U_{ad}$ there exists a sequence $\{a_h\}, h \to 0$, $a_h \in U_{ad}$, such that $a_h \to a$ in $C([0, 1])$.

**Proof.** 1° If $a = \text{const}$, then $a_h = a$. Consequently, we assume that $a \neq \text{const}$.

Let the mean value of $a$ be denoted by $S$, i.e.

$$S = \int_0^1 a(z) \, dz.$$
Let us define the sets $J = [0, 1]$, 

$$I^+ = \{ z \in [0, 1], \alpha - S \geq 0 \},$$

$$I^- = \{ z \in [0, 1], \alpha - S < 0 \},$$

and the following modified function

$$Z_{\mu \alpha} = \begin{cases} S + (1 - k \mu) (\alpha - S), & z \in I^+, \\ S + (1 - \mu) (\alpha - S), & z \in I^-, \end{cases}$$

where $k$ and $\mu$ are positive real parameters, $k$ is fixed ($k = k(\alpha)$) and $\mu \to 0$. Note that $\text{mes}(I^+) > 0$, $\text{mes}(I^-) > 0$.

It is easy to see that

(A.1) $\|Z_{\mu \alpha} - \alpha\|_{\infty, I} \leq \mu \max (1, k) \|\alpha - S\|_{\infty, I}$,

(A.2) $\left\| \frac{d}{dz} (Z_{\mu \alpha}) \right\|_{\infty, I} \leq \left\| \frac{d\alpha}{dz} \right\|_{\infty, I} \leq C_1$,

(A.3) $\alpha_{\max} - Z_{\mu \alpha} \geq \alpha_{\max} - \bar{\alpha} + k\mu(\bar{\alpha} - S) \geq k\mu(\bar{\alpha} - S),$ 

where $\bar{\alpha}$ is the maximum of $\alpha$ on $[0, 1]$,

(A.4) $Z_{\mu \alpha} - \alpha_{\min} \geq \alpha^* - \alpha_{\min} + \mu(S - \alpha^*) \geq \mu(S - \alpha^*),$ 

where $\alpha^*$ is the minimum of $\alpha$ on $[0, 1]$.

2° We apply Lemma 7.1 of [1] to obtain a sequence $\{\beta_h\}$, $h \to 0$, such that

(A.5) $\beta_h \to Z_{\mu \alpha}$ in $C([0, 1])$ for $h \to 0$, 

$$\beta_h |_{\Delta_j} \in P_1(\Delta_j), \quad j = 1, 2, \ldots, N, \quad \beta_h \in C([0, 1]),$$

(A.6) $\min_{z \in I} (Z_{\mu \alpha}) \leq \beta_h(z) \leq \max_{z \in I} (Z_{\mu \alpha}), \quad |d\beta_h/dz| \leq C_1,$

$$\int_0^1 \beta_h \, dz = \int_0^1 Z_{\mu \alpha} \, dz.$$ 

Since

$$\int_0^1 \beta_h^2 \, dz = \int_0^1 \alpha^2 \, dz$$

in general, we define a constant $a_h$ by the relation

(A.7) $\int_0^1 (\beta_h + a_h)^2 \, dz = \int_0^1 \alpha^2 \, dz$

and show that $k = k(\alpha)$ exists such that

$$\alpha_h = \beta_h + a_h$$

satisfies the conditions of the Lemma.
Denoting
\[ c_{\mu h} = \int_0^1 (\beta_h^2 - \alpha^2) \, dz, \quad \int_0^1 Z_{\mu k} \alpha \, dz = u, \]
we obtain from (A.7)
\[ (A.8) \quad a_h = -u + (u^2 - c_{\mu h})^{1/2}, \]
\[ c_{\mu h} = c_{\mu h}^{(1)} + c_{\mu h}^{(2)}, \quad c_{\mu h}^{(1)} = \int_0^1 (\beta_h^2 - (Z_{\mu k} \alpha)^2) \, dz, \]
\[ (A.9) \quad c_{\mu h}^{(2)} = \int_0^1 ((Z_{\mu k} \alpha)^2 - \alpha^2) \, dz = \int_0^1 (Z_{\mu k} \alpha - \alpha) (Z_{\mu k} \alpha + \alpha) \, dz = \]
\[ = -\mu 2S \left( k \int_{I^+} \gamma \, dz + \int_{I^-} \gamma \, dz \right) - k\mu (2 - k\mu) \int_{I^+} \gamma^2 \, dz - \mu (2 - \mu) \int_{I^-} \gamma^2 \, dz, \]
where \( \gamma = \alpha - S \).

3° Let us show that a positive constant \( k = k(\alpha) \) exists such that
\[ (A.10) \quad 0 < a_h < k\mu (\bar{\alpha} - S) \]
holds for sufficiently small \( \mu \) and \( h < h_0(\mu) \).

First we choose \( k \) such that \( c_{\mu h}^{(2)} < 0 \) for \( \mu \) sufficiently small. To this end we distinguish two cases.

1. Case:
\[ S\bar{\gamma} \geq \int_{I^+} \gamma \alpha \, dz, \quad (\bar{\gamma} = (\bar{\alpha} - S)). \]
Let us put \( k = 1 \) to obtain
\[ c_{\mu h}^{(2)} = -\mu (2 - \mu) \int_0^1 \gamma^2 \, dz < 0 \quad \forall \mu \in (0, 2). \]

2. Case:
\[ S\bar{\gamma} < \int_{I^+} \gamma \alpha \, dz. \]
Then if
\[ (A.11) \quad k > 1 - \int_0^1 \gamma^2 \, dz \left( \int_{I^+} \gamma \alpha \, dz \right)^{-1}, \]
we derive easily that
\[ (A.12) \quad \lim_{\mu \to 0^+} \frac{c_{\mu h}^{(2)}}{(2\mu)} = -S \left( k \int_{I^+} \gamma \, dz + \int_{I^-} \gamma \, dz \right) - \]
\[ - k \int_{I^+} \gamma^2 \, dz - \int_{I^-} \gamma^2 \, dz = - \int_0^1 \gamma^2 \, dz - (k - 1) \int_{I^+} \gamma \alpha \, dz < 0. \]
Consequently, we have
\[ c_{\mu}^{(2)} < 0 \quad \text{for} \quad \mu \in (0, \mu_1). \]

By virtue of (A.5) we have
\[ c_{\mu h}^{(1)} \to 0 \quad \text{for} \quad h \to 0 \]
and
\[ c_{\mu h} = c_{\mu h}^{(1)} + c_{\mu}^{(2)} < 0 \quad \forall h < h_1(\mu), \quad \mu < \mu_1. \]

Then from (A.7) we conclude that
\[ 0 < a_h < -c_{\mu h}/(2u). \]

Let us show that \( k \) exists such that
\[ 0 < -c_{\mu}^{(2)} < 2uk(\bar{a} - S)\mu \]
holds for \( \mu \) sufficiently small.

In fact, let us first consider the case
\[ S \geq \int_{I^+} y\alpha \, dz \]
and set \( k = 1. \) Then
\[ u = S, \]
\[ \lim_{\mu \to 0} \left(\mu(\bar{a} - S) + c_{\mu}^{(2)}/(2\mu)\right) = S \geq \int_0^1 \gamma^2 \, dz > \int_{I^+} \gamma\alpha \, dz + \int_{I^-} \gamma\alpha \, dz - \int_0^1 \gamma^2 \, dz = 0. \]
Consequently,
\[ 2\mu u(\bar{a} - S) + c_{\mu}^{(2)} > 0, \quad 0 < \mu < \mu_2 \]
and (A.16) is fulfilled.

Second, let
\[ S \leq \int_{I^+} y\alpha \, dz \]
and set
\[ k < -\int_{I^-} \gamma\alpha \, dz \left(\int_{I^+} \gamma\alpha \, dz - S\right)^{-1}. \]

Note, that (A.18) is compatible with (A.11), since
\[ 1 - \int_0^1 \gamma^2 \, dz \left(\int_{I^+} \gamma\alpha \, dz \right)^{-1} = \left(\int_{I^+} \gamma\alpha \, dz - \int_0^1 \gamma^2 \, dz \right). \]
\[ \left(\int_{I^+} \gamma\alpha \, dz \right)^{-1} < \left(\int_{I^+} \gamma\alpha \, dz - \int_0^1 \gamma^2 \, dz \right). \]
\[ \left(\int_{I^+} \gamma\alpha \, dz - S\right)^{-1} = \left(\int_{I^-} \gamma\alpha \, dz \right)\left(\int_{I^+} \gamma\alpha \, dz - S\right)^{-1}. \]
Then
\[
\lim_{\mu \to 0} \left[ k u (\bar{a} - S) + c_\mu^{(2)}/(2\mu) \right] = \\
= k S\bar{y} - S \left( k \int_{I_+} \gamma \, dz + \int_{I_-} \gamma \, dz \right) - k \int_{I_+} \gamma^2 \, dz - \int_{I_-} \gamma^2 \, dz = \\
= k \left( S\bar{y} - S \int_{I_+} \gamma \, dz - \int_{I_-} \gamma^2 \, dz \right) - \left( S \int_{I_-} \gamma \, dz + \int_{I_-} \gamma^2 \, dz \right) = \\
= k \left( S\bar{y} - \int_{I_+} \gamma \, dz \right) - \int_{I_-} \gamma \, dz > 0 .
\]

Consequently, we easily deduce
\[
2\mu ku (\bar{a} - S) + c_\mu^{(2)} > 0 \quad \text{for} \quad \mu < \mu_3
\]
and (A.16) holds.

Since
\[
-(c_{ph}/(2u)) = -(c_{ph}^{(1)}/(2u)) - c_\mu^{(2)}/(2\mu) < k\mu (\bar{a} - S)
\]
follows from (A.4), (A.13) and (A.16) for \( h < h_2(\mu) \), we arrive at (A.10), making use of (A.15).

4° Combining (A.10) with (A.6), (A.3), we obtain
\[
\alpha_h \equiv \beta_h + \alpha_h \leq \max \left( Z_{\mu_h} \alpha \right) + k\mu (\bar{a} - S) \leq \alpha_{\text{max}} .
\]

The lower bound is obvious.

Finally, we may write
\[
\| \alpha_h - \alpha \|_{\infty, I} \leq \alpha_h + \| \beta_h - Z_{\mu_h} \alpha \|_{\infty, I} + \| Z_{\mu_h} \alpha - \alpha \|_{\infty, I} .
\]

It is easy to see that (A.15), (A.19) imply that
\[
\lim_{h \to 0} a_h = 0 .
\]

Then the uniform convergence of \( \{ \alpha_h \} \) to \( \alpha \) follows from (A.20), (A.5) and (A.1).

References

Souhrn
OPTIMALIZACE OBLASTI V OSOVĚ SYMETRICKÝCH ELIPTICKÝCH ÚLOHÁCH METODOU KONEČNÝCH ELEMENTŮ
IVAN HLAVÁČEK

Uvažuje se osově symetrická eliptická úloha druhého řádu s kombinovanými okrajovými podmínkami. Je třeba nalézt část hranice oblasti tak, aby minimalizovala jeden ze čtyř typů účelového funkcionálu. Dokazuje se existence optimální hranice a konvergence přibližných, po částech lineárních řešení, a to prostřednictvím teorie Sobolevových prostorů s vahou.

Pešiome
OPTIMIZACIJA OBLASTI V OCESIMMETRIČESKIH ZADACHAX METODOM KONEČNYX ELEMENTOV
IVAN HLAVÁČEK

Рассматривается осесимметрическая эллиптическая задача второго порядка с смешанными краевыми условиями. Требуется найти часть границы области так, чтобы минимизировать один из четырех типов целевого функционала. Доказывается существование оптимальной границы и сходимость приближенных по частям линейных решений. В анализе используются пространства Соболева с весом.

Author's address: Ing. Ivan Hlaváček, DrSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.

244