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INTERVAL SOLUTIONS OF LINEAR INTERVAL EQUATIONS

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Summary. It is shown that if the concept of an interval solution to a system of linear interval equations given by Ratschek and Sauer is slightly modified, then only two nonlinear equations are to be solved to find a modified interval solution or to verify that no such solution exists.

Keywords: linear systems, interval arithmetic.


In this paper we shall deal with the following problem. Given a square interval matrix \( A^I = [A^-, A^+] = \{A; A^- \leq A \leq A^+\} \), where \( A^- = (a^-_{ij}) \) and \( A^+ = (a^+_{ij}) \) are \( n \times n \) matrices, and an interval vector \( b^I = [b^-, b^+] = \{b; b^- \leq b \leq b^+\} \) with \( b^- = (b^-_i) \), \( b^+ = (b^+_i) \), find an interval \( n \)-vector \( x^I = [x^-, x^+] \) such that

\[
\sum_{j=1}^{n} [a^-_{ij}, a^+_{ij}] \cdot [x^-_j, x^+_j] = [b^-_i, b^+_i] \quad (i = 1, \ldots, n)
\]

holds, where the operations involved are performed in interval arithmetic and are generally defined by

\[
[a^-, a^+] \circ [\beta^-, \beta^+] = \{a \circ \beta; \, a \in [a^-, a^+], \, \beta \in [\beta^-, \beta^+]\}
\]

for \( \circ \in \{+, -, \cdot, /\} \), which amounts to

\[
[a^-, a^+] + [\beta^-, \beta^+] = [a^- + \beta^-, a^+ + \beta^+]
\]
\[
[a^-, a^+] - [\beta^-, \beta^+] = [a^- - \beta^+, a^+ - \beta^-]
\]
\[
[a^-, a^+] \cdot [\beta^-, \beta^+] = [\min \{a^- \beta^-, a^- \beta^+, a^+ \beta^-, a^+ \beta^+\},
\max \{a^- \beta^-, a^- \beta^+, a^+ \beta^-, a^+ \beta^+\}]
\]
\[
[a^-, a^+] / [\beta^-, \beta^+] = [a^-, a^+] \cdot \frac{1}{[\beta^-, \beta^+]},
\]

where

\[
\frac{1}{[\beta^-, \beta^+]^T} = \left[\frac{1}{\beta^-}, \frac{1}{\beta^+}\right]
\]

provided \( 0 \notin [\beta^-, \beta^+] \).
(for interval arithmetic, see e.g. [4]). This concept of solution was formulated for interval systems with arbitrary \(m \times n\) interval matrices by Ratschek and Sauer [7] and solved there for the case \(m = 1\). It seems that a general solution to (1) is not yet known; cf. also Nickel [5]. In this paper we shall show that systems of type (1) with square regular interval matrices can be solved if we impose an additional restriction upon the concept of a solution in the following sense:

**Definition.** Given \(A^I\) (square) and \(b^I\), an interval vector \(x^I\) is called a **strong solution** if it satisfies (1) and if there exist \(A', A'' \in A^I\) and \(x', x'' \in x^I\) such that \(A'x' = b^-, A''x'' = b^+\) hold.

Let us first introduce

\[
A_c = \frac{1}{2}(A^- + A^+), \\
\Delta = \frac{1}{2}(A^+ - A^-),
\]

so that \(\Delta \geq 0\) and \(A^- = A_c - \Delta, A^+ = A_c + \Delta\). We shall show that the problem of finding a strong solution is closely connected with solving the nonlinear equations

\[
\begin{align*}
A_c x - \Delta |x| &= b^- , \\
A_c x + \Delta |x| &= b^+ 
\end{align*}
\]

where \(x = (x_j)\) is a real (not interval) vector and the absolute value is defined as \(|x| = (|x_j|)\). We shall need some results about solutions to (2.1), (2.2). A square interval matrix \(A^I\) is called regular if each \(A \in A^I\) is nonsingular.

**Theorem 1.** Let \(A^I\) be regular. Then the equations (2.1), (2.2) have unique solutions \(x_1\) and \(x_2\), respectively.

For the proof of this result, see [8], Theorem 1.2. To solve (2.1) and (2.2), we may observe that \(|x| = Dx\), where \(D\) is a diagonal matrix with \(D_{jj} = 1\) if \(x_j \geq 0\) and \(D_{jj} = -1\) otherwise. Then (2.1) can be written as a system of linear equations \((A_c - \Delta D)x = b^-\), where \(D\) must be found such that \(Dx(= |x|) \geq 0\). This is the underlying idea of the following algorithm:

**Algorithm 1** (for solving (2.1), (2.2)).

**Step 0.** Set \(D = E\) (unit matrix).

**Step 1.** Solve \((A_c - \Delta D)x = b^-\) (for (2.2): \((A_c + \Delta D)x = b^+)\).

**Step 2.** If \(Dx \geq 0\), set \(x_1 := x\) (or, \(x_2 := x\)) and terminate.

**Step 3.** Otherwise find \(k = \min \{j; D_{jj}x_j < 0\}\).

**Step 4.** Set \(D_{kk} := -D_{kk}\) and go to Step 1.

The algorithm is general, as the following result shows:
Theorem 2. Let $A^f$ be regular. Then Algorithm 1 is finite, passing through Step 1 at most $2^n$ times.

The proof of this theorem can be again found in [8]. Another possibility, though not general, for solving (2.1) (similarly, (2.2)) consists in reformulating (2.1) as a fixed-point equation

$$x = A_c^{-1} A|x| + A_c^{-1} b^-$ $

which may be solved iteratively by

$$x^0 = A_c^{-1} b^-,$$

$$x^{i+1} = A_c^{-1} A|x^i| + A_c^{-1} b^- \quad (i = 0, 1, \ldots),$$

but convergence of $\{x^i\}$ to $x_1$ can be established only under the assumption that $q\left(\left|A_c^{-1}\right| A\right) < 1$, which is not always the case with regular interval matrices; nevertheless, if $A$ is of small norm, then the iterative method is to be preferred.

Returning now back to our original problem of finding a strong solution, we shall show in the next theorem that if strong solutions exist at all, then one of them can be easily expressed by means of the above vectors $x_1, x_2$. Since generally neither $x_1 \leq x_2$, nor $x_1 \geq x_2$ holds, we introduce $\min \{x_1, x_2\}$ as the vector with components $\min\{(x_1)_j, (x_2)_j\} (j = 1, \ldots, n)$, and similarly for $\max \{x_1, x_2\}$.

Theorem 3. Let $A^f$ be regular and let (1) have a strong solution. Then $x^I = \left[ x^-, x^+ \right]$, given by

$$x^+ = \max \{x_1, x_2\},$$

is also a strong solution.

Proof. Let $\bar{x}^I$ be a strong solution. Then there exist $A', A'' \in A^f$ and $x', x'' \in \bar{x}^I$ such that $A'x' = b^-, A''x'' = b^+$ hold. Due to the definition of interval operations, the resulting left-hand side interval vector in (1) contains all vectors of the form $Ax', A \in A^f$. On the other hand, according to the theorem by Oettli and Prager [6], we have $\{Ax'; A \in A^f\} = \left[ A_c x' - A|x'|, A_c x' + A|x'| \right]$. Since $A'x' = b^-$, we conclude that

$$A_c x' - A|x'| = b^-$$

holds, implying $x' = x_1$ in view of the uniqueness of the solution to (2.1) stated in Theorem 1. In a similar way we would obtain that $x'' = x_2$. Now, for $x^I$ given by (3), the interval vector with the components

$$\sum_{j=1}^{n} [a_{ij}^{-}, a_{ij}^{+}] \cdot [x_j^{-}, x_j^{+}] \quad (i = 1, \ldots, n)$$

is contained in $b^I$ since $x^I \subseteq \bar{x}^I$, but also contains $b^-$ and $b^+$ since $x_1, x_2 \in x^I$; hence it equals $b^I$, so that (1) holds and $x^I$ is a strong solution. Q.E.D.
Summing up the results, we can formulate the following algorithm for solving our problem:

**Algorithm 2** (finding a strong solution)

**Step 1.** Solve (2.1), (2.2) (by Algorithm 1 or iteratively) to find $x_1, x_2$.

**Step 2.** Construct $x^I$ by (3).

**Step 3.** If $x^I$ satisfies (1), stop: $x^I$ is a strong solution.

**Step 4.** Otherwise stop: no strong solution exists.

The algorithm works provided $A^I$ is regular, which is the case e.g. if the spectral radius of $|A^{-1}| A$ is less than 1 (Beeck [2]), a condition widely satisfied in practice.

We add two small examples with regular matrices to illustrate the possible outcomes.

**Example 1** (Hansen [3]). Let

$$A^- = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}, \quad A^+ = \begin{pmatrix} 3 & 1 \\ 2 & 3 \end{pmatrix}$$

and $b^- = (0, 60)^T$, $b^+ = (120, 240)^T$. Solving (2.1), (2.2), we obtain

$$x_1 = (0, 30)^T, \quad x_2 = \left(\frac{120}{7}, \frac{480}{7}\right)^T,$$

and

$$x^I = ([0, \frac{120}{7}], [30, \frac{480}{7}])^T$$

satisfies (1), therefore it is a strong solution.

**Example 2** (Barth and Nuding [1]). Let

$$A^- = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \quad A^+ = \begin{pmatrix} 4 & 1 \\ 2 & 4 \end{pmatrix}$$

and $b^- = (-2, -2)^T$, $b^+ = (2, 2)^T$. Here $x^I$ does not satisfy (1), so that no strong solution exists.

A preliminary version of this paper appeared in [9].

**References**


Souhrn

INTERVALOVÁ ŘEŠENÍ SOUSTAV LINEÁRNÍCH INTERVALOVÝCH ROVNIC
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Je zavedeno modifikované intervalové řešení soustavy lineárních intervalových rovnic, k jehož výpočtu je třeba vyřešit dvě soustavy nelineárních rovnic.

Резюме

ИНТЕРВАЛЬНЫЕ РЕШЕНИЯ СИСТЕМ ЛИНЕЙНЫХ
ИНТЕРВАЛЬНЫХ УРАВНЕНИЙ
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В статье показано, как можно вычислить модифицированное интервальное решение системы линейных интервальных уравнений путём решения двух систем нелинейных уравнений.

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