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TIME-PERIODIC SOLUTIONS OF A QUASILINEAR BEAM EQUATION VIA ACCELERATED CONVERGENCE METHODS

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Summary. The author investigates time-periodic solutions of the quasilinear beam equation with the help of accelerated convergence methods. Using the Newton iteration scheme, the problem is approximated by a sequence of linear equations solved via the Galerkin method. The derivative loss inherent to this kind of problems is compensated by taking advantage of smoothing operators.

Provided that the right-hand side of the equation is small and smooth, the existence of at least one solution is established.

Keywords: quasilinear beam equation, periodic solutions, accelerated convergence method.

AMS Classification: 35L70, 35B10.

In his Thesis [8], M. Štědrý succeeded in proving the existence of at least one periodic solution to beam equations involving both damping terms and a small nonlinear right-hand side. Being achieved with the help of the abstract Moser theorem, his results cover many important cases except the situation when all “space” derivatives of the unknown function occur in the nonlinearity mentioned above. Consequently, no information is gained concerning e.g. quasilinear equations in spite of their frequent appearance in the so called physically nonlinear elasticity.

The forced transversal vibrations of a damped beam with simply supported ends can be modelled by the equation

\[ u_{tt} + \alpha(u_t) + \sigma(u_{xx})_{xx} = g \]
\[ u = u(x, t), \quad x \in (0, l), \quad t \in \mathbb{R} \]

with the boundary conditions

\[ u(0, t) = u(l, t) = u_{xx}(0, t) = u_{xx}(l, t) = 0, \quad t \in \mathbb{R} \]

and with the periodicity condition

\[ u(x, t + \omega) = u(x, t) \]
\[ x \in (0, l), \quad t \in \mathbb{R}. \]
In this paper, existence of at least one solution to the problem just outlined will be established if, roughly speaking, the function $g$ is small and smooth. For a more precise formulation we refer to Section 2, Theorem 2.1.

The generalized Newton iteration scheme our proof leans on requires to solve but the linearized equations related to (E). To this end, we have to restrict ourselves to functions $\alpha, \sigma$ which are smooth on an open neighbourhood of $0 \in \mathbb{R}$ and satisfy

$$
\alpha(0) = \sigma(0) = 0, \quad \alpha'(0) = d > 0, \quad \sigma'(0) = a > 0.
$$

If quasilinear equations are involved, there are other methods to solve them, see e.g. [4]. Our approach seems to have the benefit of allowing to add another nonlinear term to (E) containing e.g. all derivatives of $u$ up to the order 2. Speaking about a beam equation, we should mention the work [6] of H. Petzeltová for completeness.

To agree upon notation, let the symbol $\mathbb{R}$ stand for the set of all reals, while $\mathbb{Z}$ is the set of all integers, and $\mathbb{Z}^+$ denotes its positive part including 0. Throughout the paper, the symbols $c(v)$ are used to denote strictly positive constants depending on the quantity $v$ only.

1. Function spaces. Following [7], we determine the spaces in which the solution is to be looked for. Seeing that the concrete values of $l, \omega$ do not matter, we may put $\omega = 2\pi, l = \pi$, and $Q = (0, \pi) \times (0, 2\pi)$.

To begin with, the symbols $L_p = L_p(Q)$ are reserved for the spaces of integrable functions with the norm $\| \cdot \|_{L_p}$ defined in the standard way.

Next, we consider anisotropic Sobolev spaces of periodic functions $H^{k,l}, \{H^{k,l}\}$ determined as the completion of all smooth (real-valued) functions satisfying (P), \{(P), (B)\} with respect to the norm

$$
\|v\|_{H^{k,l}} = \max \left\{ \|\partial_{l}^{i}v\|_{L_2} \mid i = 0, 1, \ldots, j \right\}
$$

(see [9]).

Finally, the most important spaces are $U_n, n \in \mathbb{Z}^+$,

$$
U_n = \{v \mid \partial_j^i v \in H^{j,2}, j = 0, 1, \ldots, n\},
$$

with norms

$$
\|v\|_{U_n} = \max \{\|\partial_{j}^{i}v\|_{H^{j,2}} \mid j = 0, 1, \ldots, n\}.
$$

In a similar way, we set

$$
I_n = \{v \mid \partial_j^i v \in H^{j,2}, j = 0,1, \ldots, n\},
$$

$$
\|v\|_{I_n} = \max \{\|\partial_{j}^{i}v\|_{H^{j,2}} \mid j = 0, 1, \ldots, n\},
$$

$$
G_n = \{v \mid \partial_j^i v \in L_2, j = 0, 1, \ldots, n\},
$$

$$
\|v\|_{G_n} = \max \{\|\partial_{j}^{i}v\|_{L_2} \mid j = 0, 1, \ldots, n\}.
$$
Being linearized, the inverse operator related to (E) loses derivatives, which is an inherent difficulty with hyperbolic problems. To overcome it, smoothing operators are needed.

Every \( v \in H^{0,0} \) can be expressed in the form

\[
v = \sum_{i=-\infty}^{\infty} v_i(x) z_i(t), \quad v_i = \int_0^{2\pi} v(x, t) z_i(t) \, dt
\]

where

\[
z_i = \frac{1}{\sqrt{i}} \sin (it), \quad i > 0
\]

\[
1/(2\pi), \quad i = 0
\]

\[
1/(\sqrt{i}) \cos (it), \quad i < 0, \quad i \in \mathbb{Z}.
\]

For a fixed number \( r > 1 \), the sequence of linear operators \( S_n = S_n(r), n \in \mathbb{Z}^+ \) is defined as

\[
(1.1) \quad S_n v = \sum_{i \in r^n} v_i(x) z_i(t).
\]

It is a matter of routine (cf. [7]) to prove

\[
(1.2) \quad \|S_n v - v\|_{X_k} \leq r^{-n}\|v\|_{X_{k+1}},
\]

\[
(1.3) \quad \|S_n v\|_{X_{k+1}} \leq r^{n}\|v\|_{X_k}, \quad l, k \in \mathbb{Z}^+
\]

where \( U, I, \) or \( G \) can be inserted instead of \( X \).

Pursuing Hörmander's work [1], we draw from (1.2), (1.3) the interpolation inequality

\[
(1.4) \quad \|v\|_{X_n} \leq c(k, l, n) \|v\|_{X_k}^{1-\lambda} \|v\|_{X_l}^{\lambda},
\]

\[n = \lambda k + (1 - \lambda) l, \quad n, k, l \in \mathbb{Z}^+, \quad \lambda \in [0, 1].
\]

Combining (1.4) with the well known relations \( ab \leq a^p/p + b^q/q, 1/p + 1/q = 1 \), we conclude (cf. [3])

\[
(1.5) \quad \|v_1\|_{X_n}^{p_1} \|v_2\|_{X_n}^{p_2} \leq c(n_1 + m_1) (\|v_1\|_{X_n}^{p_1} \|v_2\|_{X_n}^{p_2} + \|v_1\|_{X_m} \|v_2\|_{X_m}),
\]

\[n_i, m_i \in \mathbb{Z}^+, \quad n_1 + m_1 = n_2 + m_2 = n_3 + m_3, \quad i = 1, 2, 3,
\]

\[n_1 \leq n_2 \leq n_3
\]

where the symbols \( X, Y \) stand for \( U, I, \) or \( G \).

2. MAIN RESULT

**Theorem 2.1.** Let an integer \( M \geq 10 \) be given.

If (0.1) holds, we are able to find \( \varepsilon = \varepsilon(M) > 0 \) such that the problem (E), (B), (P)
possesses at least one solution $u \in U_{[M/2]}$ (in the sense of generalized derivatives) whenever $g \in G_M$ and

\[(2.1) \quad \|g\|_{G_M} < \varepsilon.\]

If, moreover, $g$ is continuous, then the solution $u$ is a classical one.

Remark. $[M/2]$ denotes the greatest integer less or equal to $M/2$.

The main tool to prove the above theorem is the generalized Newton iteration scheme which will be introduced in Section 3. In connection with differential equations, this method is often called Nash’s or Moser’s method, see e.g. [5].

To solve linear equations, some auxiliary results formulated in Section 4 and proved in Sections 6, 7 are of interest. The proof of Theorem 2.1 will be carried out in Section 5.

3. ITERATION SCHEME

We proceed analogously as in [2], [3].

Let us put

$F(u) = u_{tt} + \alpha(u_t) + \sigma(u_{xx})_{xx},$

$F'(u) y = y_{tt} + \alpha'(u_t) y_t + (\sigma'(u_{xx}) y_{xx})_{xx}.$

The following sequence of linear equations (for $y_n$) is to be solved:

\[(3.1)_n \quad F'(S_n u_n) y_n = h_n, \quad n \in \mathbb{Z}^+\]

where the function $u_0$ is determined by the equation

\[(3.1)_{-1} \quad F'(0) u_0 = g.\]

Our aim is to obtain the solution $u$ as a limit of functions $u_n$, $u_n \to u$ where $u_n$ are given successively as

$u_n = u_0 + \sum_{k=0}^{n-1} y_k.$

The only terms we are to pick out are the functions $h_n$. Following the Taylor expansion formula, we can express

\[(3.2) \quad F(u_{n+1}) = F(u_n) + F'(u_n) y_n + e_{n+1}^1.\]

Setting

\[(3.3) \quad e_{n+1}^2 = (F'(u_n) - F'(S_n u_n)) y_n\]

we get by induction

\[(3.4) \quad F(u_{n+1}) = F(u_0) + \sum_{k=0}^{n} h_k + \sum_{k=1}^{n+1} e_k, \quad e_k = e_k^1 + e_k^2.\]
To accomplish $F(u_n) \to g$, we put
\begin{equation}
S_n(g - F(u_0)) = \sum_{k=0}^{n} h_k + S_n \sum_{k=1}^{n} e_k.
\end{equation}

Accepting the convention $e_0 = F(u_0) - g$, we easily derive
\begin{equation}
h_n = -S_n e_n - (S_n - S_{n-1}) \sum_{k=0}^{n-1} e_k.
\end{equation}

Combining (3.4) with (3.5), we conclude that
\begin{equation}
F(u_{n+1}) = g + e_{n+1} + (id - S_n) \sum_{k=0}^{n} e_k.
\end{equation}

4. AUXILIARY RESULTS

To succeed in solving (3.1), the following assertion is of interest.

**Proposition 4.1.** Given a fixed number $N \in \mathbb{Z}^+$, then $\mathcal{O}(N) > 0$ can be found in such a way that there is a unique solution $y \in U_N$ to the equation
\begin{equation}
F'(u) y = h
\end{equation}
provided that
\begin{equation}
\left\|u\right\|_{U_4} < \delta_1 , \quad u \in \bigcap_{n=0}^{\infty} U_n.
\end{equation}

Moreover, we have the estimates
\begin{equation}
\left\|y\right\|_{U_1} \leq c(l) \left( \left\|h\right\|_{Q_{l+1}} + \left\|u\right\|_{U_{l+4}} \left\|h\right\|_{C_1} \right) l = 0, 1, \ldots, N.
\end{equation}

Remark. According to embedding theorems (see [9]) $U_4 \subset C^2(\overline{\Omega})$. Consequently, $\delta_1$ being chosen close to 0, $F'(u) y$ is defined and the equation is satisfied in the sense of generalized derivatives.

We postpone the proof to Section 6.

The next proposition enables us to estimate the norm of the nonlinear terms appearing in Section 3.

**Proposition 4.2.** Let us consider a smooth function $q : (-\mu, \mu) \to \mathbb{R}$. The embedding $H^{2,2} \subset C^2(\overline{\Omega})$ being taken into account, the composition $q(v(x, t))$ makes sense provided that
\begin{equation}
\left\|v\right\|_{l_0} < \delta_2 ,
\end{equation}

$\delta_2 > 0$ being sufficiently small.

Under these circumstances, we have the estimate
\begin{equation}
\left\|q(v)\right\|_{L_1} \leq c(l, q) \left( \left\|q(v)\right\|_{L_2} + \left\|v\right\|_{L_1} \right)
\end{equation}
for any $v \in I_1, \ l \in \mathbb{Z}^+$. 

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Remark. If \( I_{k} \) were ordinary Sobolev spaces, the theorem above would coincide with that presented by Moser in [5].

For the proof we refer to Section 7.

Taking advantage of Proposition 4.2, we will estimate the quantities \( e_{n} \). To begin with, we assume

\[
\|u_{n}\|_{2} + \|y_{n}\|_{2} < \delta_{3},
\]

\( \delta_{3} > 0 \) small enough, for we want all terms involved to be well defined. Observe that the space \( H^{2,2} \) is a Banach algebra, which seems to be crucial in what follows.

Let us, for instance, treat the most difficult term \( f \) appearing in \( e_{n+1} \),

\[
f = \partial_{x}^{2} \int_{0}^{1} (1 - s) \sigma''(\partial_{x}^{2}(u_{n} + sy_{n})) (\partial_{x}^{2}y_{n})^{2} \, ds.
\]

For \( 0 \leq j \leq l \) we get

\[
\| \partial_{j}f \|_{L_{2}} \leq c(l) \sum_{j_{1} + j_{2} + j_{3} = j} \left( \| \partial_{x}^{2}y_{n} \|_{I_{j_{1}}} \| \partial_{x}^{2}y_{n} \|_{I_{j_{2}}} \int_{0}^{1} \| \sigma''(\partial_{x}^{2}(u_{n} + sy_{n})) \|_{I_{j_{3}}} \, ds \right).
\]

Using (4.4), we conclude

\[
\| f \|_{G_{1}} \leq c(l) \sum_{j_{1} + j_{2} + j_{3} = l} (1 + \| u_{n} \|_{V_{j_{1} + 2}} + \| y_{n} \|_{U_{j_{1} + 2}}) \| y_{n} \|_{V_{j_{2} + 2}} \| y_{n} \|_{U_{j_{3} + 2}}.
\]

Repeating this procedure, we obtain the estimates

\[
\| e_{0} \|_{G_{1}} \leq c(l) \sum_{j_{1} + j_{2} + j_{3} = l} (1 + \| u_{0} \|_{V_{j_{1} + 2}}) \| u_{0} \|_{V_{j_{2} + 2}} \| u_{0} \|_{V_{j_{3} + 2}};
\]

\[
\| e_{n+1} \|_{G_{1}} \leq c(l) \sum_{j_{1} + j_{2} + j_{3} = l} (1 + \| u_{n} \|_{V_{j_{1} + 2}} + \| y_{n} \|_{V_{j_{1} + 2}}) \| y_{n} \|_{V_{j_{2} + 2}} \| y_{n} \|_{V_{j_{3} + 2}},
\]

\[
\| e_{n+1} \|_{G_{1}} \leq c(l) \sum_{j_{1} + j_{2} + j_{3} = l} (1 + \| u_{n} \|_{V_{j_{1} + 2}} + \| y_{n} \|_{V_{j_{1} + 2}}) \| y_{n} \|_{V_{j_{2} + 2}} \| u_{n} - S_{n}u_{n} \|_{U_{j_{3} + 2}}
\]

where \( l \in \mathbb{Z}^{+} \), \( u_{n}, y_{n} \in U_{l+2} \).

5. PROOF OF THEOREM 2.1.

First of all, we put \( N = M - 1 \) where \( M \) appears in Theorem 2.1. Further, a number \( \beta \) is chosen such that \( 2\beta \in (N + 1, N + 2) \). Consequently, \( \beta > [M/2] \geq 5 \). Finally, let the relation (2.1) hold for some \( \epsilon > 0 \).

In view of Proposition 4.1, there exists a (unique) solution \( u_{0} \) of (3.1)\(_{n-1}\) satisfying

\[
\| u_{0} \|_{V_{l}} \leq c(l) \| g \|_{G_{1} + 1}, \quad l \leq N, \quad l \in \mathbb{Z}^{+}.
\]

In order to solve (3.1)\(_{n}\), \( n \in \mathbb{Z}^{+} \), we require both the relation (4.1) for \( S_{n}u_{n} \) and (4.5) for \( m = 0, 1, \ldots, n - 1 \) where we have set \( u_{-1} = u_{0}, y_{-1} = 0 \). To fulfill that, we
have to keep \( u_m, y_{m-1} \) small, more precisely,

\[
(5.2)_n \quad \|u_m\|_{U_4} + \|y_{m-1}\|_{U_4} \leq \delta_4 \quad \text{for all } m = 0, 1, \ldots, n
\]
is required where \( \delta_4 > 0 \) is sufficiently small.

Observe that \( (5.2)_0 \) holds provided the number \( \varepsilon > 0 \) has been chosen small enough. According to Proposition 4.1, \((3.1)_0\) is solvable and the solution \( y_0 \) satisfies

\[
\|y_0\|_{U_1} \leq c(l) (\|h_0\|_{G_{1+1}} + \|S_0 u_0\|_{U_1} \|h_0\|_{G_1}), \quad l \leq N.
\]

For an arbitrarily chosen \( \delta > 0 \), we are able to find \( \varepsilon > 0, \varepsilon \leq \delta \) such that

\[
(5.3)_0 \quad \|y_0\|_{U_1} \leq \delta \quad \text{for all } l = 0, 1, \ldots, N
\]
and \( \varepsilon \) appearing in \((2.1)\). To see that, we combine \((4.6), (5.1)\) with the above estimate.

Following \([3]\), our goal is to choose \( \delta > 0 \) so small that all equations \((3.1)_n\) may be solvable and

\[
(5.3)_n \quad \|y_n\|_{U_1} \leq \delta r^{(l-\beta)n}
\]
may hold for all \( l = 0, 1, \ldots, N \).

At this stage, we proceed by induction. We intend to prove \((5.3)_{n+1}\) having already shown \((5.3)_m\) for all \( m = 0, 1, \ldots, n \).

To this end, we estimate \( u_{m+1} \) as follows:

\[
\|u_{m+1}\|_{U_1} \leq \|u_0\|_{U_1} + \sum_{k=0}^m \|y_k\|_{U_i}
\]
(according to \((5.1), (5.3)\))

\[
\leq \delta c(l) (1 + \sum_{k=0}^m r^{(l-\beta)k}).
\]

Summing up the series on the right-hand side, we deduce

\[
(5.4) \quad \|u_{m+1}\|_{U_1} \leq \delta c(l) (1 + r^{(l-\beta)(m+1)}) \quad \text{for all } l \leq N, \quad m \leq n.
\]
Consequently, \((5.2)_{n+1}\) is satisfied and we are able to solve \((3.1)_{n+1}\).

The unique solution \( y_{n+1} \) fulfils

\[
\|y_{n+1}\|_{U_1} \leq c(l) (\|h_{n+1}\|_{G_{1+1}} + \|S_{n+1} u_{n+1}\|_{U_1} \|h_{n+1}\|_{G_1}).
\]
To accomplish \((5.3)_{n+1}\), we are to treat but the term \( \|h_{n+1}\|_{G_1} \), namely, we are going to prove

\[
(5.5) \quad \|h_{n+1}\|_{G_1} \leq \delta^2 c(l) r^{(l-2\beta+4)(n+1)}, \quad l \in \mathbb{Z}^+.
\]
Indeed, we can pick out \( \delta > 0 \) such that \( \delta c(l) \leq 1 \) for all \( l \leq N \). Since

\[
\|S_{n+1} u_{n+1}\|_{U_1} \leq r^{(n+1)} \delta_4,
\]
we get

\[
\|y_{n+1}\|_{U_1} \leq \delta r^{(l+5-2\beta)(n+1)}.
\]
Since \( \beta > 5 \), \((5.3)_{n+1}\) follows.
We focus our attention upon (5.5). To estimate $e_{m+1}$, we derive

$$\|(id - S_{m+1}) u_{m+1}\| u_i \leq r^{(l-N)(m+1)} \|u_{m+1}\| u_N.$$

In accordance with (5.4), we have

$$\|(id - S_{m+1}) u_{m+1}\| u_i \leq \delta c(l) r^{(l-\beta)(m+1)} \quad \text{for all } l \leq N, \quad m \leq n. \quad (5.6)$$

Combining (4.6)-(4.8) with (5.4)-(5.6) we obtain

$$\|e_{m+1}\| g_i \leq \delta^2 c(l) r^{(l-2\beta+4)(m+1)} \quad \text{for all } l \leq N - 2, \quad m \leq n. \quad (5.7)$$

Reasoning as in (5.4) we get

$$\sum_{k=0}^{m} e_k \| g_i \| \leq \delta^2 c(l) (1 + r^{(l-2\beta+4)(m+1)}), \quad l \leq N - 2. \quad (5.8)$$

Due to $N + 2 > 2\beta$, we obtain similarly as in (5.6)

$$\|(id - S_j) \sum_{k=0}^{n} e_k\| g_i \leq \delta^2 c(l) r^{(l-2\beta+4)(n+1)} \quad \text{for all } l \leq N - 2, \quad j = n, \quad n + 1. \quad (5.9)$$

The hardest term to estimate appearing in $h_{n+1}$ is

$$\|(S_{n+1} - S_n) \sum_{k=0}^{n} e_k\| g_i \leq \delta^2 c(l) r^{(l-2\beta+4)(n+1)} \quad (5.10)$$

where $l$ is supposed to satisfy $l \leq N - 2$.

On the other hand, if $l > N - 2$, we have

$$\|(S_{n+1} - S_n) \sum_{k=0}^{n} e_k\| g_i \leq \|(S_{n+1} \sum_{k=0}^{n} e_k\| g_i + \|S_n \sum_{k=0}^{n} e_k\| g_i.$$

According to (1.3), (5.8), one has (5.10) even for $l > N - 2$. Consequently, (5.5) has been proved.

Solving the equations (3.1)$_n$, $n \in Z^+$, we obtain the sequence $\{u_n\}_{n=0}^{\infty}$. Moreover, according to (5.3), this sequence admits a limit $u \in \hat{U}_{[M/2]}$ and, a fortiori, $u_n \to u$ in $\hat{U}_5$. Using (3.7), we get the estimate

$$\|F(u_{n+1}) - g\| \leq \|e_{n+1}\| g_o + \|(id - S_n) \sum_{k=0}^{n} e_k\| g_o \leq \delta^2 c r^{(-2\beta+4)(n+1)}.$$

Consequently, $F(u_{n+1}) \to g$ in $G_0$.

On the other hand, $U_5 \subset H^{3,7}$. In view of embedding theorems (see [9]), all derivatives of $u_n$ up to the order 3 are continuous and converge to the corresponding derivatives of $u$ in $C(\bar{Q})$. Thus $F(u_{n+1}) \to F(u)$ in $G_0$, and we get

$$F(u) = g. \quad (5.11)$$
If \( g \in C(Q) \), then \( u_{xxxx} \in C(Q) \) due to (5.11). Theorem 2.1 has been proved.

6. Linear equations. In this section, our aim is to prove Proposition 4.1. The problem consists in solving the linear equation

\[
Ly + (b(x, t) y_{xx})_{xx} + w(x, t) y_t = h
\]

where a solution \( y \) is to satisfy the conditions (B), (P). We have denoted \( Ly = y_{tt} + dy_t + ay_{xxxx} \), \( b = \sigma'(u_{xx}) - a \), \( w = \alpha'(u_t) - d \). According to (4.1), we have

\[
b, w \in \bigcap_{n=0}^{\infty} I_n.
\]

To begin with, we claim that there is an equivalent norm on \( \tilde{H}^{4,2} \) given by

\[
\|v\|_{\tilde{H}^{4,2}} = \max \{ \|\partial_x^4 v\|_{L_2}, \|\partial_x^2 v\|_{L_2} \}
\]

(see [9]).

Seeing that the standard Galerkin method is applicable to our situation, we are going to derive \( a \) priori estimates only. For this purpose, we are allowed to assume \( y \) to be smooth.

We put (cf. [6])

\[
\Theta(y) = dy + 2y_t - dy_{tt} - 2y_{ttt} + \frac{da}{2} y_{xxxx}.
\]

For \( t \in \mathbb{Z}^+ \) we set \( \Xi(y, l) = \Theta((-1)^l \partial_t^l y) \).

Let us multiply the equation (L) successively by \( \Xi(y, l) \) and integrate over \( Q \). Using integration by parts, the following relations are obtained:

\[
\int_Q L y \Xi(y, l) = d \|\partial_t^{l+1} y\|^2_{L_2} + d \|\partial_t^{l+2} y\|^2_{L_2} + da \|\partial_x^2 \partial_t^l y\|^2_{L_2} +
\]

\[
+ \frac{da}{2} \|\partial_x^2 \partial_t^{l+1} y\|^2_{L_2} + \frac{da^2}{2} \|\partial_x^4 \partial_t^l y\|^2_{L_2},
\]

(6.3)

\[
|\int_Q h \Xi(y, l)| \leq c(d, a) \|h\|_{\tilde{H}^{1,1}} \|y\|_{L_2}.
\]

Next, we are going to show that

(6.4)

\[
|\int_Q w y_t \Xi(y, l)| \leq c(l) \sum_{k=0}^{l} \|w\|_{L_1} \|y\|_{L_2} \|y\|_{L_2}.
\]

The most difficult term is \( A \),

\[
A = \int_Q w y_t \partial_t^{l+1} y = (-1)^{l+1} \sum_{k=0}^{l+1} \int_Q \binom{l+1}{k} \partial_t^{l+1-k} w \partial_t^k y \partial_t^{l+2-k} y.
\]

With the help of the Hölder inequality, we get

\[
|A| \leq c(l) \sum_{k=0}^{l} \|\partial_t^{l+1-k} w\|_{L^2(Q)} \|y\|_{L_2} \|y\|_{L_2} + \|w\|_{L^2(Q)} \|y\|_{L_2}^2.
\]

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Owing to $H^{2,2} \cap C(\bar{Q})$, (6.4) follows. Finally, we estimate

\[(6.5) \quad |\int_Q (by_{xx})_{xx} \psi(y, l)| \leq c(l) \sum_{k=0}^{l} \|b\|_{I_{l+2-k}} \|y\|_{L_4} \|u_k\|_4 \|y\|_{H^4}.
\]

The hardest terms seem to be

\[B = \int_Q (by_{xx})_{xx} \partial_t^{2l+3}y, \quad D = \int_Q (by_{xx})_{xx} \partial_t^{2l}y_{xxxx} \cdot
\]

To cope with $B$, we use the embedding relation (see [9])

\[(6.6) \quad \|v_{xxx}\|_{L_2} \leq c\|\psi\|_{H^{4,2}}.
\]

We have

\[B = (-1)^{l+2} \sum_{k=0}^{l+2} \int_Q \left( \binom{l+2}{k} \partial_t^{l+2-k}b \partial_t^k y_{xx} \partial_t^l y_{xxx} \right)
\]

and, consequently,

\[|B| \leq c(l) \sum_{k=0}^{l} \|\partial_t^{l+2-k}b\|_{C(\bar{Q})} \|y\|_{L_4} \|u_k\|_4 \|y\|_{H^4} + \|\partial_t b\|_{C(\bar{Q})} \|y\|_{H^4} \|\psi\|_{H^4}.
\]

As to the term $D$, we use the estimates (see [9])

\[(6.7) \quad \|v_{xxx}\|_{L_4} \leq c\|\psi\|_{H^{4,2}},
\]

\[(6.8) \quad \|v_{xxx}\|_{C(\bar{Q})} \leq c\|\psi\|_{H^{4,2}}.
\]

We can decompose

\[D = \int_Q b_{xx}y_{xx} \partial_t^{2l}y_{xxxx} + 2b_{xx}y_{xxx} \partial_t^{2l}y_{xxxx} + by_{xxxx} \partial_t^{2l}y_{xxxx}.
\]

In view of (6.8), we deduce

\[D_1 = \int_Q b_{xx}y_{xx} \partial_t^{2l}y_{xxxx} = (-1)^l \sum_{k=0}^{l} \int_Q \left( \binom{l}{k} \partial_t^{l-k}b_{xx} \partial_t^k y_{xx} \partial_t^l y_{xxxx} \right),
\]

\[|D_1| \leq c(l) \sum_{k=0}^{l} \|\partial_t^{l-k}b_{xx}\|_{L_2} \|\partial_t^k y_{xx}\|_{C(\bar{Q})} \|\partial_t^l y_{xxxx}\|_{L_2} \leq c(l) \sum_{k=0}^{l} \|b\|_{I_{l-k}} \|y\|_{L_4} \|u_k\|_4 \|y\|_{H^4}.
\]

The second term is treated in the following way:

\[D_2 = \int_Q b_{xx}y_{xxx} \partial_t^{2l}y_{xxxx} = (-1)^l \sum_{k=0}^{l} \int_Q \left( \binom{l}{k} \partial_t^{l-k}b_{xx} \partial_t^k y_{xxx} \partial_t^l y_{xxxx} \right),
\]

\[|D_2| \leq c(l) \sum_{k=0}^{l} \|\partial_t^{l-k}b_{xx}\|_{L_4} \|\partial_t^k y_{xxx}\|_{L_4} \|\partial_t^l y_{xxxx}\|_{L_2} \leq c(l) \sum_{k=0}^{l} \|b\|_{I_{l-k}} \|y\|_{L_4} \|u_k\|_4 \|y\|_{H^4}.
\]

Summarizing the results just achieved, we get by (6.1)

\[(6.9) \quad \|y\|_{H^4} \leq c(l) \left( \sum_{k=0}^{l} \|w\|_{I_{l+1-k}} + \|b\|_{I_{l+2-k}} \right) \|y\|_{L_4} \|u_k\|_4 \|y\|_{H^4} + \|h\|_{G_{l+1}} \|y\|_{H^4},
\]

\[l = 0, 1, \ldots.
\]
According to (1.5), we obtain

$$\| y \|_{l^2} \leq c(l) \left( \| w \|_{l_{t+1}} + \| b \|_{l_{t+2}} \right) \| y \|_{x_0} \| y \|_{u_1} + \| h \|_{l_{t+1}} \| y \|_{u_1} + c(l) \left( \| w \|_{l_1} + \| b \|_{l_2} \right) \| y \|_{l^2}, \quad l = 0, 1, \ldots.$$

For $N \geq 0$ there exists $\delta_5 > 0$ such that

$$\| y \|_{u_1} \leq c(l) \left( \| w \|_{l_{t+1}} + \| b \|_{l_{t+2}} \right) \| h \|_{l_1} + \| h \|_{l_{t+1}}$$

holds for all $l = 0, 1, \ldots, N$ whenever

$$\| w \|_{l_1} + \| b \|_{l_2} < \delta_5.$$

To conclude this section, we claim that the a priori estimates (6.10) imply (4.2), while (6.11) corresponds to (4.1). Indeed, we can use Proposition 4.2 together with the relations

$$\| v_{x_0} \|_{l_1} \leq c(l) \left\| v \right\|_{l_{t+2}}, \quad \| v \|_{l_1} \leq c(l) \left\| v \right\|_{l_{t+2}}.$$

7. PROOF OF PROPOSITION 4.2

We are to estimate $\| \partial_{x}^{j} Q(v) \|_{H^{2,2}}$, $j \leq l$.

First of all, suppose $1 \leq j \leq l$. The term $\partial_{x}^{j} Q(v)$ can be expressed as the sum of quantities

$$(\partial^{\alpha_0} Q(v)) (\partial_{x} v)^{a_1} \cdots (\partial_{x}^{l} v)^{a_j}$$

where $\sum_{i=1}^{j} \alpha_i \leq j$ and at least one $\alpha_i \neq 0$.

Since the space $H^{2,2}$ is a Banach algebra, we get

$$\| \partial_{x}^{j} Q(v) \|_{H^{2,2}} \leq c(l, \alpha) \sum_{i=1}^{j} \| \partial^{\alpha_0} Q(v) \|_{H^{2,2}} \| (\partial_{x} v)^{a_1} \|_{l_{t+2,2}} \cdots \| (\partial_{x}^{l} v)^{a_j} \|_{l_{t+2,2}} \leq c(l) \sum_{i=1}^{j} \| \partial^{\alpha_0} Q(v) \|_{H^{2,2}} \| v \|_{l_{t+2}} \leq c(l, \delta_2) \sum_{i=1}^{j} \| \partial^{\alpha_0} Q(v) \|_{H^{2,2}} \| v \|_{l_{t+2}}.$$

If $j = 0$, we estimate

$$\| Q(v) \|_{H^{2,2}} \leq \| Q(v) \|_{L_2} + \| \partial_{x} Q(v) \|_{L_2} + \| \partial_{x}^{2} Q(v) \|_{L_2} + \| \partial_{x}^{2} Q(v) \|_{L_2}.$$

Let us treat $\| \partial_{x}^{2} Q(v) \|_{L_2}$, for example:

$$\| \partial_{x}^{2} Q(v) \|_{L_2} \leq \| Q^{0}(v) \|_{L_2} v_{x}^{2} \|_{L_2} + \| Q(v) \|_{C(\overline{Q})} v_{x} \|_{L_2} \leq \| Q^{0}(v) \|_{C(\overline{Q})} \| v \|_{L_2} + \| Q^{0}(v) \|_{C(\overline{Q})} \| v \|_{H^{2,2}}.$$

To complete the proof, we have but to realize that $H^{2,2} \subseteq C(\overline{Q})$. 372
References


Souhrn

ČASOVÉ PERIODICKÁ ŘEŠENÍ KVASILINEÁRNÍ ROVNICE TYČE — POUŽITÍ METOD URYCHLENÉ KONVERGENCE

Eduard Feireisl


Je dokázána existence alespoň jednoho (klasického) řešení úlohy za předpokladu, že pravá strana rovnice je dostatečně malá a hladká.

Резюме

ИЗУЧЕНИЕ ПЕРИОДИЧЕСКИХ ВО ВРЕМЕНИ РЕШЕНИЙ КВАЗИЛИНЕЙНОГО УРАВНЕНИЯ СТЕРЖНЯ ПРИ ПОМОЩИ МЕТОДОВ УСКОРЕННОЙ СХОДИМОСТИ

Eduard Feireisl

Автор изучает периодические во времени решения квазилинейного уравнения стержня, пользуясь методом ускоренной сходимости. На основе итерационного метода Ньютона задача аппроксимируется последовательностью линейных уравнений, которые решаются методом Галеркина. Типичная при решении гиперболических задач трудность — потеря производных — решается при помощи регулирующих операторов. Показано, что задача обладает по крайней мере одним решением, если правая часть уравнения является малой гладкой функцией.

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