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THE ROTHE METHOD AND TIME PERIODIC SOLUTIONS  
TO THE NAVIER-STOKES EQUATIONS AND EQUATIONS  
OF MAGNETOHYDRODYNAMICS

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*Summary.* The existence of a periodic solution of a nonlinear operator equation  $z' + A_0z + B_0z = F$  is proved. The theory developed may be used to prove the existence of a periodic solution of the variational formulation of the Navier-Stokes equations or the equations of magnetohydrodynamics. The proof of the main existence theorem is based on Rothe method in combination with the Galerkin method, using the Brouwer fixed point theorem.

*Keywords:* Navier-Stokes equations, periodic solutions, existence of generalized solutions.

*AMS Classifications:* 65N40, 35Q10.

1. INTRODUCTION

In this paper Rothe method is applied to prove the existence of a periodic solution of a nonlinear operator equation

$$(1) \quad u' + A_0u + B_0u = F,$$

where  $A_0$  is a linear differential operator continuous as an operator from the separable Hilbert space  $L_2(0, T; V)$  into his dual  $L_2(0, T; V')$ ,  $B_0$  is a nonlinear differential operator of a special type operating on  $L_2(0, T; V)$  and  $F \in L_\infty(0, T; V')$ .

Special properties of the nonlinear operator  $B_0$  arise as a consequence of the fact that the equation (1) represents a general form of the variational formulation of the Navier-Stokes equations or the equations of magnetohydrodynamics. In particular,  $B_0$  does not fulfil the condition of monotonicity, therefore it is not possible to use Rothe method in such a way as it was done e.g. in [1–4]. The most similar problems to that given by (1) were solved by R. Temam in [5] but without periodicity assumptions.

In solving our problem we first apply time-discretization to the equation (1) (with a partition of interval  $\langle 0, T \rangle$  in  $N$  subintervals of length  $h = T/N$ ) and then we look for a solution of the  $m$ -dimensional approximation of this semi-discretized problem

(the principle of the Galerkin method). A solution of the equation (1) is then obtained by simultaneous limiting processes  $m \rightarrow \infty$  and  $h = h_m \rightarrow 0$ .

## 2. DEFINITIONS AND NOTATION

Let two real separable Hilbert spaces  $V, H$  be given,  $V$  being compactly and densely embedded in  $H$ . Identifying  $H$  with its dual  $H'$ , we may write  $V \subset H \subset V'$ . The inner products in  $H$  and  $V$  are denoted by  $(\cdot, \cdot)$  and  $[[\cdot, \cdot]]$ , and the norms induced by these products are denoted by  $|\cdot|$ , and  $[[\cdot]]$ , respectively. Further, let a real separable Hilbert space  $V_s$  be given,  $s \in \mathbb{N}$  ( $s$  fixed), with an inner product  $[[\cdot, \cdot]]_s$  and the norm  $[[\cdot]]_s$ , such that  $V_s$  is continuously and densely embedded in  $V$ . The dual pairing between  $V$  and  $V'$  as well as between  $V_s$  and  $V'_s$  is denoted by  $\langle \cdot, \cdot \rangle$ .

In the sequel,  $B$ -spaces  $L_2(0, T; X)$ ,  $L_\infty(0, T; X)$  will be used, where  $X$  is a  $B$ -space; their definitions may be found e.g. in [3].

We shall denote by  $\mathcal{D}'(0, T; V'_s)$  the space of distributions on  $\langle 0, T \rangle$  with values in  $V'_s$ . We shall understand under the derivative  $z'$  of a function  $z \in L_2(0, T; V) \subset \mathcal{D}'(0, T; V'_s)$  the derivative in the sense of distributions such that  $z' \in L_2(0, T; V'_s)$ .

Let the operator  $A_0: V \rightarrow V'$  be defined by the relation

$$(2) \quad \langle A_0 z, \tilde{z} \rangle = [[z, \tilde{z}]], \quad z, \tilde{z} \in V.$$

$A_0$  is a linear continuous operator on  $V$ .

Let the operator  $B_0: V \rightarrow V'_s$  be defined by the relation

$$(3) \quad B_0 z = \tilde{B}_0(z, z),$$

where  $\tilde{B}_0: H \times V \rightarrow V'_s$  is a bilinear continuous operator such that

$$(4) \quad \langle \tilde{B}_0(z, \tilde{z}), \hat{z} \rangle = -\langle \tilde{B}_0(z, \hat{z}), \tilde{z} \rangle, \quad z, \tilde{z} \in V, \quad \hat{z} \in V_s$$

and

$$(5) \quad \langle \tilde{B}_0(z, \tilde{z}), \hat{z} \rangle \leq c|z| [[\tilde{z}]]_s [[\hat{z}]]_s, \quad z \in H, \quad \tilde{z} \in V, \quad \hat{z} \in V_s$$

hold ( $c$  is positive constant).

It follows from (4) that

$$(6) \quad \langle B_0 z, z \rangle = 0, \quad z \in V_s.$$

We shall assume the operators  $A_0, B_0$  to be defined on  $L_2(0, T; V)$  by the relations

$$(A_0 z)(t) = A_0 z(t)$$

$$(B_0 z)(t) = B_0 z(t)$$

for a.e.  $t \in \langle 0, T \rangle$ .

Analogously, the operator  $\tilde{B}_0$  is supposed to be defined on  $L_2(0, T; H) \times L_2(0, T; V)$  by the relation

$$(\tilde{B}_0(z, \dot{z}))(t) = \tilde{B}_0(z(t), \dot{z}(t))$$

for a.e.  $t \in \langle 0, T \rangle$ .

### 3. EXISTENCE THEOREM

**Theorem 1.** *Let a function  $F \in L_\infty(0, T; V')$  be given. Then there exists at least one function  $z$  satisfying the relations*

$$(7) \quad z \in L_2(0, T; V) \cap L_\infty(0, T; H),$$

$$z' \in L_2(0, T; V_s'),$$

$$(8) \quad z' + A_0 z + B_0 z = F,$$

$$(9) \quad z(0) = z(T).$$

*Proof.* We shall apply Rothe method in the following way: Let  $\{t_p\}_{p=1}^N$  be a uniform partition of  $\langle 0, T \rangle$ ,  $h = T/N$ ,  $N \in \mathbb{N}$ ,  $t_p = ph$ . Semidiscretizing (8) we get a system of  $N$  equations

$$(10) \quad \frac{z^p - z^{p-1}}{h} + A_0 z^p + B_0 z^p = F^p, \quad p = 1, \dots, N,$$

where  $z^p = z(t_p)$ ,

$$F^p = \frac{1}{h} \int_{I_p} F(t) dt, \quad I_p = (t_{p-1}, t_p), \quad p = 1, \dots, N.$$

It is evident that

$$(11) \quad \|F^p\|_{V'} \leq \|F\|_{L_\infty(0, T; V')}.$$

We shall use the Galerkin method to solve (10).

The space  $V_s$  being compactly embedded in  $H$ , there exists an orthonormal base  $\{\omega_i\}_{i=1}^\infty$  in  $V_s$  and the set of numbers  $\gamma_i > 0$ ,  $i = 1, 2, \dots$  such that

$$(12) \quad \llbracket \omega_i, v \rrbracket_s = \gamma_i (\omega_i, v)$$

for every  $v \in V_s$ .

For every  $m \in \mathbb{N}$  we look for a sequence  $\{z_m^p\}$ ,  $p = 0, 1, \dots, N$  of the form

$$(13) \quad z_m^p = \sum_{i=1}^m \xi_i^p \omega_i$$

so that the following equations are satisfied:

$$(14) \quad \left( \frac{z_m^p - z_m^{p-1}}{h}, \omega_j \right) + \llbracket z_m^p, \omega_j \rrbracket + \langle B_0 z_m^p, \omega_j \rangle = \\ = \langle F^p, \omega_j \rangle, \quad j = 1, \dots, m,$$

$$(15) \quad z_m^0 = z_m^N.$$

To prove the existence of a solution of (14) and (15) we formulate the next two lemmas:

**Lemma 1.** *There exists a number  $R > 0$  independent of  $m, h, p$  and such that the following assertion holds: If  $z_m^{p-1} \in B_m(0, R)$  for some  $p \in \{1, \dots, N\}$  ( $B_m(0, R)$  denotes the closed ball with center 0 and radius  $R$  in the space  $\text{lin} \{\omega_i\}_{i=1}^m$  with the norm  $|\cdot|$ ), and if  $z_m^{p-1}, z_m^p$  satisfy the equation (14), then  $z_m^p \in B_m(0, R)$  as well.*

*Proof of Lemma 1.* Using (6) we obtain from (14) ( $\varepsilon > 0$ )

$$(16) \quad |z_m^p|^2 + h \llbracket z_m^p \rrbracket^2 \leq \frac{|z_m^{p-1}|^2}{2} + \frac{|z_m^p|^2}{2} + \frac{h \|F^p\|_{V'}^2}{2\varepsilon} + \frac{\varepsilon h \llbracket z_m^p \rrbracket^2}{2}.$$

For  $0 < \varepsilon < 2$  and  $R$  such that

$$(17) \quad R \geq c_1 \|F\|_{L_\infty(0, T; V')} \sqrt{\left( \frac{1}{\varepsilon(2 - \varepsilon)} \right)}$$

we get easily from (16) that the desired implication

$$(18) \quad |z_m^{p-1}| \leq R \Rightarrow |z_m^p| \leq R$$

holds ( $c_1$  denotes the constant of the embedding of  $V$  in  $H$ ).

**Lemma 2.** *Let  $R$  satisfy (17). Then for every  $m \in \mathbb{N}$  there exists  $h = h_m$  such that the following assertion holds: If  $z_m^{p-1} \in B_m(0, R)$  then there exists a unique function  $z_m^p \in B_m(0, R)$  such that  $z_m^{p-1}, z_m^p$  satisfy the equation (14) (with  $h = h_m$ ). Moreover,  $z_m^p$  depends continuously on  $z_m^{p-1}$ .*

To prove Lemma 2 we shall use the theorem on local existence of a solution of an equation depending on a parameter (Theorem 3.4.1, Chap. I, [6]) which we present here as Lemma 3:

**Lemma 3.** *Let  $(X_1, d_1), (X_2, d_2)$  be two complete metric spaces, let  $x_0 \in X_1, p_0 \in X_2, \alpha, \beta > 0, 0 \leq \lambda < 1$  and suppose that*

- (i)  $G: B(x_0, \alpha; X_1) \times B(p_0, \beta; X_2) \rightarrow X_1$  is continuous;
- (ii)  $d_1(G(x_1, p), G(x_2, p)) \leq \lambda d_1(x_1, x_2)$  for  $x_1, x_2 \in B(x_0, \alpha; X_1)$  and  $p \in B(p_0, \beta; X_2)$ ;
- (iii)  $d_1(G(x_0, p), x_0) < \alpha(1 - \lambda)$  for  $p \in B(p_0, \beta; X_2)$ .

Then the equation  $x = G(x, p)$  has a unique solution  $x = x^*(p) \in B(x_0, \alpha; X_1)$  for any  $p \in B(p_0, \beta; X_2)$ . Moreover,  $x^*: B(p_0, \beta; X_2) \rightarrow B(x_0, \alpha; X_1)$  is continuous.

Proof of Lemma 2. Using (13) we can rewrite the equation (14) in the form

$$(19) \quad \xi^p = \xi^{p-1} + hA^{-1}[\varphi^p - \hat{A}\xi^p - B(\xi^p)\xi^p]$$

where  $\xi^p$  and  $\varphi^p$  are vectors, the coordinates of which are respectively  $\xi_i^p$  and  $(F^p, \omega_i)$ ,  $i = 1, \dots, m$ , while  $\hat{A}$ ,  $A$  and  $B(\xi^p)$  are matrices, the elements of which are respectively  $(\hat{A})_{ij} = \llbracket \omega_j, \omega_i \rrbracket$ ,  $(A)_{ij} = (\omega_j, \omega_i)$  and  $B(\xi^p)_{ij} = \langle \tilde{B}_0(z_m^p, \omega_j), \omega_i \rangle$ ,  $i = 1, \dots, m$ ,  $j = 1, \dots, m$ . To prove the existence of a solution of the equation (19) we use Lemma 3:

Let the mapping  $G: \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  be defined by the relation

$$(20) \quad G(\xi^p, \xi^{p-1}) = \xi^{p-1} + hA^{-1}[\varphi^p - \hat{A}\xi^p - B(\xi^p)\xi^p].$$

Without going in details, we may ensure that the conditions (i)–(iii) of Lemma 3 be fulfilled by making  $h$  dependent on  $m$ . In particular, it can be easily shown that for every  $m \in \mathbb{N}$  there exists  $h = h_m$  sufficiently small ( $h_m \rightarrow 0$ ,  $m \rightarrow \infty$ ) such that  $G$  defined by (20) satisfies the conditions (i)–(iii) of Lemma 3 (with  $\alpha > \beta > 0$ ). Moreover, we choose (for every  $m$ )  $h = h_m$  so that  $h_m = T/N_m$ , where  $N_m \in \mathbb{N}$ . Thus, the partition of the interval  $\langle 0, T \rangle$  is made dependent on  $m$ .

For  $G$  defined by (20) the assertion of Lemma 3 can be formulated as follows:

For every  $\xi^{p-1} \in B(0, \beta, \mathbb{R}^m)$  ( $B(0, \beta, \mathbb{R}^m)$  denotes the closed ball with center 0 and radius  $\beta$  in the space  $\mathbb{R}^m$ ) there exists one and only one  $\xi^p \in B(0, \alpha, \mathbb{R}^m)$  ( $\alpha > \beta$ ) such that  $\xi^{p-1}$ ,  $\xi^p$  satisfy (19) (with  $h = h_m$ ) and, moreover,  $\xi^p$  depends continuously on  $\xi^{p-1}$ .

To interpret this result in the space of functions  $z_m^{p-1}$ ,  $z_m^p$ , we choose  $\beta$  sufficiently large so that

$$B(0, R, \mathbb{R}_\gamma^m) \subset B(0, \beta, \mathbb{R}^m)$$

where  $\mathbb{R}_\gamma^m$  denotes the space  $\mathbb{R}^m$  with the norm  $|\varrho|_{m,\gamma} = \sqrt{(\sum_{i=1}^m (\varrho_i^2/\gamma_i))}$ , ( $\gamma_i$  were defined in (12)).

Thus,  $\xi^{p-1} \in B(0, \beta, \mathbb{R}^m)$  for  $z_m^{p-1} \in B_m(0, R)$ , and we may define a continuous mapping  $\mathcal{F}_m^p$  ( $m \in \mathbb{N}$ ,  $p \in \{1, \dots, N_m\}$ ) in the following way:

$$\mathcal{F}_m^p: B_m(0, R) \rightarrow B_m(0, k\alpha)$$

$$z_m^{p-1} \mapsto z_m^p,$$

where  $k$  is the constant of the embedding of  $V_s$  in  $H$ ,  $\alpha > \beta > 0$ , and  $z_m^p$  is the unique solution of the equation (14). From Lemma 1 we obtain that  $z_m^p = \mathcal{F}_m^p(z_m^{p-1}) \in B_m(0, R)$ . Thus, the mapping  $\mathcal{F}_m^p$  maps the closed ball  $B_m(0, R)$  continuously into itself. This completes the proof of Lemma 2.

Define a mapping

$$\begin{aligned}\mathcal{P}_m &= \mathcal{F}_m^{N_m} \circ \dots \circ \mathcal{F}_m^1 \\ \mathcal{P}_m(z_m^0) &= z_m^{N_m}\end{aligned}$$

such that in the sequence

$$(21) \quad \{z_m^0, \dots, z_m^{N_m}\}$$

each two neighbouring elements satisfy the equation (14) (with  $h = h_m$ ), each element is uniquely determined by its predecessor and depends on it continuously.  $\mathcal{P}_m$  maps the closed ball  $B_m(0, R)$  continuously into itself. From the Brouwer fixed point theorem we obtain the existence of a function  $z_{0m}$  such that  $\mathcal{P}_m(z_{0m}) = z_{0m}$ . We put  $z_m^0 = z_{0m}$  and for this initial function there exists a unique sequence (21) with the properties mentioned above. The following estimate takes place:

$$(22) \quad |z_m^p| \leq R, \quad p = 0, 1, \dots, N_m, \quad m \in \mathbb{N}.$$

#### 4. APRIORI ESTIMATES AND LIMITING PROCESSES

Using the sequence (21) with  $z_m^0 = z_{0m}$  we define functions

$$(23) \quad \begin{aligned}z_m(t) &= z_m^p, \quad t \in I_p, \quad I_p = ((p-1)h_m, ph_m], \quad p = 1, \dots, N_m, \\ z_m(0) &= z_m^0,\end{aligned}$$

$$(24) \quad \begin{aligned}Z_m(t) &= \left( \frac{t - (p-1)h_m}{h_m} \right) z_m^p + \left( \frac{ph_m - t}{h_m} \right) z_m^{p-1}, \\ t \in I_p, \quad p &= 1, \dots, N_m, \\ Z_m(0) &= z_m^0.\end{aligned}$$

Further, we define

$$F_m(t) = F^p, \quad t \in I_p, \quad p = 1, \dots, N_m.$$

In the following lemma standard methods are used to prove some of the apriori estimates (cf. e.g. [4], [5]).

**Lemma 4.** *The sequences of functions  $\{z_m\}_{m=1}^\infty, \{Z_m\}_{m=1}^\infty$  are bounded in the space  $L_2(0, T; V) \cap L_\infty(0, T; H)$ . Moreover,*

$$(25) \quad z_m - Z_m \rightarrow 0 \quad \text{in } L_2(0, T; H), \quad m \rightarrow \infty.$$

*Proof.* From (22) we immediately obtain

$$(26) \quad \|z_m\|_{L_\infty(0, T; H)} \leq R$$

$$(27) \quad \|Z_m\|_{L_\infty(0, T; H)} \leq R$$

The equation (14) implies

$$\begin{aligned} & \sum_{p=1}^{N_m} (|z_m^p|^2 - |z_m^{p-1}|^2 + |z_m^p - z_m^{p-1}|^2) + 2h_m \sum_{p=1}^{N_m} \llbracket z_m^p \rrbracket^2 \leq \\ & \leq \sum_{p=1}^{N_m} h_m (\|F^p\|_{V'}^2 + \llbracket z_m^p \rrbracket^2). \end{aligned}$$

Using (11) and the periodicity condition (15), we conclude that

$$(28) \quad \sum_{p=1}^{N_m} |z_m^p - z_m^{p-1}|^2 \leq d_1,$$

$$(29) \quad \|z_m\|_{L_2(0,T;V)}^2 = \sum_{p=1}^{N_m} h_m \llbracket z_m^p \rrbracket^2 \leq d_1,$$

$$(30) \quad \|Z_m\|_{L_2(0,T;V)}^2 \leq 2 \sum_{p=1}^{N_m} h_m \llbracket z_m^p \rrbracket^2 \leq 2d_1$$

where  $d_1 = T \|F\|_{L_\infty(0,T;V')}^2$ .

The condition (25) is easily proved if we notice that

$$z_m(t) - Z_m(t) = \frac{ph_m - t}{h_m} (z_m^p - z_m^{p-1}), \quad t \in I_p,$$

and that (28) holds.

**Lemma 5.** *The sequence  $\{dZ_m/dt\}_{m=1}^\infty$  is bounded in the space  $L_2(0, T; V'_s)$ .*

*Proof.* The proof of this lemma is based on a special choice of the base  $\{\omega_i\}_{i=1}^\infty$  in the space  $V_s$  (relation (12)): For the functions (23), (24) the equation (14) is of the form

$$(31) \quad \left( \frac{dZ_m(t)}{dt}, \omega_j \right) + \llbracket z_m(t), \omega_j \rrbracket + \langle \tilde{B}_0(z_m(t), z_m(t)), \omega_j \rangle = \langle F_m(t), \omega_j \rangle,$$

for a.e.  $t \in \langle 0, T \rangle$ ,  $j = 1, \dots, m$ .

Define a projector

$$(32) \quad \begin{aligned} P_m: H & \rightarrow \text{lin} \{ \omega_i \}_{i=1}^m, \\ P_m(v) & = \sum_{i=1}^m \gamma_i(v, \omega_i) \omega_i. \end{aligned}$$

For  $v \in V_s$  we have from (31)

$$(33) \quad \left( \frac{dZ_m(t)}{dt}, v \right) = \langle F_m(t), P_m v \rangle - \llbracket z_m(t), P_m v \rrbracket - \langle \tilde{B}_0(z_m(t), z_m(t)), P_m v \rangle.$$



It follows from (32) that

$$(34) \quad \llbracket P_m v \rrbracket_s \leq \llbracket v \rrbracket_s$$

and, consequently,

$$(35) \quad \llbracket P_m v \rrbracket \leq c_3 \llbracket v \rrbracket_s.$$

Thus, from (33) we get

$$|(Z'_m(t), v)|^2 \leq \tilde{c} \{ \|F_m(t)\|_{V'}^2 + \llbracket z_m(t) \rrbracket^2 + |z_m(t)|^2 \llbracket z_m(t) \rrbracket^2 \} \llbracket v \rrbracket_s^2$$

and this implies, by virtue of (11), (26) and (29), the assertion of Lemma 5.

Lemmas 4 and 5 enable us to assert that there exist subsequences  $\{z_\mu\} \subset \{z_m\}$ ,  $\{Z_\mu\} \subset \{Z_m\}$  such that

$$(36) \quad z_\mu \rightharpoonup z \quad (\text{weak convergence in } L_2(0, T; V) \text{ and} \\ \text{*weak convergence in } L_\infty(0, T; H)),$$

$$(37) \quad Z_\mu \rightharpoonup Z \quad (\text{weak convergence in } L_2(0, T; V) \text{ and} \\ \text{*weak convergence in } L_\infty(0, T; H)),$$

$$(38) \quad Z'_\mu \rightharpoonup Z' \quad (\text{weak convergence in } L_2(0, T; V'_s)).$$

The theorem on compact embedding (Theorem 5.1, Chap. I, [7]) implies that

$$(39) \quad Z_\mu \rightarrow Z \quad \text{strongly in } L_2(0, T; H).$$

From (25) and (36)–(39) it follows that

$$(40) \quad z = Z$$

and

$$(41) \quad Z'_\mu \rightharpoonup z' \quad (\text{weak convergence in } L_2(0, T; V'_s)).$$

From the definition of  $A_0$  we have

$$(42) \quad A_0 z_\mu \rightharpoonup A_0 z \quad (\text{weak convergence in } L_2(0, T; V'_s)).$$

**Lemma 6.** *There exists a subsequence of  $\tilde{B}_0(z_\mu, z_\mu)$  that converges weakly to  $\tilde{B}_0(z, z)$  in  $L_2(0, T; V'_s)$ .*

*Proof.* Using the properties of  $\tilde{B}_0$  and the fact that  $z_\mu \rightarrow z$  in  $L_2(0, T; H)$ , it can be easily proved that

$$(43) \quad \langle \tilde{B}_0(z_\mu, z_\mu), v \rangle \rightarrow \langle \tilde{B}_0(z, z), v \rangle \quad \text{for } v \in L_\infty(0, T; V_s).$$

Since (26) and (29) hold, the sequence  $\tilde{B}_0(z_\mu, z_\mu)$  is bounded in  $L_2(0, T; V'_s)$ . Thus, there exists a subsequence of  $\tilde{B}_0(z_\mu, z_\mu)$  that converges weakly in the space  $L_2(0, T; V'_s)$ . For simplicity it will be denoted by  $\tilde{B}_0(z_\mu, z_\mu)$  again. By virtue of (43) the assertion of Lemma 6 is proved.

**Lemma 7.** *The sequence  $\{F_\mu\}$  converges to  $F$  in  $L_2(0, T; V')$ .*

The proof of this lemma is quite analogous to that of Lemma 4.9, Chap. III, [5], therefore we do not present it here.

The system of equations (31) can be rewritten in a form suitable for the limiting process:

$$(44) \quad \left( \frac{dZ_\mu(t)}{dt}, \omega_j \right) + \langle A_0 z_\mu(t), \omega_j \rangle + \langle \tilde{B}_0(z_\mu(t), z_\mu(t)), \omega_j \rangle = \\ = \langle F_\mu(t), \omega_j \rangle, \quad j = 1, \dots, \mu.$$

Let  $\{g_{ij}\}_{i=1}^\infty$  denote the base in  $L_2(0, T)$ . From (44), using Lemmas 6 and 7 and the relations (41) and (42), we obtain by the limiting process  $\mu \rightarrow \infty$ , for every  $i, j \in \mathbb{N}$ , the equation

$$(45) \quad \langle z', g_i w_j \rangle_t + \langle A_0 z, g_i w_j \rangle_t + \langle B_0 z, g_i w_j \rangle_t = \langle F, g_i w_j \rangle_t,$$

where  $\langle \cdot, \cdot \rangle_t$  denotes the dual pairing between  $L_2(0, T; V_s)$  and  $L_2(0, T; V'_s)$ . This implies that (8) is fulfilled. The periodicity condition (9) is also satisfied since it follows from (37), (40) and (41) that

$$(46) \quad Z_\mu(t) \rightarrow z(t) \quad (\text{weak convergence in } V'_s) \quad \text{for every } t \in \langle 0, T \rangle,$$

and, simultaneously,

$$Z_\mu(0) = Z_\mu(T) \quad \text{for all } \mu.$$

This completes the proof of Theorem 1.

*Remark.* Specifying the operators  $A_0, B_0$  and the spaces  $V_s, V$  and  $H$ , we can use Theorem 1 to prove the existence of a periodic solution of a variational formulation of the Navier-Stokes equations and the equations of magnetohydrodynamics, for details see [8].

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Souhrn

ROTHEHO METODA A PERIODICKÉ ŘEŠENÍ NAVIER-STOKESOVÝCH ROVNIC  
A ROVNIC MAGNETICKÉ HYDRODYNAMIKY

DANA LAUEROVÁ

V článku je dokázána existence periodického řešení nelineární operátorové rovnice  $z' + A_0z + B_0z = F$ , která vzniká zobecněním variační formulace úlohy Navier-Stokesových rovnic nebo rovnic magnetické hydrodynamiky. Vhodným spojením Rotheho metody, Galerkinovy metody a Brouwerovy věty je dokázána hlavní existenční věta.

Резюме

МЕТОД РОТЕ И ПЕРИОДИЧЕСКИЕ РЕШЕНИЯ УРАВНЕНИЙ  
НАВИЕРА-СТОКСА И УРАВНЕНИЙ МАГНЕТОГИДРОДИНАМИКИ

DANA LAUEROVÁ

В статье доказывается существование периодического решения нелинейного операторного уравнения  $z' + A_0z + B_0z = F$ . Сформулированную теорему существования можно непосредственно применить к доказательству существования периодического решения вариационной формулировки уравнений Навьера-Стокса или уравнений магнетогидродинамики. Главным методом доказательства является метод Роте в комбинации с методом Галеркина при использовании теоремы Броуэра.

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