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ON MEAN VALUE IN F-QUANTUM SPACES

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Summary. The paper deals with a new mathematical model for quantum mechanics based on the fuzzy set theory [1]. The indefinite integral of observables is defined and some basic properties of the integral are examined.

Keywords: Quantum mechanics, observables, states, probability, fuzzy sets.

AMS Classification: 81C20.

1. INTRODUCTION

A new model for mechanics was suggested by A. Dvurečenskij and the author in [1] and [2]. This model was further developed e.g. in [3–5]. In [6–8], a calculus for observables was constructed. There are three basic notions in the F-quantum space theory: F-quantum space, F-observable and F-state.

F-quantum space is a family $F \subset \langle 0, 1\rangle^X$ of real functions satisfying the following properties: 1. If $f \in F$, then $f' = 1 - f \in F$. 2. If $f_n \in F$ $(n = 1, 2, \ldots)$, then $\bigvee_n f_n = \sup_n f_n \in F$.

F-observable is a σ-homomorphism from the σ-algebra B of Borel subsets of R to F, i.e. a mapping with the following two properties: 1. $x(E') = x(E)'$ for every $E \in B$. 2. $x(\bigcup_n E_n) = \bigvee_n x(E_n)$ for every $E_n \in B$ $(n = 1, 2, \ldots)$.

F-state is a mapping $m: F \to \langle 0, 1\rangle$ defined on an F-quantum space F and satisfying the following two conditions: 1. $m(a \vee a') = 1$ for every $a \in F$. 2. If $a_n \in F$ $(n = 1, 2, \ldots)$ and $a_i \leqq a'_j$ $(i \neq j)$, then $m(\bigvee_n a_n) = \sum_n m(a_n)$. Recall that the definition due to Piasecki [9] inspired our investigations.

A classical analogue of a state is a probability measure, a classical analogue of an observable is a random variable ξ defined on a probability space (Ω, S, P). To every random variable ξ an F-observable x can be assigned by the formula $x(E) = \xi^{-1}(E)$.

If x is an F-observable and m is an F-state, then the composite mapping $m \circ x$ is a probability measure on the σ-algebra B. We shall denote it by m_x, hence $m_x(E) = m(x(E))$, $E \in B$.

In a framework of the calculus constructed in [6–8], we shall construct the indefinite integral of an observable and prove its σ-additivity. Another approach to the problem is given in [10].

Recall that an F-observable x is called integrable, if the integral $\int_R t \, \mathrm{d}m_x(t)$ exists. It is then denoted by $m(x)$ and called the mean value of x. This definition is also in a full agreement with the classical one.

2. INDEFINITE INTEGRAL

Our aim is to define the indefinite integral $\int_a x \, \mathrm{d}m$, $a \in F$. This integral presents the crucial point in the concept of conditional probability. We shall follow again the classical case, where $\int_A \xi \, \mathrm{d}P = \int \chi_A \xi \, \mathrm{d}P$. Therefore we must investigate the preimages $(\xi\chi_A)^{-1}(E)$, $E \in B$. This investigation leads to the following definition.

Definition 1. *If $x: B \to F$ is an F-observable, then for every $a \in F$ and every Borel set $E \in B$ we define*

$$x_a(E) = \begin{cases} a \wedge (x(E) \vee a'), & \text{if } 0 \notin E \\ a' \vee (x(E) \wedge a), & \text{if } 0 \in E \end{cases}$$

Proposition 1. *The mapping $x_a: B \to F$ is an F-observable for any $a \in F$. If x is integrable, then x_a is integrable, too.*

Proof. If $0 \notin E$, then $0 \in E'$. Therefore

$$x_a(E') = a' \vee (x(E') \wedge a) = a' \vee (x(E)' \wedge a) = \\ = (a \wedge (x(E) \vee a'))' = (x(E))'.$$

The case $0 \in E$ can be examined similarly.

If A, B are disjoint Borel sets and $0 \notin A$, $0 \in B$, then $0 \in A \cup B$ and

$$x_a(A) \vee x_a(B) = [a \wedge (x(A) \vee a')] \vee a' \vee (a \wedge x(B)) = \\ = a' \vee (a \wedge (x(A))) \vee (a \wedge (x(B))) = \\ = a' \vee (a \wedge (x(A) \vee x(B))) = \\ = a' \vee [a \wedge x(A \cup B)] = x_a(A \cup B).$$

The case $0 \notin A$, $0 \notin B$ can be examined similarly. Now, if $A_n \in B$ $(n = 1, 2, \ldots)$ and A_n are disjoint, then 0 belongs at most to one set, say $0 \in A_1$. Then by the above

$$x_a(\bigcup_n A_n) = x_a(A_1) \vee x_a(\bigcup_{n \neq 1} A_n) = x_a(A_1) \vee (a \wedge ((\bigvee_{n \neq 1} x(A_n)) \vee a') = \\ = x_a(A_1) \vee \bigvee_{n \neq 1} (a \wedge (x(A_n) \vee a')) = \\ = x_a(A_1) \vee \bigvee_{n \neq 1} x_a(A_n) = \bigvee_n x_a(A_n).$$

The case when $0 \notin \bigcup A_n$ can be examined similarly.

Let x be integrable. Put $G(t) = m(x((-\infty, t)))$, $H(t) = m(x_a((-\infty, t)))$. Then $H(t) \leqq G(t) + 1$. Since x is integrable, the integral $\int_R |t| \, \mathrm{d}m_x(t)$ exists. Therefore, $\int_R |t| \, \mathrm{d}H(t)$ and hence also $\int_R t \, \mathrm{d}H(t) = \int_R t \, \mathrm{d}m_{x_a}(t)$ exists.

Definition 2. *Let x be an integrable F-observable, $a \in F$. Then we define*

$$\int_a x \, \mathrm{d}m = m(x_a) = \int_R t \, \mathrm{d}m_{x_a}(t) .$$

3. SUM OF OBSERVABLES

Since our next step is the proof of the σ-additivity of the mapping $a \mapsto \int_a x \, \mathrm{d}m$, in the connection with the relation $\chi_{A \cup B} = \chi_A + \chi_B$ $(A \cap B = \emptyset)$, we must first study the sum of observables. The sum was defined in [6–8] as an F-observable $z: B \to F$ by the formula

$$z((-\infty, t)) = \bigvee_{r \in Q} [x((-\infty, r)) \wedge y((-\infty, t - r))] , \quad t \in R .$$

Of course, it was proved that by this formula an F-observable z is uniquely determined. It is denoted by $z = x + y$.

Proposition 2. *If $a, b \in F$ are orthogonal elements (i.e. $a \leqq b'$), then $m(x_{a \vee b}) = m(x_a + x_b)$.*

Proof. First observe that $m(b) = 1$ implies $m(b \wedge c) = m(c)$ and $m(b) = 0$ implies $m(b \vee c) = m(c)$. Denote $z = x_a + x_b$. Let $t \leqq 0$. Then

$$m(z(-\infty, t)) = m(\bigvee_{r<t} (a \wedge (x((-\infty, r)) \vee a')) \wedge$$
$$\wedge (b' \vee (x(-\infty, t - r) \wedge b)) \vee \bigvee_{t \leqq r \leqq 0} (a \wedge (x(-\infty, r) \vee a')) \wedge$$
$$\wedge b \wedge (x(-\infty, t - r) \vee b')) \vee \bigvee_{r>0} (a' \vee ((x(-\infty, r)) \wedge a)) \wedge$$
$$\wedge b \wedge (x(-\infty, t - r) \vee b'))) = m(\bigvee_{r<t} (a \wedge x((-\infty, r))) \wedge$$
$$\wedge (b' \vee x((-\infty, t - r))) \vee \bigvee_{t \leqq r < 0} (a \wedge (x(-\infty, r)) \wedge$$
$$\wedge b \wedge x((-\infty, t - r)) \vee \bigvee_{r \geqq 0} (a' \vee x((-\infty, r))) \wedge$$
$$\wedge b \wedge x((-\infty, t - r))) = m(((a \wedge x((-\infty, t))) \vee$$
$$\vee (b \wedge x(((-\infty, t)))) = m((a \vee b) \wedge x((-\infty, t))) =$$
$$= m(x_{a \vee b}((-\infty, t))) .$$

If $t > 0$, then

$$m(z((-\infty, t))) = m(\bigvee_{r \leqq 0} ((a \wedge (x((-\infty, r)) \vee a')) \wedge$$

$$\wedge (b' \vee (x(-\infty, t - r) \wedge b))) \vee \bigvee_{0 < r < t} ((a' \vee (x((-\infty, r)) \wedge a) \wedge$$

$$\wedge (b' \vee (x((-\infty, t - r)) \wedge b))) \vee \bigvee_{r \geqq t} ((a' \vee x((-\infty, r)) \wedge a) \wedge$$

$$\wedge (b \wedge x((-\infty, t - r)) \vee b'))) = m((a' \wedge b') \vee x((-\infty, t))) =$$

$$= m((a \vee b)' \vee x((-\infty, t))) = m(x_{a \vee b}((-\infty, t))) .$$

Since the equalities hold for every $t \in R$, we have $m(x_{a \vee b}(D)) = m(x_a + x_b(D))$ for every $D \in B$.

Proposition 3. *If x is an integrable F-observable and a, b are two orthogonal elements of F, then*

$$m(x_{a \vee b}) = m(x_a) + m(x_b) .$$

Proof. For every $c \in F$ we define $Q_c: B \to \langle 0, 1 \rangle$ by the equality $Q_c(D) = m(x_c(D \setminus \{0\}))$. Since $0 \notin D \setminus \{0\}$, we have

$$Q_c(D) = m(c \wedge x(D \setminus \{0\})) ,$$

hence

$$Q_{a \vee b}(D) = m((a \vee b) \wedge x(D \setminus \{0\})) = Q_a(D) + Q_b(D) .$$

Moreover,

$$m(x_c) = \int_R t \, \mathrm{d}m_{x_c}(t) = \int_{R \setminus \{0\}} t \, \mathrm{d}m_{x_c}(t) + \int_{\{0\}} t \, \mathrm{d}m_{x_c}(t) =$$

$$= \int_{R \setminus \{0\}} t \, \mathrm{d}m_{x_c}(t) = \int_R t \, \mathrm{d}Q_c(t)$$

for every $c \in F$, hence

$$m(x_{a \vee b}) = \int_R t \, \mathrm{d}Q_{a \vee b}(t) = \int_R t \, \mathrm{d}Q_a(t) + \int_R t \, \mathrm{d}Q_b(t) = m(x_a) + m(x_b) .$$

4. PROPERTIES OF THE INDEFINITE INTEGRAL

Proposition 4. *If $a_n \in F$ $(n = 1, 2, \ldots)$, $a_n \nearrow a$, $a \in F$ and x is an integrable observable, then*

$$\int_{a_n} x \, \mathrm{d}m \to \int_a x \, \mathrm{d}m .$$

Proof. Put $\mu_n = m_{x_{a_n}}$ $(n = 1, 2, \ldots)$, $\mu = m_{x_a}$, i.e.

$$\mu_n(E) = \begin{cases} m(a_n \wedge x(E)) , & \text{if } \ 0 \notin E \\ m(a_n' \vee x(E) , & \text{if } \ 0 \in E , \end{cases}$$

and a similar rule holds for μ. Evidently $\mu_n(E) \nearrow \mu(E)$ for $0 \notin E$ and $\mu_n(E) \searrow \mu(E)$ if $0 \in E$. Moreover, $\mu_n(E) \leqq \mu(E)$ in the former case and $\mu_n(E) \leqq \mu_1(E)$ in the latter.

Since the integrals $\int_R t \, d\mu_1(t)$ and $\int_R t \, d\mu(t)$ exist, for every $\varepsilon > 0$ there is an interval $\langle a, b \rangle$ such that

$$\int_{R \setminus \langle a,b \rangle} |t| \, d\mu_1(t) < \varepsilon \, , \quad \int_{R \setminus \langle a,b \rangle} |t| \, d\mu(t) < \varepsilon \, .$$

It is not difficult to see that

$$\lim_{n \to \infty} \int_{\langle a,b \rangle} t \, d\mu_n(t) = \int_{\langle a,b \rangle} t \, d\mu(t) \, .$$

Therefore

$$\left| \int_{a_n} x \, dm - \int_a x \, dm \right| = \left| \int_R t \, d\mu_n(t) - \int_R t \, d\mu(t) \right| \leqq$$
$$\leqq \int_{R \setminus \langle a,b \rangle} |t| \, d\mu_n(t) + \int_{R \setminus \langle a,b \rangle} |t| \, d\mu_1(t) +$$
$$+ \left| \int_{\langle a,b \rangle} t \, d\mu_n(t) - \int_{\langle a,b \rangle} t \, d\mu(t) \right| < 3\varepsilon \, .$$

Theorem. *Let x be an integrable observable. For any $a \in F$ put $v(a) = \int_a x \, dm$. Then v has the following two properties:*

1. $v(a \vee a') = v(1)$ *for every* $a \in F$.
2. *If* $a_n \in F$ $(n = 1, 2, \ldots)$, $a_n \leqq a'_m$ $(n \neq m)$, *then* $\mu\left(\bigvee_{n=1}^{\infty} a_n\right) = \sum_{n=1}^{\infty} \mu(a_n)$.

Proof. $v(a \vee a') = \int_R t \, d\mu(t)$, where $\mu(E) = m((a \vee a') \wedge x(E))$ or $\mu(E) = m((a \vee a')' \vee x(E)) = m(x(E))$. Similarly $v(1) = \int_R t \, d\varkappa(t)$, where $\varkappa(E) = m(x(E))$ in both cases. Therefore $\mu = \varkappa$ and $v(a \vee a') = v(1)$ for any $a \in F$.

If c, d are pairwise orthogonal, then by Proposition 2 and Proposition 3

$$v(c \vee d) = m(x_{c \vee d}) = m(x_c + x_d) = m(x_c) + m(x_d) = v(c) + v(d) \, .$$

Hence, by induction,

$$v\left(\bigvee_{i=1}^{n} a_i\right) = \sum_{i=1}^{n} v(a_i) \, .$$

If we now put $b_n = \bigvee_{i=1}^{n} a_i$, then $b = \bigvee_{n=1}^{\infty} b_n = \bigvee_{i=1}^{\infty} a_i$. Therefore by Proposition 4

$$v\left(\bigvee_{i=1}^{\infty} a_i\right) = v(b) = \lim_{n \to \infty} v(b_n) = \lim_{n \to \infty} \sum_{i=1}^{n} v(a_i) = \sum_{i=1}^{\infty} v(a_i) \, .$$

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Súhrn

O STREDNEJ HODNOTE V F-KVANTOVOM PRIESTORE

BELOSLAV RIEČAN

Práca sa zaoberá novým matematickým modelom pre kvantovú mechaniku, ktorý je založený na teórii fuzzy množín [1]. Definuje sa neurčitý integrál z pozorovateľnej a skúmajú sa jeho základné vlastnosti.

Резюме

О СРЕДНЕМ ЗНАЧЕНИИ В F-КВАНТОВОМ ПРОСТРАНСТВЕ

BELOSLAV RIEČAN

В работе рассматривается новая математическая модель квантовой механики, основанная на теории нечетких множеств. Опрeделяется неопределенный интеграл от измерения, рассматриваются его основные свойства.

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