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A SUFFICIENT CONDITION FOR ADMISSIBILITY IN LINEAR ESTIMATION

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Summary. It was recently shown that all estimators which are locally best in the relative interior of the parameter set, together with their limits constitute a complete class in linear estimation, both unbiased and biased. However, not all these limits are admissible. A sufficient condition for admissibility of a limit was given by the author (1986) for the case of unbiased estimation in a linear model with the natural parameter space. This paper extends this result to the general linear model and to biased estimation.

Keywords: Linear model, linear estimation, admissibility.

AMS Subject Classifications: primary 62C15, secondary 62J05.

1. INTRODUCTION

Necessary and sufficient conditions for admissibility of a linear estimator in the Gauss-Markov model were given by Cohen (1966), Shinozaki (1975) and Rao (1976); see also Stępniak (1984). This paper concerns the general linear model with unknown variance components.

Olsen, Seely and Birkes (1976) have shown that the set of all locally best estimators constitutes a complete class, while the set of all admissible locally best estimators constitutes a minimal complete class in linear unbiased estimation. A more efficient characterization of the minimal complete class in linear estimation, both unbiased and biased, was given by LaMotte (1982). He presented a procedure by which we can verify, in a finite number of steps, whether a linear estimator is admissible or not. However, this condition is not suitable for practical use.

It is easy to prove that linear estimators which are locally best in the relative interior of the parameter set are admissible. Stępniak (1987) has shown that all such estimators and their limits constitute a complete class in linear estimation, both unbiased and biased. This class is not greater, and usually smaller, than the class of all locally best estimators. However, not all limits are admissible. A sufficient condition for admissibility of a limit was given by Stępniak (1986) for the case of unbiased estimation in a linear model with the natural parameter space.

This paper extends this result to the general linear model and to biased estimation.

2. THE RESULTS

In this paper the usual vector-matrix notation will be used. By $\mathscr{S}_n$ we will denote the space of all symmetric matrices of $n \times n$, by $\mathscr{S}_n^+$ – the cone of all non-negative definite matrices in $\mathscr{S}_n$. Moreover, the symbol $ri(K)$, where K is a nonempty set in the Euclidean space $\mathbb{R}^m$, stands for the interior of K relative to the minimal affine set including K, that is,

$$ri(K) = \{x \in K: S(x, \varepsilon) \cap \operatorname{aff}(K) \subseteq K \text{ for some } \varepsilon > 0\},$$

where $S(x, \varepsilon)$ is the open ball in $\mathbb{R}^m$ centered at x and of radius ε (cf. Rockafellar, 1970).

Let Y be a random vector of dimension $n \times 1$ with the expectation $\mathsf{E}Y = X\beta$ and the variance-covariance matrix $\operatorname{Cov} Y = \Sigma$, where X is a known $n \times p$ matrix and (β, Σ) is a unknown element of a given set Ω in the product $\mathbb{R}^p \times \mathscr{S}_n^+$. Consider the estimation of a parameter $\Phi = c'\beta$, $c \in \mathbb{R}^p$, by estimators of the form $\hat{\Phi} = d'Y$, where d belongs to a given closed convex set D in $\mathbb{R}^n$. A particular case is the linear unbiased estimation specified by $D = \{d = d_1 + d_2: X'd_1 = c$ and $d_2'X\beta = 0$ for all $\beta\}$.

The mean square error of an estimator $d = d'Y$ under a parameter (β, Σ) is defined by

$$MSE(d \mid \beta, \Sigma) = \mathsf{E}(d'Y = c'\beta)^2 = \operatorname{tr}(\Sigma dd') + \operatorname{tr}\{\beta\beta'(c - X'd)(c - X'd)'\}.$$

We note that the MSE is a linear function of $\theta = (\Sigma, \beta\beta')$. For convenience we shall write θ_1 and θ_2 instead of Σ and $\beta\beta'$, respectively. Defining the inner product $\langle \cdot, \cdot \rangle$ in the space $\mathscr{S}_n \times \mathscr{S}_p$ by $\langle (A_1, A_2), (B_1, B_2) \rangle = \operatorname{tr}(A_1 B_1) + \operatorname{tr}(A_2 B_2)$ we may identify the set of the possible values of $\theta = (\theta_1, \theta_2)$ with a subset of $\mathbb{R}^m$, where $m = \frac{1}{2}n(n+1) + \frac{1}{2}p(p+1)$. Further, using some arguments in Olsen, Seely, Birkes (1976) and LaMotte (1982) we may reduce our consideration to some compact convex set Θ of θ. Thus the problem of linear estimation of Φ in the class D reduces to the statistical game (Θ, D, R) with the parameter set Θ, the decision set D and the risk function

$$R(\theta, d) = \operatorname{tr}(\theta_1 dd') + \operatorname{tr}\{\theta_2(c - X'd)(c - X'd)'\}. \tag{1}$$

It is well known (cf. Blyth, 1951) that any decision rule d which is Bayes relative to a prior distribution τ on Θ with the support Θ is admissible. We can also exploit the fact that the risk function (1) satisfies the condition

$$R(\lambda\theta + (1 - \lambda)\bar{\theta}, d) = \lambda R(\theta, d) + (1 - \lambda) R(\bar{\theta}, d) \tag{2}$$

for all $\theta, \bar{\theta} \in \Theta$ and $\lambda \in [0, 1]$. By this condition and by Jensen's inequality the Bayes risk of a rule d relative to a prior τ depends on τ only through the expectation E_τ corresponding to this distribution. Moreover, a rule d is Bayes relative to τ if and only if it is locally best at the point $\theta = E_\tau$. Thus, in linear estimation the term "Bayes" may be replaced by "locally best" while the condition on τ given by Blyth may be replaced by the corresponding condition on E_τ. To this aim we state the following lemma.

Lemma 1. *Let K be a nonempty closed convex set in $\mathbb{R}^m$ and let τ be a distribution on K with the support K. Then the expectation of τ, if it exists, belongs to $ri(K)$. Conversely, for each $k_0 \in ri(K)$ there exists a distribution τ in R^m with the support K and the expectation k_0.*

Proof. Without loss of generality we may assume that $\dim(K) = m$. Then the interior $\text{int}(K)$ relative to $\mathbb{R}^m$ is nonempty and coincides with $ri(K)$.

Let π be a distribution on $\text{int}(K)$ such that $\text{supp}(\pi) = K$. Then $E_\pi \in \text{int}(K)$. Suppose, on the contrary, that $E_\pi = k \neq k_0$. As $k + (k_0 - k) \in \text{int}(K)$, there exists a scalar $\lambda > 1$ such that $k + \lambda(k_0 - k) \in \text{int}(K)$. Define

$$\tau = \frac{\lambda - 1}{\lambda} \pi + \frac{\pi_0}{\lambda}$$

where π_0 is the distribution concentrated at the point $k + \lambda(k_0 - k)$. Then $E_\tau = k_0$ and $\text{supp}(\tau) = K$.

Now let τ be a distribution with the support K. Then $0 < \tau(\text{int}(K)) \leqq 1$. If $\tau(\text{int}(K)) = 1$ then $E_\tau \in \text{int}(K)$. Otherwise let us define distributions

$$\pi_1(B) = \frac{\tau(B \cap \text{int}(K))}{\tau(\text{int}(K))}$$

and

$$\pi_2(B) = \frac{\tau(B - \text{int}(K))}{1 - \tau(\text{int}(K))}$$

in K. We notice that $E_{\pi_1} \in \text{int}(K)$ and $E_{\pi_2} \in K$. Thus $E_\tau = \tau(\text{int}(K)) E_{\pi_1} + (1 - \tau(\text{int}(K)) E_{\pi_2} \in \text{int}(K)$ which completes the proof.

From this lemma we get □

Corollary 1. *If $\theta \in ri(\Theta)$ then any linear θ-best estimator is admissible in D.*

Olsen, Seely, Birkes (1976) and LaMotte (1982) proved that all θ-best estimators, $\theta \in \Theta$, constitute a complete class. It was recently shown by Stępniak (1987) that all θ-best estimators for $\theta \in ri(\Theta)$ and their limits also constitute a complete class. This class is not greater, and usually smaller, than the class of all locally best estimators. However, not all limits are admissible. A sufficient condition for the admissibility of a limit was given by Stępniak (1986) for the case of unbiased estimation in a linear model with the so called natural parameter space.

Considering statistical games with a convex set Θ we will extend this result to the general linear model and to biased linear estimation.

Denote by Θ^* the set of all distributions τ on Θ such that the Bayes risk

$$r(\tau, d) = \int_\Theta R(\theta, d)\, \mathrm{d}\tau(\theta)$$

of a rule d relative to τ is finite for all $d \in D$. It is easy to prove

Lemma 2. *Let Θ be a convex compact set in $\mathbb{R}^m$ and let $R(\theta, d)$ be a continuous function of θ for each $d \in D$. Then for each inadmissible rule d in the statistical game (Θ, D, R) there exists a rule d_0 such that $r(\tau, d_0) \leqq r(\tau, d)$ for* all $\tau \in \Theta^*$ *with the strict inequality for all distributions τ with the support Θ.*

As a consequence of Lemmas 1 and 2 we obtain

Corollary 2. *Under the additional condition* (2), *for each inadmissible rule d there exists a rule d_0 such that $R(\theta, d_0) \leqq R(\theta, d)$ for all $\theta \in \Theta$, with the strict inequality for $\theta \in ri(\Theta)$.*

Now we are ready to prove the main result of this paper.

For arbitrary θ and $\bar{\theta}$ in Θ such that $\theta \in ri(\Theta)$, and for an arbitrary sequence $\{c_n\}$ of scalars such that $0 < c_n < 1$, $n = 1, 2, \ldots$, define

$$\theta_n = c_n\theta + (1 - c_n)\,\bar{\theta}\,, \quad n = 1, 2, \ldots. \tag{3}$$

Theorem. *Let d_n, $n = 1, 2, \ldots$, be a θ_n-best linear estimator of Φ within D under the risk* (1). *Then $\lim_{n\to\infty} d_n$, if it exists, is admissible.*

Proof. First, notice that $\lambda\theta + (1 - \lambda)\,\bar{\theta} \in ri(\Theta)$ for all $\lambda \in (0, 1)$ by virtue of the relation $S(\lambda\theta + (1 - \lambda)\,\bar{\theta}, \lambda\varepsilon) = \lambda S(\theta, \varepsilon) + (1 - \lambda)\,\bar{\theta}$.

Suppose on the contrary that $d = \lim d_n$ is inadmissible. Then, by Corollary 2, there exists a $d_0 \in D$ such that $R(\lambda\theta + (1 - \lambda)\bar{\theta}, d_0) < R(\lambda\theta + (1 - \lambda\,\bar{\theta}), d)$ for all $\lambda \in (0, 1)$. In particular, d is inadmissible under the restricted parameter set $\Theta_0 = \{\theta, \bar{\theta}\}$.

On the other hand, in this restricted case, the estimator d_n, $n = 1, 2, \ldots$, is admissible because it is Bayes relative to prior τ_n on Θ_0 defined by $\tau_n(\theta) = c_n$ and $\tau_n(\bar{\theta}) = 1 - c_n$. Thus, by Theorem 2 in Stępniak (1986), the limit $d = \lim d_n$ is also admissible. This contradiction completes the proof. □

From this theorem we immediately obtain Theorem 4 in Stępniak (1986).

References

[1] *C. R. Blyth:* On minimax statistical decision procedures and their admissibility. Ann. Math. Statist. 22 (1951), 22—44.

[2] *A. Cohen:* All admissible linear estimates of the mean vector. Ann. Math. Statist. 37 (1966), 458—463.

[3] *L. R. LaMotte:* Admissibility in linear estimation. Ann. Statist. 10 (1982), 245—255.

[4] *A. Olsen, J. Seely, D. Birkes:* Invariant quadratic unbiased estimation for two variance components. Ann. Statist. 4 (1976), 878—890.
[5] *C. R. Rao:* Estimation of parameters in a linear model. Ann. Statist. 4 (1976), 1023—1037.
[6] *R. T. Rockafellar:* Convex Analysis. Princeton Univ. Press, 1970.
[7] *N. Shinozaki:* A study of generalized inverse of matrix and estimation with quadratic loss. Ph. D. Thesis, Keio University, 1975.
[8] *C. Stępniak:* On admissible estimators in a linear model. Biom. J. 7 (1984), 815—816.
[9] *C. Stępniak:* Admissible linear estimators in a linear model with the natural parameter space. Bull. Informatics and Cybernetics. 22 (1986), 71—77.
[10] *C. Stępniak:* A complete class for linear estimation in a general linear model. Ann. Inst. Statist. Math. A. 39 (1987), 563—573.

Souhrn

POSTAČUJÍCÍ PODMÍNKA PŘÍPUSTNOSTI PŘI LINEÁRNÍM ODHADOVÁNÍ

Nedávno bylo dokázáno, že všechny odhady, které jsou lokálně nejlepší v relativním vnitřku parametrického prostoru, tvoří spolu se svými limitami úplnou třídu v lineárních odhadech, jak vychýlených, tak nevychýlených. Ne všechny tyto limity jsou však přípustné. Autor podal postačující podmínku přípustnosti limity pro případ nevychýleného odhadování v lineárním modelu s přirozeným parametrickým prostorem (1986). V tomto článku se uvedený výsledek zobecňuje na obecný lineární model a na vychýlené odhadování.

Резюме

ДОСТАТОЧНОЕ УСЛОВИЕ ДОПУСТИМОСТИ ПРИ ЛИНЕЙНОМ ОЦЕНИВАНИИ

Czesław Stępniak

Недавно выло доказано, что все оценки, которые являются наилучшими в относительной внутренности пространства парамeтров, образуют вместе со своими пределами полный класс в линейных оценках, как смещённых так и несмещённых. Однако не все эти пределы допустимы. Автор статьи нашёл (в 1986 г.) достаточное условие допустимости предела в случае несмещённого оценивания в линейной модели с естественным пространством параметров. В настоящей статье этот результат обобщается на общую линейную модель и смещённое оценивание.

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