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Petr Přikryl; Jiří Taufer; Emil Vitásek

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TRANSFER OF CONDITIONS FOR SINGULAR BOUNDARY VALUE PROBLEMS

PETR PŘIKRYL, JIŘÍ TAUFER, EMIL VITÁSEK

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Summary. Numerical solution of linear boundary value problems for ordinary differential equations by the method of transfer of conditions consists in replacing the problem under consideration by a sequence of initial value problems. The method of transfer for systems of equations of the first order with Lebesgue integrable coefficients was studied by one of the authors before. The purpose of this paper is to extend the idea of the transfer of conditions to singular boundary value problems for a linear second-order differential equation.

Key words: numerical analysis, transfer of conditions, invariant imbedding, singular boundary value problems.

AMS Classification: 65L10, 34B05.

1. INTRODUCTION

The boundary value problems for ordinary differential equations of the type

$$-(p(t)\,y')' + q(t)\,y = f(t) \tag{1.1}$$

where q or $1/p$ has a singularity at one of the endpoints of the interval of integration are subject of great interest for physicists and mathematicians. Such a problem arises, for example, when using the Fourier method for the Poisson equation in polar coordinates. Another example of a problem of this type is the eigenvalue problem

$$-y''(t) + \left[\frac{k}{t^2} + w(t) - \lambda\right] y(t) = 0\,, \tag{1.2}$$

$$y(0) = 0\,,$$

$$\alpha\,y(b) + \beta\,y'(b) = 0\,,\quad \alpha^2 + \beta^2 \neq 0$$

for the radial Schrödinger equation.

A very efficient method for solving boundary value problems for ordinary differential equations is the invariant imbedding [1], [2], [3]. In the linear case, one possible derivation of the method is based on the idea of transferring the boundary conditions. The idea of transfer of conditions consists, roughly speaking, in the observation that the function y which satisfies the differential equation (1.1) and the condition of the type

$$\alpha\, y(t_0) + \beta\, p(t_0)\, y'(t_0) = \gamma \tag{1.3}$$

should satisfy a linear differential equation of the first order, that is, a condition of the type (1.3), at any point of the interval of integration. This approach was presented for very general multipoint boundary value problems by one of the authors in [2], [3], and also various algorithms resulting from it have been studied there. However, the results of [2], [3] apply only to such equations of type (1.1) that the functions $1/p$ and q are Lebesgue integrable in the interval of integration. The purpose of this paper is to extend the method of transfer to equations for which the above assumption is violated.

For the reader's convenience, the next part of the paper surveys the method (and the notation used) in the regular case. The only difficulty of the adaptation of the method to the singular case consists in transferring the boundary condition prescribed at the point where $1/p$ or q has a singularity. This transfer is studied in the third section, namely in Theorems 3.1 and 3.2.

2. THE METHOD OF TRANSFER OF CONDITIONS

This section describes the transfer of boundary conditions in the regular case and the resulting methods for the numerical solution of boundary value problems as developed by Taufer in [2]. All the proofs are omitted here as they can be found in [2].

First we give the definition of the boundary value problem we will be concerned with in the present section.

Problem 1. Let $[a, b]$ be a closed finite interval and let p, q, and f be functions from $[a, b]$ to $\mathbb{R}$ such that $1/p$, q, and f are Lebesgue integrable on $[a, b]$. Find a function $y: [a, b] \to \mathbb{R}$ such that

(i) y and py' are absolutely continuous functions on $[a, b]$;

(ii) y satisfies the equation

$$-(p(t)\, y'(t))' + q(t)\, y(t) = f(t) \tag{2.1}$$

a.e. (almost everywhere) on $[a, b]$;

(iii) y satisfies the boundary conditions

$$\alpha_1\, y(a) - \beta_1\, p(a)\, y'(a) = \gamma_1\,, \tag{2.2}$$

$$\alpha_2\, y(b) + \beta_2\, p(b)\, y'(b) = \gamma_2\,, \tag{2.3}$$

where $\alpha_i, \beta_i, \gamma_i \in \mathbb{R}$ are such that $\alpha_i^2 + \beta_i^2 \neq 0$, $i = 1, 2$.

The fundamental idea of the method of transfer consists in constructing functions α, β, and γ (all from $[a, b]$ to $\mathbb{R}$) such that every solution of (2.1) satisfying the boundary condition (2.2) satisfies the relation

$$\alpha(t)\, y(t) - \beta(t)\, p(t)\, y'(t) = \gamma(t) \tag{2.4}$$

at every point $t \in [a, b]$. We then say that the condition (2.2) has been ***transferred*** to the point t and call (2.4) a *transferred condition.*

Similarly, we try to find functions $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$: $[a, b] \to \mathbb{R}$ such that every solution of (2.1) satisfying the boundary condition (2.3) satisfies

$$\hat{\alpha}(t)\, y(t) + \hat{\beta}(t)\, p(t)\, y'(t) = \hat{\gamma}(t) \tag{2.5}$$

for any $t \in [a, b]$.

For a fixed $t \in [a, b]$ the transferred conditions (2.4) and (2.5) represent a system of two linear algebraic equations (with unknowns $y(t)$ and $p(t)\, y'(t)$) that the solution of Problem 1 – if there is any – should satisfy. Hence, if the transfer is practicable (i.e. if we are able to construct the functions α, β, γ and $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$) and if Problem 1 has a solution, then the value $y(t_0)$ of the solution at a point $t_0 \in [a, b]$ may be found by solving the system of equations (2.4), (2.5) with $t = t_0$.

It has been shown [1], [2] that appropriate functions α, β, γ and $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ in (2.4), (2.5) may be found as solutions of initial value problems for certain nonlinear ordinary differential equations of the first order. In such a way we arrive at a method which reduces the boundary value problem under consideration to solving several initial value problems and certain linear algebraic systems. Since the solution of initial value problems seems today to be a relatively easy task due to the existence of very good and well-programmed methods, the above procedure for solving boundary value problems may be quite useful.

Observing that multiplication of equation (2.4) or (2.5) by an arbitrary function different from zero leads to the same condition, we can see that there exist infinitely many ways of transferring boundary conditions. Hence, the general idea of the transfer of boundary conditions may lead to algorithms with substantially different properties. For example, the well-known simple shooting method is also tractable in terms of a transfer of boundary conditions. However, this algorithm is known to be numerically unstable in general. The questions of stability were studied in detail by Taufer in [2] and an example of a numerically stable transfer of the left boundary condition is given in the next theorem.

Theorem 2.1. *Consider Problem* 1 *and suppose, in addition, that* $p(t) > 0$ *and* $q(t) \geqq 0$ *a.e. on* $[a, b]$ *and that* $\alpha_i \geqq 0$, $\beta_i \geqq 0$ *for* $i = 1, 2$. *Then:*

(i) *If* $\alpha_1 > 0$ *then there are absolutely continuous functions* η, ζ: $[a, b] \to \mathbb{R}$ *which are uniquely determined by differential equations*

$$\eta'(t) = q(t)\, \eta^2(t) - \frac{1}{p(t)} \quad \text{a.e. on} \quad [a, b], \tag{2.6}$$

$$\zeta'(t) = q(t)\,\eta(t)\,\zeta(t) - \eta(t) f(t) \quad \text{a.e. on} \quad [a, b] \tag{2.7}$$

with initial conditions

$$\eta(a) = -\beta_1/\alpha_1\,, \tag{2.8}$$

$$\zeta(a) = \gamma_1/\alpha_1\,. \tag{2.9}$$

These functions possess the property that every solution of (2.1) *that satisfies the left boundary condition* (2.2) *satisfies also the transferred condition*

$$y(t) + \eta(t)\,p(t)\,y'(t) = \zeta(t) \tag{2.10}$$

for any $t \in [a, b]$.

(ii) *If* $\beta_1 > 0$ *then there are absolutely continuous functions* $\eta, \zeta: [a, b] \to \mathbb{R}$ *which are uniquely determined by differential equations*

$$\eta'(t) = \frac{1}{p(t)}\,\eta^2(t) - q(t) \quad \text{a.e. on} \quad [a, b]\,, \tag{2.11}$$

$$\zeta'(t) = \frac{1}{p(t)}\,\eta(t)\,\zeta(t) - f(t) \quad \text{a.e. on} \quad [a, b] \tag{2.12}$$

with initial conditions

$$\eta(a) = -\alpha_1/\beta_1\,, \tag{2.13}$$

$$\zeta(a) = -\gamma_1/\beta_1\,. \tag{2.14}$$

These functions possess the property that every solution of (2.1) *that satisfies the left boundary condition* (2.2) *satisfies also the transferred condition*

$$\eta(t)\,y(t) + p(t)\,y'(t) = \zeta(t) \tag{2.15}$$

for any $t \in [a, b]$.

Remark 2.1. Completely analogous statements are true regarding the transfer of the right boundary condition (2.3), provided $\alpha_2 > 0$ or $\beta_2 > 0$. Thus, Theorem 2.1 yields a procedure for constructing functions α, β, γ and $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ in the transferred conditions (2.4), (2.5). In addition, if these functions are constructed according to Theorem 2.1 it may be shown [2] that the following is true:

(i) The system (2.4), (2.5) for $y(t)$ and $p(t)\,y'(t)$ has a solution for any $t \in [a, b]$ if and only if Problem 1 has a solution.

(ii) The system (2.4), (2.5) has a unique solution for any $t \in [a, b]$ if and only if Problem 1 has a unique solution.

Remark 2.2. If the hypotheses of Theorem 2.1 are satisfied and, in addition, $p(t) \geqq p_0 > 0$ for all $t \in [a, b]$ and either $\alpha_1 + \alpha_2 > 0$ or $q(t) \not\equiv 0$, then Problem 1 has a unique solution.

Remark 2.3. The generalization of the transfer of conditions described in Theorem 2.1 to systems of linear differential equations and to multipoint boundary value problems with internal conditions of various types was studied by one of the authors in [2].

Theorem 2.1 assumes Lebesgue integrability of $1/p$, q, and f on the whole interval $[a, b]$. The rest of the paper is devoted to the study of boundary value problems with q or $1/p$ having a singularity at an endpoint of $[a, b]$ (see, for example, (1.2)). Our aim is to obtain a modification of Theorem 2.1 in which the transfer of the boundary condition from the singular point will be described.

3. TRANSFER OF THE BOUNDARY CONDITION FROM A SINGULAR POINT

We first describe the singular boundary value problems we will be concerned with in what follows. The problems are similar to Problem 1 but the assumptions on q or $1/p$ are weakened.

Problem 2. Let $[a, b]$ be a closed bounded interval and let p, q, and f be functions from $[a, b]$ to $\mathbb{R}$ such that $1/p \in \mathscr{L}([a, b])$, $f \in \mathscr{L}([a, b])$. Suppose further that $q \in \mathscr{L}([a + \varepsilon, b])$ for any $0 < \varepsilon < b - a$ and $q \notin \mathscr{L}([a, b])$. Find a function $y: [a, b] \to \mathbb{R}$ such that

(i) y and py' are absolutely continuous on $[a, b]$;

(ii) y satisfies the equation (2.1) a.e. on $[a, b]$;

(iii) y satisfies the boundary condition

$$\alpha\, y(b) + \beta\, p(b)\, y'(b) = \gamma\,, \quad \alpha^2 + \beta^2 \neq 0\,. \tag{3.1}$$

Problem 3. Let $[a, b]$ be a closed bounded interval and let p, q, and f be functions from $[a, b]$ to $\mathbb{R}$ such that $q \in \mathscr{L}([a, b])$, $f \in \mathscr{L}([a, b])$. Suppose further that $1/p \in \mathscr{L}([a + \varepsilon, b])$ for any $0 < \varepsilon < b - a$ and $1/p \notin \mathscr{L}([a, b])$. Find a function $y: [a, b] \to \mathbb{R}$ such that

(i) y and py' are absolutely continuous on $[a, b]$;

(ii) y satisfies the equation (2.1) a.e. on $[a, b]$;

(iii) y satisfies the boundary condition (3.1).

At first sight it seems that Problems 2 and 3 are not boundary value problems since no conditions are prescribed at the left boundary point $t = a$, The following two lemmas, however, show that this is not true and that some special conditions of the type (2.2) are automatically satisfied at the point $t = a$. Before formulating them, it will be useful to say what we mean under a solution of the differential equation (2.1) in general. In what follows, a solution of (2.1) on $[a, b]$ is any function y such that y and py' are absolutely continuous on $[a, b]$ and y satisfies the equation a.e. on $[a, b]$.

Lemma 3.1. *Let y be a solution of Problem 2. Then*

$$y(a) = 0\,. \tag{3.2}$$

Proof is by contradiction. First, suppose that $y(a) > 0$. This and the continuity of y imply the existence of $\delta > 0$ and $y_0 > 0$ such that

$$y(t) \geqq y_0 \quad \text{for all} \quad t \in [a, a + \delta]\,. \tag{3.3}$$

From (2.1) we easily obtain that $qy \in \mathscr{L}([a, b])$ for any solution of Problem 2. Further, (3.3) implies that $1/y$ is bounded on $[a, a + \delta]$ and since y is continuous, $1/y$ is also integrable on this interval. Thus, $qy \,.\, (1/y) = q \in \mathscr{L}([a, a + \delta])$, which contradicts the assumption that $q \notin \mathscr{L}([a, b])$.

Assuming $y(a) < 0$ we can proceed analogously. Hence $y(a) = 0$.

Lemma 3.2. *Let y be a solution of Problem 3. Then*

$$p(a)\, y'(a) = 0\,. \tag{3.4}$$

Proof of this lemma is completely analogous to that of Lemma 3.1.

Since the functions q and $1/p$ are Lebesgue integrable on $[a + \varepsilon, b]$, $0 < \varepsilon < b - a$, the transfer of the right boundary condition from b may be performed in the usual way based on the analogue of Theorem 2.1 for the point $t = b$. Our task here is to describe the transfer of the left boundary conditions (3.2) and (3.4). This may be performed with the help of the next two theorems.

Theorem 3.1. *Consider Problem 2 and suppose that $p(t) > 0$ and $q(t) \geqq 0$ a.e. on $[a, b]$. Then there is a unique pair of absolutely continuous functions η, ζ: $[a, b] \to \mathbb{R}$ which satisfy the differential equations*

$$\eta'(t) = q(t)\, \eta^2(t) - \frac{1}{p(t)} \quad \text{a.e. on} \quad [a, b]\,, \tag{3.5}$$

$$\zeta'(t) = q(t)\, \eta(t)\, \zeta(t) - \eta(t) f(t) \quad \text{a.e. on} \quad [a, b] \tag{3.6}$$

and the initial conditions

$$\eta(a) = 0\,, \tag{3.7}$$

$$\eta(t) \leqq 0 \quad \text{in a right neighbourhood} \quad U_+(a) \quad \text{of} \quad a\,, \tag{3.8}$$

and

$$\zeta(a) = 0\,. \tag{3.9}$$

The functions η and ζ possess the property that any solution y of the equation (2.1) on $[a, b]$ satisfies the condition

$$y(t) + \eta(t)\, p(t)\, y'(t) = \zeta(t) \tag{3.10}$$

for any $t \in [a, b]$.

Theorem 3.2. *Consider Problem* 3 *and suppose that* $p(t) > 0$ *and* $q(t) \geqq 0$ *a.e. on* $[a, b]$. *Then there is a unique pair of absolutely continuous functions* η, ζ: $[a, b] \to \mathbb{R}$ *which satisfy the differential equations*

$$\eta'(t) = \frac{1}{p(t)}\eta^2(t) - q(t) \quad \text{a.e. on} \quad [a, b], \tag{3.11}$$

$$\zeta'(t) = \frac{1}{p(t)}\eta(t)\,\zeta(t) - f(t) \quad \text{a.e. on} \quad [a, b] \tag{3.12}$$

and the initial conditions (3.7)–(3.9). *The functions* η, ζ *possess the property that any solution* y *of the equation* (2.1) *on* $[a, b]$ *satisfies the condition*

$$\eta(t)\,y(t) + p(t)\,y'(t) = \zeta(t) \tag{3.13}$$

for any $t \in [a, b]$.

Thus, the conditions (3.10) and (3.13) may be viewed as the transferred conditions (3.2) and (3.4), respectively.

To prove Theorems 3.1 and 3.2 we need several simple lemmas.

Lemma 3.3. *Let* $P, Q \in \mathscr{L}([a, b])$ *be nonnegative a.e. on* $[a, b]$ *and let* $\eta_0 \leqq 0$. *Then there exists a unique function* η *which is absolutely continuous on* $[a, b]$ *and satisfies the differential equation*

$$\eta'(t) = Q(t)\,\eta^2(t) - P(t) \quad \text{a.e. on} \quad [a, b] \tag{3.14}$$

along with the initial condition

$$\eta(a) = \eta_0\,. \tag{3.15}$$

Moreover, for the function η *and all* $t \in [a, b]$ *we have the bound*

$$\eta_0 - \int_a^t P(s)\,ds \leqq \eta(t) \leqq 0\,. \tag{3.16}$$

Proof. The existence and uniqueness of η has been proved in [2], Lemma 1.1. The proof of $\eta(t) \leqq 0$ may be found there as well. The remaining part of (3.16) follows from (3.14) by integration. For the reader's convenience we briefly sketch the existence proof as the uniqueness part follows from the well-known general theorems on ordinary differential equations.

The solution of (3.14) may be found in the form $-u/v$ where u and v are solutions of the linear system

$$\begin{aligned} u'(t) &= P(t)\,v(t)\,, \\ v'(t) &= Q(t)\,u(t) \end{aligned} \tag{3.17}$$

with the initial conditions

$$u(a) = -\eta_0\,, \quad v(a) = 1\,. \tag{3.18}$$

For the system (3.17), (3.18) one can easily show that $u(t)\,v(t) \geqq 0$ for any $t \in [a, b]$ and, moreover, that $v(t) > 0$ in $[a, b]$. The last inequality shows that the ratio $-u/v$ is well defined in the whole interval $[a, b]$.

Lemma 3.4. *Let $P, Q \in \mathscr{L}([a, b])$ be nonnegative a.e. on $[a, b]$ and suppose that η_1, η_2 are two nonpositive solutions of (3.14) on $[a, b]$. Let $\eta_1(t_0) \leqq \eta_2(t_0)$ for some $t_0 \in [a, b]$. Then*

$$\eta_1(t) \leqq \eta_2(t) \leqq 0 \quad \textit{for all} \quad t \in [a, b] \tag{3.19}$$

and for any pair $t', t'' \in [a, b]$ such that $t' \leqq t''$ we have

$$\eta_2(t') - \eta_1(t') \geqq \eta_2(t'') - \eta_1(t'') . \tag{3.20}$$

Proof. The inequality (3.19) may be proved by contradiction using the continuity of η_1, η_2 and the uniqueness statement of Lemma 3.3. To prove (3.20) we first use (3.14) to write

$$(\eta_2(t) - \eta_1(t))' = Q(t)\,(\eta_2(t) + \eta_1(t))\,(\eta_2(t) - \eta_1(t)) . \tag{3.21}$$

Since Q is nonnegative a.e. on $[a, b]$ and, furthermore, $\eta_2(t) + \eta_1(t) \leqq 0$ for all $t \in [a, b]$ we find from (3.19) and (3.21) that $(\eta_2(t) - \eta_1(t))' \leqq 0$ a.e. on $[a, b]$. This inequality and the absolute continuity of η_1, η_2 yield (3.20).

Lemma 3.5. *Let $P, Q \in \mathscr{L}([a, b])$ be nonnegative a.e. on $[a, b]$ and let $\eta_0 \geqq 0$. Then there exists a unique function η which is absolutely continuous on $[a, b]$ and satisfies the differential equation (3.14) a.e. on $[a, b]$ along with the initial condition*

$$\eta(b) = \eta_0 . \tag{3.22}$$

Moreover, for the function η and all $t \in [a, b]$ we have the bound

$$0 \leqq \eta(t) \leqq \eta_0 + \int_t^b P(s)\,ds . \tag{3.23}$$

Proof follows from Lemma 3.3 by substitution.

Lemma 3.6. *Let $P, Q \in \mathscr{L}([a, b])$ be nonnegative a.e. on $[a, b]$ and suppose that η_1, η_2 are two nonnegative solutions of (3.14) on $[a, b]$. Let $\eta_2(t_0) \geqq \eta_1(t_0)$ for some $t_0 \in [a, b]$. Then*

$$\eta_2(t) \geqq \eta_1(t) \geqq 0 \quad \textit{for all} \quad t \in [a, b] \tag{3.24}$$

and for any pair $t', t'' \in [a, b]$ such that $t' \leqq t''$ we have

$$\eta_2(t') - \eta_1(t') \leqq \eta_2(t'') - \eta_1(t'') . \tag{3.25}$$

Proof is quite analogous to that of Lemma 3.4.

Lemma 3.7. *Let $P \in \mathscr{L}([a, b])$. Further, let $Q \in \mathscr{L}([a + \varepsilon, b])$ for any $0 < \varepsilon < b - a$ and suppose $Q \notin \mathscr{L}([a, b])$. Finally let P and Q be nonnegative a.e. on $[a, b]$. Then there exists one and only one function $\eta: [a, b] \to \mathbb{R}$ which is absolutely continuous on $[a, b]$, satisfies the differential equation (3.14) a.e. on $[a, b]$ and is nonpositive in some right neighbourhood $U_+(a)$ of the point a.*

Proof. Let $\tau \in (a, b]$ and $\eta_\tau \geqq 0$. Let c be an arbitrary number satisfying $a < c < \tau$. Then, by Lemma 3.5 there exists a solution $\eta(\cdot; \tau, \eta_\tau)$ of (3.14) in $[c, \tau]$ satisfying the initial condition

$$\eta(\tau; \tau, \eta_\tau) = \eta_\tau \,. \tag{3.26}$$

Since $c > a$ was arbitrary, the function $\eta(\cdot; \tau, \eta_\tau)$ is well defined in $(a, \tau]$. Let us now investigate its limit for $t \to a+$.

From (3.14) we have

$$\begin{aligned} \eta(t; \tau, \eta_\tau) &= \eta_\tau - \textstyle\int_t^\tau [Q(s)\, \eta^2(s; \tau, \eta_\tau) - P(s)]\, \mathrm{d}s = \\ &= \eta_\tau - \textstyle\int_t^\tau Q(s)\, \eta^2(s; \tau, \eta_\tau)\, \mathrm{d}s + \int_t^\tau P(s)\, \mathrm{d}s \end{aligned} \tag{3.27}$$

for any $t \in (a, \tau]$ as the right-hand side of (3.27) has sense. Since $Q(\cdot)\, \eta^2(\cdot; \tau, \eta_\tau)$ is, obviously, measurable and nonnegative on $(a, \tau]$ and $P \in \mathscr{L}([a, b])$ we can pass in (3.27) to the limit as $t \to a+$ obtaining

$$\lim_{t \to a+} \eta(t; \tau, \eta_\tau) = \eta_\tau - \textstyle\int_a^\tau Q(s)\, \eta^2(s; \tau, \eta_\tau)\, \mathrm{d}s + \int_a^\tau P(s)\, \mathrm{d}s = A \,. \tag{3.28}$$

Lemma 3.5 yields that $\eta(t; \tau, \eta_\tau)$ is nonnegative and therefore A is also nonnegative. This fact and the summability of P imply that A is finite and $Q(\cdot)\, \eta^2(\cdot; \tau, \eta_\tau) \in \mathscr{L}([a, \tau])$.

Thus, we have proved that there exists a finite limit of $\eta(t; \tau, \eta_\tau)$ as $t \to a+$. We define the value of the function η at the point a by this limit. In this way we obviously get a function which is continuous on $[a, \tau]$. Further, we necessarily have

$$\eta(a; \tau, \eta_\tau) = 0 \tag{3.29}$$

since in the opposite case the function Q would be integrable on $[a, a + \delta]$ for some $\delta > 0$. (Indeed, if (3.29) were not satisfied then $1/\eta^2$ would be measurable and bounded in some neighbourhood of a and it would be sufficient to take into account that $Q\eta^2$ is integrable.)

Let us now consider the function $\eta(\cdot; \tau)$ which is defined as $\eta(t; \tau, 0)$ on $[a, \tau]$ (hence nonnegative there) and as the solution of (3.14) with zero initial condition at the point τ on $[\tau, b]$ (such a function is defined uniquely and is nonpositive on $[\tau, b]$ according to Lemma 3.3). Therefore, the function $\eta(\cdot; \tau)$ represents a solution of (3.14) on any interval $[a + \varepsilon, b]$ with $0 < \varepsilon < b - a$. It is, moreover, continuous on $[a, b]$ and satisfies the conditions

$$\eta(a; \tau) = \eta(\tau; \tau) = 0 \,. \tag{3.30}$$

Further, for η we have the estimates

$$0 \leqq \eta(t; \tau) \leqq \int_t^\tau P(s)\,ds \quad \text{for} \quad t \in [a, \tau]\,, \tag{3.31}$$

and

$$-\int_\tau^t P(s)\,ds \leqq \eta(t; \tau) \leqq 0 \quad \text{for} \quad t \in [\tau, b]\,, \tag{3.32}$$

which follow from (3.27) and Lemma 3.3, respectively.

Choose now $a < \tau_1 \leqq \tau_2 \leqq b$ and investigate the difference $\eta(t; \tau_2) - \eta(t; \tau_1)$ for $t \in (a, b]$. For $t \in (a, \tau_1]$ both functions $\eta(\cdot; \tau_1)$ and $\eta(\cdot; \tau_2)$ are nonnegative according to (3.31). From this and (3.32) we see that the assumptions of Lemma 3.6 are satisfied on any interval $[a + \varepsilon, \tau_1]$. Thus, supposing $t \in [a + \varepsilon, \tau_1]$ we have

$$0 \leqq \eta(t; \tau_2) - \eta(t; \tau_1) \leqq \eta(\tau_1; \tau_2) - \eta(\tau_1; \tau_1) = \eta(\tau_1; \tau_2) \leqq \int_{\tau_1}^{\tau_2} P(s)\,ds\,, \tag{3.33}$$

where the last inequality follows from (3.31). From the continuity of $\eta(\cdot; \tau_2) - \eta(\cdot; \tau_1)$ it follows that the estimate

$$0 \leqq \eta(t; \tau_2) - \eta(t; \tau_1) \leqq \int_{\tau_1}^{\tau_2} P(s)\,ds \tag{3.34}$$

holds for any $t \in [a, \tau_1]$. Using inequalities (3.31), (3.32) and Lemma 3.4 in a similar way we can conclude that (3.34) holds on the whole interval $[a, b]$.

Finally, put

$$\eta_n(t) = \eta\left(t; a + \frac{b - a}{n}\right), \quad n = 1, 2, \ldots\,. \tag{3.35}$$

From (3.34) it follows immediately that the sequence η_n forms a fundamental sequence in the space C of functions continuous on $[a, b]$. Thus, $\{\eta_n\}$ is uniformly convergent. Denote its limit by $\hat{\eta}$. The sequence $\eta_n(t)$ is nonincreasing for any t as follows from Lemmas 3.4 and 3.6. Hence, we have

$$\eta_1(t) \geqq \eta_2(t) \geqq \ldots \geqq \eta_n(t) \geqq \ldots \geqq \hat{\eta}(t)\,, \quad t \in [a, b]\,. \tag{3.36}$$

Since $\eta_n(t) \leqq 0$ for $t \geqq a + (b - a)/n$, we have

$$\hat{\eta}(t) \leqq 0 \quad \text{for} \quad t \in [a, b]\,. \tag{3.37}$$

Every function η_n satisfies (3.14) on any $[a + \varepsilon, b]$ and, moreover, it satisfies the condition $\eta_n(a) = 0$ (cf. (3.30)). Thus

$$\eta_n(t) = \int_a^t Q(s)\,\eta_n^2(s)\,ds - \int_a^t P(s)\,ds \tag{3.38}$$

for any $t \in [a, b]$. Using (3.38) with $t = b$ and taking into account that $\eta_n(b) \leqq 0$ we obtain

$$\int_a^b Q(s)\,\eta_n^2(s)\,ds \leqq \int_a^b P(s)\,ds\,. \tag{3.39}$$

The Fatou lemma implies that

$$\int_a^b Q(s)\,\hat{\eta}^2(s)\,ds \leqq \liminf_{n \to \infty} \int_a^b Q(s)\,\eta_n^2(s)\,ds\,, \tag{3.40}$$

which together with (3.39) proves that

$$Q\hat{\eta}^2 \in \mathscr{L}([a, b]) . \tag{3.41}$$

We have $0 \leqq \eta_n(t) \leqq \eta_1(t)$ for $t \in [a, a + (b - a)/n]$ and $0 \geqq \eta_n(t) \geqq \hat{\eta}(t)$ for $t \in [a + (b - a)/n, b]$, which yields

$$Q(t)\, \eta_n^2(t) \leqq Q(t)\, \eta_1^2(t) + Q(t)\, \hat{\eta}^2(t) \quad \text{a.e. on} \quad [a, b] . \tag{3.42}$$

Since $Q\eta_1^2 + Q\hat{\eta}^2$ is integrable we can perform the passage to the limit as $n \to \infty$ under the integration sign in (3.38) obtaining an identity which proves that $\hat{\eta}$ is a solution of (3.14) on $[a, b]$.

It remains to prove the uniqueness. We will prove it by contradiction. Thus, let η_1 and η_2 be two different solutions of (3.14) which are nonpositive in $[a, a + \delta]$ for some $\delta > 0$. For any $t > a$ we have, say,

$$\eta_1(t) < \eta_2(t) \tag{3.43}$$

as a consequence of the uniqueness theorem for the nonsingular case. Hence, we have

$$(\eta_2(t) - \eta_1(t))' = Q(t)\,(\eta_2(t) + \eta_1(t))\,(\eta_2(t) - \eta_1(t)) \leqq 0 \tag{3.44}$$

a.e. on $[a, b]$. But (3.44) implies $\eta_2(t) \leqq \eta_1(t)$ on $[a, a + \delta]$ which contradicts (3.43). Lemma is proved.

Remark 3.1. It follows from the proof of Lemma 3.7 that in the singular case the differential equation (3.14) has infinitely many solutions satisfying the zero initial condition but only one of them is nonpositive in the whole interval $[a, b]$.

Lemma 3.8. *Let $P \in \mathscr{L}([a, b])$. Further, let $Q \in \mathscr{L}([a + \varepsilon, b])$ for any $0 < \varepsilon < b - a$ and suppose $Q \notin \mathscr{L}([a, b])$. Finally, let Q be nonpositive a.e. on $[a, b]$. Then there exists one and only one absolutely continuous function $\zeta: [a, b] \to \mathbb{R}$ such that*

$$\zeta'(t) = Q(t)\, \zeta(t) + P(t) \quad \textit{a.e. on} \quad [a, b] \tag{3.45}$$

and

$$\zeta(a) = 0 . \tag{3.46}$$

Proof. To prove the existence it is sufficient to observe that the function φ given by the formula

$$\varphi(t) = \int_a^t P(s) \exp\left(\int_s^t Q(u)\, du\right) ds \tag{3.47}$$

is a solution of (3.45), (3.46) since

$$\left|P(s) \exp\left(\int_s^t Q(u)\, du\right)\right| \leqq |P(s)| \in \mathscr{L}([a, b]) . \tag{3.48}$$

The uniqueness follows from the relation $(d/dt)(\zeta_2 - \zeta_1)^2 = 2Q(\zeta_2 - \zeta_1)^2$, which holds a.e. on $[a, b]$ for any two solutions of (3.45) as can immediately be seen.

Remark 3.2. If $Q \in \mathscr{L}([a, b])$, then the assertion of Lemma 3.8 is obviously true even without the assumption $Q \leqq 0$.

Proof of Theorem 3.1. Existence and uniqueness of functions η and ζ follow immediately from Lemmas 3.7 and 3.8 or Remark 3.2, respectively. To prove (3.10) put

$$\varphi(t) = y(t) + \eta(t)\, p(t)\, y'(t) - \zeta(t)\,. \tag{3.49}$$

It is clear that the function φ is absolutely continuous and by direct computation we obtain

$$\varphi'(t) = q(t)\, \eta(t)\, \varphi(t) \quad \text{a.e. on} \quad [a, b]\,. \tag{3.50}$$

Since any (absolutely continuous) solution of (2.1) with $q \notin \mathscr{L}([a, b])$ satisfies $y(a) = 0$ according to Lemma 3.1 we have $\varphi(a) = 0$. But from this and from Lemma 3.8 or Remark 3.2 we obtain that $\varphi \equiv 0$ on $[a, b]$. Thus, (3.10) is proved and Theorem 3.1 holds.

Proof of Theorem 3.2 is completely analogous to that of Theorem 3.1 and is omitted.

On the basis of Theorems 3.1 and 3.2 we can develop algorithms completely similar to that mentioned in Section 2. One must only carefully choose the numerical method for solving equations (3.5) or (3.11) to obtain really the nonpositive solutions. Also, studying in more detail the properties of functions η and ζ realizing the transfer of boundary conditions one could obtain existence and uniqueness theorems for singular boundary value problems 2 and 3. Both these topics will be dealt with in further papers.

References

[1] *G. H. Meyer:* Initial Value Methods for Boundary Value Problems — Theory and Application of Invariant Imbedding. Academic Press, New York 1973.

[2] *J. Taufer:* Lösung der Randwertprobleme für Systeme von linearen Differentialgleichungen. Rozpravy ČSAV 83 (1973), No. 5.

[3] *J. Taufer:* Numerical Solution of Boundary Value Problems by Stable Methods Based on the Transfer of Conditions. In: Numerical Solutions of Boundary Value Problems for Ordinary Differential Equations (A. K. Aziz, ed.), Academic Press, New York—San Francisco—London 1975, pp. 317—330.

Souhrn

METODA PŘESUNU PODMÍNEK PRO SINGULÁRNÍ OKRAJOVÉ ÚLOHY

PETR PŘIKRYL, JIŘÍ TAUFER, EMIL VITÁSEK

Metoda přesunu podmínek převádí řešení lineárních okrajových úloh pro obyčejné diferenciální rovnice na řešení jistých počátečních úloh a řešení soustav lineárních algebraických rovnic. Pro soustavy rovnic prvního řádu s lebesgueovsky integrovatelnými koeficienty popsal metodu jeden z autorů již dříve. Cílem předkládaného článku je úprava metody pro řešení okrajových úloh pro lineární diferenciální rovnici druhého řádu se singularitami v koeficientech.

Резюме

МЕТОД ПЕРЕНОСА УСЛОВИЙ ДЛЯ СИНГУЛЯРНЫХ КРАЕВЫХ ЗАДАЧ

PETR PŘIKRYL, JIŘÍ TAUFER, EMIL VITÁSEK

Численное решение линейных краевых задач для обыкновенных дифференциальных уравнений методом переноса условий состоит в приведении рассматриваемой краевой задачи к последовательности задач Коши. Применение этого метода к решению краевых задач для систем уравнений первого порядка с суммируемыми коэффициентами было рассмотрено одним из авторов уже раньше. Цель настоящей статьи — обобщить идею переноса условий на случай краевых задач для линейного уравнения второго порядка с несуммируемыми особенностями в коэффициентах.

Authors' address: RNDr. *Petr Přikryl*, CSc., RNDr. *Jiří Taufer*, CSc., RNDr. *Emil Vitásek*, CSc., Matematický ústav ČSAV, Žitná 25, 115 67 Praha 1.