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ON MEAN VALUE IN F-QUANTUM SPACES

BELOSLAV RIEČAN

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Summary. The paper deals with a new mathematical model for quantum mechanics based on the fuzzy set theory [1]. The indefinite integral of observables is defined and some basic properties of the integral are examined.

Keywords: Quantum mechanics, observables, states, probability, fuzzy sets.

AMS Classification: 81C20.

1. INTRODUCTION

A new model for mechanics was suggested by A. Dvurečenskij and the author in [1] and [2]. This model was further developed e.g. in [3–5]. In [6–8], a calculus for observables was constructed. There are three basic notions in the F-quantum space theory: F-quantum space, F-observable and F-state.

F-quantum space is a family $F \subseteq \langle 0, 1 \rangle^x$ of real functions satisfying the following properties: 1. If $f \in F$, then $f' = 1 - f \in F$. 2. If $f_n \in F$ ($n = 1, 2, \ldots$), then $\bigvee f_n = \sup f_n \in F$.

F-observable is a $\sigma$-homomorphism from the $\sigma$-algebra $B$ of Borel subsets of $R$ to $F$, i.e. a mapping with the following two properties: 1. $x(E') = x(E)'$ for every $E \in B$. 2. $x(\bigcup E_n) = \bigvee x(E_n)$ for every $E_n \in B$ ($n = 1, 2, \ldots$).

F-state is a mapping $m: F \to \langle 0, 1 \rangle$ defined on an F-quantum space $F$ and satisfying the following two conditions: 1. $m(a \lor a') = 1$ for every $a \in F$. 2. If $a_n \in F$ ($n = 1, 2, \ldots$) and $a_i \leq a'_j$ ($i \neq j$), then $m(\bigvee a_n) = \sum m(a_n)$. Recall that the definition due to Piasecki [9] inspired our investigations.

A classical analogue of a state is a probability measure, a classical analogue of an observable is a random variable $\xi$ defined on a probability space $(\Omega, S, P)$. To every random variable $\xi$ an F-observable $x$ can be assigned by the formula $x(E) = \xi^{-1}(E)$.

If $x$ is an F-observable and $m$ is an F-state, then the composite mapping $m \circ x$ is a probability measure on the $\sigma$-algebra $B$. We shall denote it by $m_x$, hence $m_x(E) = m(x(E))$, $E \in B$. 

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In a framework of the calculus constructed in [6—8], we shall construct the indefinite integral of an observable and prove its \( G \)-additivity. Another approach to the problem is given in [10].

Recall that an \( F \)-observable \( x \) is called integrable, if the integral \( \int_R t \, dm_x(t) \) exists. It is then denoted by \( m(x) \) and called the mean value of \( x \). This definition is also in full agreement with the classical one.

2. INDEFINITE INTEGRAL

Our aim is to define the indefinite integral \( \int_a x \, dm, a \in F \). This integral presents the crucial point in the concept of conditional probability. We shall follow again the classical case, where \( \int_A \xi \, dP = \int_{A}^{A} \xi \, dP \). Therefore we must investigate the preimages \( (\xi_A)^{-1}(E), E \in B \). This investigation leads to the following definition.

**Definition 1.** If \( x: B \to F \) is an \( F \)-observable, then for every \( a \in F \) and every Borel set \( E \in B \) we define

\[
x_a(E) = \begin{cases} 
    a \land (x(E) \lor a'), & \text{if } 0 \notin E \\
    a' \lor (x(E) \land a), & \text{if } 0 \in E
\end{cases}
\]

**Proposition 1.** The mapping \( x_a: B \to F \) is an \( F \)-observable for any \( a \in F \). If \( x \) is integrable, then \( x_a \) is integrable, too.

**Proof.** If \( 0 \notin E \), then \( 0 \in E' \). Therefore

\[
x_a(E') = a' \lor (x(E') \land a) = a' \lor (x(E') \land a) = (a \land (x(E) \lor a'))'.
\]

The case \( 0 \in E \) can be examined similarly.

If \( A, B \) are disjoint Borel sets and \( 0 \notin A, 0 \in B \), then \( 0 \in A \cup B \) and

\[
x_a(A) \lor x_a(B) = [a \land (x(A) \lor a')] \lor a' \lor (a \land x(B)) = a' \lor (a \land (x(A))) \lor (a \land (x(B))) = a' \lor (a \land (x(A) \lor x(B))) = a' \lor [a \land x(A \cup B)] = x_a(A \cup B).
\]

The case \( 0 \notin A, 0 \notin B \) can be examined similarly. Now, if \( A_n \in B \) \( (n = 1, 2, \ldots) \) and \( A_n \) are disjoint, then \( 0 \) belongs at most to one set, say \( 0 \in A_1 \). Then by the above

\[
x_a(\bigcup_{n} A_n) = x_a(A_1) \lor x_a(\bigcup_{n=1}^{\infty} A_n) = x_a(A_1) \lor (a \land (\bigvee_{n=1}^{\infty} x(A_n)) \lor a') = x_a(A_1) \lor \bigvee_{n=1}^{\infty} (a \land (x(A_n) \lor a')) = x_a(A_1) \lor \bigvee_{n=1}^{\infty} x_a(A_n).
\]

The case when \( 0 \notin \bigcup_{n} A_n \) can be examined similarly.
Let \( x \) be integrable. Put \( G(t) = m(x((-\infty, t))) \), \( H(t) = m(x_a((-\infty, t))) \). Then \( H(t) \leq G(t) + 1 \). Since \( x \) is integrable, the integral \( \int_R |t| \, dH(t) \) exists. Therefore, \( \int_R |t| \, dH(t) \) and hence also \( \int_R t \, dH(t) = \int_R t \, dm_x(t) \) exists.

**Definition 2.** Let \( x \) be an integrable \( F \)-observable, \( a \in F \). Then we define

\[
\int_a x \, dm = m(x_a) = \int_R t \, dm_x(t) .
\]

### 3. SUM OF OBSERVABLES

Since our next step is the proof of the \( \sigma \)-additivity of the mapping \( a \mapsto \int_a x \, dm \), in the connection with the relation \( \chi_{A \cup B} = \chi_A + \chi_B \ (A \cap B = \emptyset) \), we must first study the sum of observables. The sum was defined in [6—8] as an \( F \)-observable \( z: B \to F \) by the formula

\[
z((-\infty, t)) = \bigvee_{r < t} x((-\infty, r)) \land y((-\infty, t - r)) , \quad t \in R .
\]

Of course, it was proved that by this formula an \( F \)-observable \( z \) is uniquely determined. It is denoted by \( z = x + y \).

**Proposition 2.** If \( a, b \in F \) are orthogonal elements (i.e. \( a \perp b \)), then \( m(x_{a \lor b}) = = m(x_a + x_b) \).

**Proof.** First observe that \( m(b) = 1 \) implies \( m(b \land c) = m(c) \) and \( m(b) = 0 \) implies \( m(b \lor c) = m(c) \). Denote \( z = x_a + x_b \). Let \( t \leq 0 \). Then

\[
m(z((-\infty, t))) = m(\bigvee_{r < t} (a \land (x((-\infty, r)) \lor a'))) \land
\]

\[
\land (b' \lor (x((-\infty, t - r) \land b)) \lor \bigvee_{t \leq r \leq 0} (a \land (x((-\infty, r)) \lor a')) \land
\]

\[
\land (b \land (x((-\infty, t - r) \lor b')) \lor \bigvee_{r > 0} (a' \lor ((x((-\infty, r)) \land a)) \land
\]

\[
\land (b \land (x((-\infty, t - r)) \lor b'))) = m(\bigvee_{r < t} (a \land (x((-\infty, r))))) \land
\]

\[
\land (b' \lor x((-\infty, t - r))) \lor \bigvee_{r \leq t} (a \land (x((-\infty, r)) \land
\]

\[
\land (b \land x((-\infty, t - r)) \lor \bigvee_{t \geq r} (a \lor x((-\infty, r)))) \land
\]

\[
\land (b \land x((-\infty, t - r))) \lor \bigvee_{r \geq 0} (a' \lor x((-\infty, r))) \land
\]

\[
\land (b \land x((-\infty, t - r))) = m((a \land x((-\infty, t)))) \lor
\]

\[
\lor (b \land x((-\infty, t))) = m((a \lor b) \land x((-\infty, t))) =
\]

\[
= m(x_{a \lor b}((-\infty, t))).
\]
If \( t > 0 \), then

\[
m(z((-\infty, t))) = m(\bigvee_{r \leq 0} ((a \land (x((-\infty, r)) \lor a'))) \land \\
\land (b' \lor (x(-\infty, t - r) \land b))) \lor \bigvee_{0 < r < t} ((a' \lor x((\infty, r)) \land a) \land \\
\land (b' \lor (x((-\infty, t - r)) \land b))) \lor \bigvee_{r \geq t} ((a' \lor x((\infty, r)) \land a) \land \\
\land (b \land x((-\infty, t - r)) \lor b'))) = m((a' \lor b') \lor x((-\infty, t))) = \\
= m((a \lor b)' \lor x((-\infty, t))) = m(x_{a\lor b}((-\infty, t))).
\]

Since the equalities hold for every \( t \in R \), we have \( m(x_{a\lor b}(D)) = m(x_a + x_b(D)) \) for every \( D \in B \).

**Proposition 3.** If \( x \) is an integrable \( F \)-observable and \( a, b \) are two orthogonal elements of \( F \), then

\[
m(x_{a\lor b}) = m(x_a) + m(x_b).
\]

**Proof.** For every \( c \in F \) we define \( Q_c : B \to \langle 0, 1 \rangle \) by the equality \( Q_c(D) = m(x_c(D \setminus \{0\})) \). Since \( 0 \notin D \setminus \{0\} \), we have

\[
Q_a(D) = m(c \land x(D \setminus \{0\})),
\]

hence

\[
Q_{a\lor b}(D) = m((a \lor b) \land x(D \setminus \{0\})) = Q_a(D) + Q_b(D).
\]

Moreover,

\[
m(x_c) = \int_R t \, dm_{x_c}(t) = \int_{R \setminus \{0\}} t \, dm_{x_c}(t) + \int_{\{0\}} t \, dm_{x_c}(t) =
\]

\[
= \int_{R \setminus \{0\}} t \, dm_{x_c}(t) = \int_R t \, dQ_c(t)
\]

for every \( c \in F \), hence

\[
m(x_{a\lor b}) = \int_R t \, dQ_{a\lor b}(t) = \int_R t \, dQ_a(t) + \int_R t \, dQ_b(t) = m(x_a) + m(x_b).
\]

4. **Properties of the Indefinite Integral**

**Proposition 4.** If \( a_n \in F \) (\( n = 1, 2, \ldots \)), \( a_n \succ a \), \( a \in F \) and \( x \) is an integrable observable, then

\[
\int_{s_n} x \, dm \to \int_a x \, dm.
\]

**Proof.** Put \( \mu_n = m_{x_{a_n}} (n = 1, 2, \ldots) \), \( \mu = m_{x_a} \), i.e.

\[
\mu_n(E) = \begin{cases} 
m(a_n \land x(E)), & \text{if } 0 \notin E \\
m(a_n' \lor x(E)), & \text{if } 0 \in E,
\end{cases}
\]

and a similar rule holds for \( \mu \). Evidently \( \mu_n(E) \prec \mu(E) \) for \( 0 \notin E \) and \( \mu_n(E) \succ \mu(E) \) if \( 0 \in E \). Moreover, \( \mu_n(E) \leq \mu(E) \) in the former case and \( \mu_n(E) \leq \mu_1(E) \) in the latter.

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Since the integrals \( \int R \mu_{n}(t) \) and \( \int R \mu(t) \) exist, for every \( \varepsilon > 0 \) there is an interval \( (a, b) \) such that
\[
\int_{R \setminus (a, b)} \mu_{n}(t) < \varepsilon, \quad \int_{R \setminus (a, b)} \mu(t) < \varepsilon.
\]

It is not difficult to see that
\[
\lim_{n \to \infty} \int_{(a, b)} \mu_{n}(t) = \int_{(a, b)} \mu(t).
\]

Therefore
\[
\left| \int_{a} \mu_{n}(t) - \int_{a} \mu(t) \right| \leq \left| \int_{R \setminus (a, b)} \mu_{n}(t) \right| + \left| \int_{R \setminus (a, b)} \mu(t) \right| < 3\varepsilon.
\]

**Theorem.** Let \( x \) be an integrable observable. For any \( a \in F \) put \( v(a) = \int_{a} x \, dm \).

Then \( v \) has the following two properties:

1. \( v(a \lor a') = v(1) \) for every \( a \in F \).

2. If \( a_{n} \in F \) (\( n = 1, 2, \ldots \)), \( a_{n} \subseteq a_{n}' (n \neq m) \), then \( \mu \left( \bigvee_{n=1}^{\infty} a_{n} \right) = \sum_{n=1}^{\infty} \mu(a_{n}) \).

**Proof.** \( v(a \lor a') = \int_{R} \mu(t) \), where \( \mu(E) = m((a \lor a') \land x(E)) \) or \( v(E) = m((a \lor a') \lor x(E)) = m(x(E)) \).

Similarly \( v(1) = \int_{R} x \, dx(t) \), where \( x(E) = m(x(E)) \) in both cases. Therefore \( \mu = \varepsilon \) and \( v(a \lor a') = v(1) \) for any \( a \in F \).

If \( c, d \) are pairwise orthogonal, then by Proposition 2 and Proposition 3
\[
v(c \lor d) = m(x_{c \lor d}) = m(x_{c} + x_{d}) = m(x_{c}) + m(x_{d}) = v(c) + v(d).
\]

Hence, by induction,
\[
v(\bigvee_{i=1}^{n} a_{i}) = \sum_{i=1}^{n} v(a_{i}).
\]

If we now put \( b_{n} = \bigvee_{i=1}^{n} a_{i} \), then \( b = \bigvee_{n=1}^{\infty} a_{i} \). Therefore by Proposition 4
\[
v(\bigvee_{i=1}^{\infty} a_{i}) = v(b) = \lim_{n \to \infty} v(b_{n}) = \lim_{n \to \infty} \sum_{i=1}^{n} v(a_{i}) = \sum_{i=1}^{\infty} v(a_{i}).
\]

**References**


Súhrn

O STREDNEJ HODNOTE V F-KVANTOVOM PRIESTORE

BELOSLAV RIEČAN

Práca sa zaobiera novým matematickým modelom pre kvantovú mechaniku, ktorý je založený na teórii fuzzy množín [1]. Definuje sa neurčitý integrál z pozorovateľnej a skúmajú sa jeho základné vlastnosti.

Резюме

О СРЕДНЕМ ЗНАЧЕНИИ В F-КВАНТОВОМ ПРОСТРАНСТВЕ

BELOSLAV RIEČAN

В работе рассматривается новая математическая модель квантовой механики, основанная на теории нечетких множеств. Определяется неопределенный интеграл от измерения, рассматриваются его основные свойства.

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