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Aplikace matematiky, Vol. 35 (1990), No. 3, 178--183

Persistent URL: http://dml.cz/dmlcz/104401

35 (1990) APLIKACE MATEMATIKY No. 3, 178—183

PROPERTIES OF SPACE MOTIONS WITH TWO STRAIGHT TRAJECTORIES

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Received November 21, 1988

Summary. The paper is devoted to Euclidean space motions with two straight trajectories on two given skew straight lines. It describes all motions from this class which have one more planar trajectory in a plane not parallel to the given lines. In the conclusion it gives conditions under which such motions have further planar trajectories in planes not parallel to the given skew straight lines.

Keywords: Special space motions.

AMS Classification: 53A17.

Every one-parametric motion in the Euclidean space E_3 can be expressed in the form

$$(1)\quad \varphi(t) = \begin{pmatrix} 1, & 0, & 0, & 0 \\ t_1, & a_0^2 + a_1^2 - a_2^2 - a_3^2, & 2(a_1a_2 + a_0a_3), & 2(a_1a_3 - a_0a_2) \\ t_2, & 2(a_1a_2 - a_0a_3), & a_0^2 - a_1^2 + a_2^2 - a_3^2, & 2(a_2a_3 + a_0a_1) \\ t_3, & 2(a_0a_2 + a_1a_3), & 2(a_2a_3 - a_0a_1), & a_0^2 - a_1^2 - a_2^2 + a_3^2 \end{pmatrix}$$

where $a_i = a_i(t)$, $t_i = t_i(t)$ and $\sum_{i=0}^{3} a_i^2 = 1$ (see [1] or [2]).

We shall consider only motions which have two straight lines as trajectories, assuming these lines to be skew. Then we have

Proposition 1. *Consider a space motion $\varphi(t)$ such that the points $\bar{X}_\varepsilon = (1, 0, 0, \varepsilon_m)^\mathsf{T}$, $\varepsilon = \pm 1$ have the straight lines $z = \varepsilon r$, $x + \varepsilon y = 0$ as trajectories. Then*

$$(2)\qquad t_1 = -\,2\lambda m(a_2a_3 + a_0a_1)\,, \quad t_2 = -\,\frac{2m}{\lambda}(a_1a_3 - a_0a_2)\,, \quad t_3 = 0\,,$$

$$(\omega_1)\colon\ (a_0^2 + a_3^2)\left(1 - \frac{r}{m}\right) - (a_1^2 + a_2^2)\left(1 + \frac{r}{m}\right) = 0$$

where $2r$ is the distance of the trajectories $(m > r > 0)$ and $\lambda = -\varepsilon \operatorname{cotg} \beta/2$ (β is the angle of the trajectories).

Proof. Let us choose the frame in the fixed space as the symmetry frame of the trajectories (this means that the z-axis is the common perpendicular of the two trajectories, the origin is in the middle of the common perpendicular, the x-axis has the same angle with both trajectories), the $\bar{z}$-axis in the moving space lies on the line $\bar{X}_1\bar{X}_{-1}$, the origin is in the middle between the points. The trajectory of $\bar{X}$ must satisfy the equation of the line trajectory. We get

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & a_{11} & a_{12} & a_{13} \\ t_2 & a_{21} & a_{22} & a_{23} \\ t_3 & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \varepsilon m \end{pmatrix} = \begin{pmatrix} 1 \\ t_1 + a_{13}\ \varepsilon m \\ t_2 + a_{23}\ \varepsilon m \\ t_3 + a_{33}\ \varepsilon m \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix}$$

where a_{ij} are given by (1), $i, j = 1, 2, 3$.

Therefore we have

$$t_3 + a_{33}\varepsilon m = \varepsilon r ,$$

$$t_1 + a_{13}\varepsilon m + \varepsilon\lambda(t_2 + a_{23}\varepsilon m) = 0 .$$

This yields the statement.

Proposition 2. *Let us have a motion from Proposition* 1. *Then the trajectory of any point on* $\bar{X}_1\bar{X}_{-1}$ *different from* $\bar{X}_1, \bar{X}_{-1}$ *is an ellipse in a plane parallel to both straight trajectories.*

Proof. The frames are chosen as before. For the trajectory of the point $\bar{X} = (1, 0, 0, \bar{z}_0)^\mathsf{T}$ we have $z = (r/m)\,\bar{z}_0$, so it lies in a plane.

Further,

$$x = +a_{13}\bar{z}_0 - a_{23}\lambda m ,$$

$$y = -a_{13}\frac{m}{\lambda} + a_{23}\bar{z}_0 .$$

We express x and y using (1) and obtain

$$x^2\left(\bar{z}_0^2 + \frac{m^2}{\lambda^2}\right) + y^2(\lambda^2 m^2 + \bar{z}_0^2) + 2xy\left(\lambda m\bar{z}_0 + \bar{z}_0\frac{m}{\lambda}\right) =$$

$$= (\bar{z}_0^2 - m^2)^2\left(1 - \frac{r^2}{m^2}\right),$$

which is an ellipse as $\bar{z}_0 \neq \varepsilon m$, $r < m$.

Denote

$$\varkappa = C\bar{z}\frac{r}{m} + D\,,$$

$$\mu^2 = \frac{1 + \frac{r}{m}}{1 - \frac{r}{m}}\,.$$

Proposition 3. *Let us have a motion $\varphi(t)$ as in Proposition* 1. *Let us suppose that this motion has another point $(1, \bar{x}, 0, \bar{z})^{\mathrm{T}}$ with a plane trajectory, which lies in the plane $Ax + By + Cz + D = 0$. Then $\varphi(t)$ lies on a quadratic surface*

$$\omega_2\colon\ (A\bar{x} + \varkappa)(a_1^2 + a_0^2) + (-A\bar{x} + \varkappa)(a_2^2 + a_3^2) +$$

$$+ 2a_0a_1(B\bar{z} - A\lambda m) + 2a_0a_2\left(-A\bar{z} + B\frac{m}{\lambda} + C\bar{x}\right) -$$

$$- 2a_0a_3B\bar{x} + 2a_1a_3\left(A\bar{z} - B\frac{m}{\lambda}C\bar{x}\right) +$$

$$+ 2a_1a_2B\bar{x} + 2a_2a_3(B\bar{z} - A\lambda m) = 0\,.$$

Proof. For the trajectory we obtain

$$A(t_1 + a_{11}\bar{x} + a_{13}\bar{z}) + B(t_2 + a_{21}\bar{x} + a_{23}\bar{z}) +$$

$$+ C(t_3 + a_{31}\bar{x} + a_{33}\bar{z}) + D = 0\,.$$

Substitution from (1) and (2) gives the equation of ω_2. In what follows we shall consider only algebraic motions. Such motions are given by algebraic equation in parameters a_0, a_1, a_2, a_3. In Proposition 3 we have shown that there exist motions with two straight and one plane trajectory. For such motions we prove the following theorem.

Theorem. *Let us have an algebraic motion from Proposition* 3. *Then there exist further points with plane trajectories iff $B = \varepsilon A\lambda$. If $A^2 + B^2 \neq 0$, then these points lie on the line $\bar{z}\bar{x}_0 = \bar{z}_0\bar{x} + \varepsilon m(\bar{x}_0 - \bar{x})$, $\bar{y} = 0$, if $A = B = 0$, they are all the points of the plane $\bar{y} = 0$.*

Proof. Let us choose a point $\overline{M} = (1, \bar{x}_1, \bar{y}_1, \bar{z}_1)^{\mathrm{T}}$ and let its trajectory lie in the plane $A_1x + B_1y + C_1z + D_1 = 0$. We express the trajectory of $\overline{M}$ and obtain a quadratic equation ω_3 given by the symmetric matrix

$$\begin{pmatrix} A_1\bar{x}_1 + B_1\bar{y}_1 + \varkappa_1 & -A_1\lambda m + B_1\bar{z}_1 - C_1\bar{y}_1 & C_1\bar{x}_1 - A_1\bar{z}_1 + B_1 m/\lambda & -B_1\bar{x}_1 + A_1\bar{y}_1 \\ x & A_1\bar{x}_1 - B_1\bar{y}_1 + \varkappa_1 & A_1\bar{y}_1 + B_1\bar{x}_1 & C_1\bar{x}_1 + A_1\bar{z}_1 - B_1 m/\lambda \\ x & x & -A_1\bar{x}_1 + B_1\bar{y}_1 + \varkappa_1 & C_1\bar{y}_1 - A_1\lambda m + B_1\bar{z}_1 \\ x & x & x & -A_1\bar{x}_1 + B_1\bar{y}_1 + \varkappa_1 \end{pmatrix}.$$

The motion must satisfy all three quadratic equations $\omega_1, \omega_2, \omega_3$. Therefore $\omega_3 = p\omega_1 + \gamma\omega_2$ with $p, \gamma \in \mathscr{R}$. If $\gamma = 0$, we get a point on the $\bar{z}$ axis. So $\gamma \neq 0$; let $\gamma = 1$, hence $\omega_3 = p\omega_1 + \omega_2$. We obtain a system of ten equations:

$$\begin{aligned}
A\bar{x} + \varkappa + p &= A_1\bar{x}_1 + B_1\bar{y}_1 + \varkappa_1 , \\
A\bar{x} + \varkappa - \mu^2 p &= A_1\bar{x}_1 - B_1\bar{y}_1 + \varkappa_1 , \\
-A\bar{x} + \varkappa - \mu^2 p &= -A_1\bar{x}_1 + B_1\bar{y}_1 + \varkappa_1 , \\
-A\bar{x} + \varkappa + p &= -A_1\bar{x}_1 - B_1\bar{y}_1 + \varkappa_1 , \\
-A\bar{z} + B\frac{m}{\lambda} + C\bar{x} &= -A_1\bar{z}_1 + B_1\frac{m}{\lambda} + C_1\bar{x}_1 , \\
B\bar{z} - A\lambda m &= -A_1\lambda m + B_1\bar{z}_1 - C_1\bar{y}_1 , \\
-B\bar{x} &= A_1\bar{y}_1 - B_1\bar{x}_1 , \\
B\bar{x} &= A_1\bar{y}_1 + B\bar{x}_1 , \\
A\bar{z} - B\frac{m}{\lambda} + C\bar{x} &= A_1\bar{z}_1 - B_1\frac{m}{\lambda} + C_1\bar{x}_1 , \\
B\bar{z} - A\lambda m &= -A_1\lambda m + B_1\bar{z}_1 + C_1\bar{y}_1 .
\end{aligned}$$

By taking linear combinations of these equations we obtain $A_1\bar{y}_1 = B_1\bar{y}_1 = C_1\bar{y}_1 = 0$, so $\bar{y}_1 = 0$.

Further,

$$\left.\begin{aligned} \varkappa + p &= \varkappa_1 \\ \varkappa - \mu^2 p &= \varkappa_1 \end{aligned}\right\} \Rightarrow p = 0 , \quad \varkappa = \varkappa_1 .$$

As a consequence we obtain

(2)
$$\begin{aligned}
A\bar{x} &= A_1\bar{x}_1 \quad A\bar{z} - B\frac{m}{\lambda} = A_1\bar{z}_1 - B_1\frac{m}{\lambda} , \\
B\bar{x} &= B_1\bar{x}_1 \quad B\bar{z} - A\lambda m = -A_1\lambda m + B_1\bar{z}_1 , \\
C\bar{x} &= C_1\bar{x}_1 .
\end{aligned}$$

As $\bar{x} \neq 0$ and $A^2 + B^2 + C^2 \neq 0$ we have $\bar{x}_1 \neq 0$. Denote $\bar{x}/\bar{x}_1 = \delta$. Then $A_1 = A\delta$, $B_1 = B\delta$, $C_1 = C\delta$, and from (2) we obtain

$$(4) \qquad -A(1-\delta)\lambda m + B(\bar{z} - \delta\bar{z}_1) = 0\,,$$

$$-A(\bar{z} - \delta\bar{z}_1) + B\frac{m}{\lambda}(1-\delta) = 0\,.$$

a) The determinant is $-m^2(\delta - 1)^2 + (\bar{z} - \delta\bar{z}_1)^2$. Let it be nonzero. Then $A = B = 0$.

b) Let $m^2(\delta - 1)^2 = (\bar{z} - \delta\bar{z}_1)^2$, then

$$\delta - 1 = \frac{1}{m}(\bar{z} - \delta\bar{z}_1)$$

and

$$\bar{z}_\varepsilon = \frac{\bar{z}_0 + \varepsilon m(\delta - 1)}{\delta}\,, \quad \bar{x}_1 = \frac{\bar{x}}{\delta}\,,$$

from (4) we have $B = \varepsilon A\lambda$ and $\bar{z}_1\bar{x}_0 = \bar{z}_0\bar{x}_1 + \varepsilon m(\bar{x}_0 - \bar{x}_1)$.

Remarks. If we consider a_0, a_1, a_2, a_3 as homogeneous coordinates in the elliptic space, we see that the spherical motion which corresponds to the motion with two straight trajectories lies on the quadratic surface ω_1. This quadric is exceptional in the elliptic geometry, it is called Clifford's quadric and has two rotational axes. It contains two systems of straight lines and these lines are images of elliptic motions.

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Souhrn

VLASTNOSTI PŘÍMKOVÝCH POHYBŮ SE DVĚMA PŘÍMKOVÝMI TRAJEKTORIEMI

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Práce se zabývá euklidovskými pohyby v prostoru, které mají dvě přímkové trajektorie ležící na daných mimoběžných přímkách. Jsou nalezeny ty pohyby, které mají ještě navíc jednu rovinnou trajektorii v rovině, která není rovnoběžná s danými mimoběžkami. Nakonec jsou nalezeny podmínky, za kterých takový pohyb má ještě další rovinné trajektorie neležící v rovinách rovnoběžných s danými mimoběžkami.

Резюме

СВОЙСТВА ПРОСТРАНСТВЕННОГО ДВИЖЕНИЯ С ДВУМЯ ЛИНЕЙНЫМИ ТРАЕКТОРИЯМИ

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Показано, что в E_3 существуют движения с двумя линейными траекториями, лежащими на заданных скрещивающихся прямых, и найдены все движения из этого класса, обладающие плоской траекторией, лежащей в плоскости, которая не параллельна этим прямым. Приведены также условия, при которых такое движение обладает ещё другими плсокими траекториями этого рода.

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