Jan Franců

Monotone operators. A survey directed to applications to differential equations

Aplikace matematiky, Vol. 35 (1990), No. 4, 257–301

Persistent URL: http://dml.cz/dmlcz/104411

35 (1990) APLIKACE MATEMATIKY No. 4, 257—301

MONOTONE OPERATORS

A SURVEY DIRECTED TO APPLICATIONS TO DIFFERENTIAL EQUATIONS

JAN FRANCŮ

Dedicated to my teachers Kornélia Kropiláková and Matylda Ziková — whom I am grateful for teaching me the foundations of mathematics — on the occasion of their 75^{th} *birthday.*

(Received January 10, 1988)

Summary. The paper deals with the existence of solutions to equations of the form $Au = b$ with operators monotone in a broader sense, including pseudomonotone operators and operators satisfying conditions S and M. The first part of the paper which has a methodical character is concluded by the proof of an existence theorem for the equation on a reflexive separable Banach space with a bounded demicontinuous coercive operator satisfying condition $(M)_0$. The second part which has a character of a survey compares various types of continuity and monotony and introduces further results. Application of this theory to proofs of existence theorems for boundary value problems for ordinary and partial differential equations is illustrated by examples.

Keywords: monotone, pseudomonotone operators, operators satisfying S, M conditions, existence theorems for boundary value problems for differential equations.

AMS Classification: 35-02, 35A05, 47H05, 35F30.

INTRODUCTION

Theory of monotone operators represents — besides variational methods — an essential functional-analytical method for the investigation of nonlinear equations. In the paper we give a survey of the theory of monotone operators in a broader sense including also the generalization of the concept of monotony, e.g. pseudomonotony, S — conditions and M — conditions. The survey is directed to existence theorems for boundary value problems for nonlinear ordinary and partial differential equations.

The literature on monotone operators is very extensive, containing hundreds of papers and many monographs. One can find various formulations of existence theorems. Writing the paper I aimed to find a general theorem which would imply the other theorems, using comparison of various types of continuity and monotony.

The aim required some restriction of the problems considered. Therefore we shall confine ourselves to the case of the equations $Au = b$ on a reflexive Banach space V with a coercive (single-valued) operator A acting from V to its dual space V'. Thus we do not deal with multivalued monotone mappings. Further, we omit monotone operator theorems for variational inequalities, Hammerstein integral equations, non-stationary problems and other domains, where the concept of the monotone operator is used.

The first five sections are rather of a methodical character. The existence theorems are treated starting with equations in a one-dimensional space and ending with the main theorem for abstract equations in infinite-dimensional Banach spaces. A special case of strongly monotone Lipschitz-continuous operators is studied in Section 4.

The next two sections have a suveying character. In Section 6 various types of continuity and monotony are compared. Since the terminology is not unified, definitions of the concepts considered are introduced. The "graphical form" of some comparison theorems was inspired by the book [14]. A lemma on pseudomonotony of operators with monotony in the principal part is added. Section 7 contains some consequences to the main theorem and some special results provided the assumptions are stronger. Theorem 7.5 (which is not a consequence of the main theorem) is introduced for its elegant proof using the Minty lemma. In the end some remarks on variational inequalities and maximal monotone multivalued mappings are introduced without proofs.

The last Section 8 contains four examples of applications of the theory to boundary value problems for differential equations. Due to the limited extent of the paper the presentation of the examples is by no means exhaustive or complete. The reformulation of differential equations into abstract operator equations is only outlined, the excellent text book [4] may be recommended for details. Historical remarks close the paper.

Textbooks dealing with monotone operators are [2], [4], [5] in Czech; [14] in German and e.g. [1], [4], [11], [14] in English. The book [12] is a comprehensive monograph.

Although most definitions, properties and theorems are taken from [4], [6], [7], [8], [11], [12], [14] and [15], some results seem to be new – namely condition $(M)_0$ in 6.6, Theorem 6.8 (b), (c) and the comprehensive formulation of comparison theorems 6.2, 6.5, 6.7. The second example in the remark following Lemma 6.2 as well as Example 8.18 were also constructed for this paper.

1. MOTIVATION – ONE DIMENSIONAL CASE

The main subject of this paper is to examine conditions ensuring the existence of a solution of an abstract equation $Au = b$ with a monotone operator A.

We start with the simplest case. A monotone operator on R – the real line – is any non-decreasing function $f: R \to R$, and a strictly monotone operator is any increasing function. A prototype of theorems on monotone operators is the following theorem (see Fig. 1):

1.1. Theorem. *Let f be a real monotone continuous function on an interval (a, b) $[-\infty \leqq a < b \leqq +\infty]$. Then the equation*

$$(1.1) \qquad f(x) = y$$

has a solution for all $y \in (A, B)$, where

$$A = \lim_{x \to a+} f(x) \quad \text{and} \quad B = \lim_{x \to b-} f(x) .$$

If the function f is strictly monotone then the solution is unique.

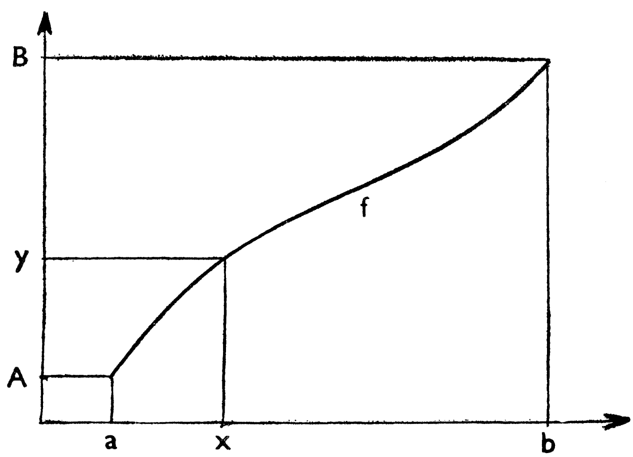


Fig. 1

The theorem implies

1.2. Theorem. *Let $f: R \to R$ be a continuous monotone function satisfying*

$$(1.2) \qquad \lim_{x \to -\infty} f(x) = -\infty , \quad \lim_{x \to +\infty} f(x) = +\infty .$$

Then the equation (1.1) has a solution for each $y \in R$.

1.3. Remarks.

(a) Provided the limits A, B exist, the assumption of monotony can be omitted.

(b) In order to generalize the condition of monotony

$$x_1 < x_2 \Rightarrow f(x_1) \leqq f(x_2)$$

to more-dimensional cases, we rewrite it in the form

$$(1.3) \qquad (f(x_2) - f(x_1))(x_2 - x_1) \geqq 0 .$$

The condition (1.2), which can be rewritten in the form

$$\lim_{|x| \to \infty} \frac{f(x)\, x}{|x|} = \infty ,$$

is called the coercivity and plays an important role in existence theorems.

2. FINITE DIMENSIONAL CASE

We will consider a mapping $f: R^n \to R^n$. We rewrite the definitions of monotony and coercivity in terms of the scalar product denoted by (x, y).

2.1. Definition. *The mapping $f: R^n \to R^n$ is called*
monotone iff

$$(2.1) \qquad (f(x_1) - f(x_2),\ x_1 - x_2) \geqq 0 \quad \forall x_1, x_2 \in R^n ,$$

strictly monotone iff

$$(2.2) \qquad (f(x_1) - f(x_2),\ x_1 - x_2) > 0 \quad \forall x_1, x_2 \in R^n , \quad x_1 \neq x_2 ,$$

coercive iff

$$(2.3) \qquad \lim_{|x| \to \infty} \frac{(f(x), x)}{|x|} = +\infty .$$

Let us start with an existence theorem for the closed ball $B_r = \{x \in R^n, |x| \leqq r\}$.

2.2. Theorem. *Let $f: B_r \to R^n$ be a continuous mapping satisfying on the boundary the condition*

$$(2.4) \qquad (f(x), x) > 0 \quad \forall x,\ |x| = r .$$

Then there exists at least one solution $x \in B_r$ of the equation

$$(2.5) \qquad f(x) = 0 .$$

2.3. Remarks.

(a) The condition (2.4) has an intuitive geometrical meaning. The values of the mapping f – the vector field in Fig. 2 – are directed outwards of the ball B_r on the boundary of this ball. Indeed, the scalar product $(f(x), x) = |f(x)|\,|x| \cos \varphi > 0$ implies that the vectors $f(x)$, x form an acute angle φ. From Figure 2 it is obvious that the continuous vector field must contain a zero vector in the ball.

(b) The theorem holds with a weakened condition (2.4) $(f(x), x) \geqq 0\ \forall x,\ |x| = r$.

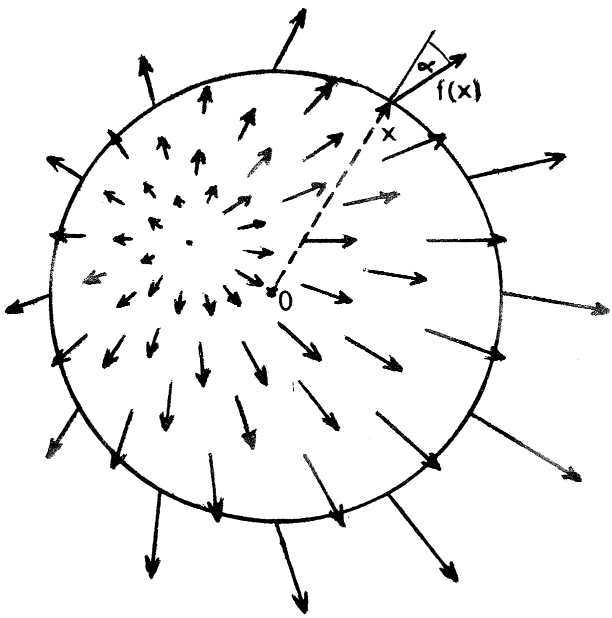


Fig. 2

We introduce a proof based on the Brouwer fixed point theorem.

2.4. Theorem (Brouwer). *Let g be a continuous mapping from B_r into itself – $g: B_r \to B_r$. Then the mapping has a fixed point, i.e. there exists $x \in B_r$ such that $g(x) = x$.*

The proof of this fundamental theorem is not simple, therefore we only refer e.g. to [4] for a proof via the topological degree or to [13] for a classical proof using homological algebra.

Proof of Theorem 2.2. We convert the problem of solving equation (2.5) into a fixed point problem. An element x is a solution of the equation $f(x) = 0$ iff x is a fixed point of a mapping g_ε ($\varepsilon > 0$) defined by

$$g_\varepsilon(x) = x - \varepsilon f(x)\,. \tag{2.6}$$

The mapping is continuous. In order to be able to use the Brouwer theorem we find a constant $\varepsilon > 0$ such that g_ε maps B_r into itself.

The mapping is continuous on the compact ball B_r, therefore it is bounded on B_r, i.e.

$$|f(x)| \leqq L \quad \forall x,\ |x| \leqq r\,.$$

Moreover, the condition (2.4) on the boundary ∂B_r for a continuous mapping on a compact implies $(f(x), x) \geqq K$ on ∂B_r for a constant $K > 0$, and further the same inequality with a smaller constant in a neighbourhood of B_r, i.e.

$$(f(x), x) \geqq K/2 \quad \forall x,\ |x| \in (\varrho, r]\,,\quad \varrho < r\,.$$

The estimates

$$\begin{aligned} |g_\varepsilon(x)|^2 &= |x|^2 - 2\varepsilon(f(x), x) + \varepsilon^2|f(x)|^2 \leqq \\ &\leqq r^2 - 2\varepsilon K/2 + \varepsilon^2 L^2 \quad \forall x,\ |x| \in (\varrho, r] \\ &\leqq \varrho^2 + 2\varepsilon\varrho L \ + \varepsilon^2 L^2 \quad \forall x,\ |x| \in [0, \varrho] \end{aligned}$$

imply the existence of a constant $\varepsilon > 0$ such that

$$|g_\varepsilon(x)| \leqq r \quad \forall x \in B_r\,.$$

The Brouwer fixed point theorem implies the existence of a fixed point of g_ε. Thus equation (2.5) has a solution. □

The finite-dimensional version of Theorem 1.2 is a simple consequence of Theorem 2.2:

2.5. Theorem. *Let $f: R^n \to R^n$ be a continuous coercive mapping. Then the equation*

$$f(x) = y \tag{2.7}$$

has at least one solution for arbitrary $y \in R^n$.

Proof. Let $y \in R^n$. The mapping f si coercive, which implies

$$(f(x) - y, x) > 0 \quad \forall x,\ |x| = r$$

for a sufficiently large $r > 0$, and the result follows. □

3. INFINITE DIMENSIONAL CASE — INTRODUCTION

Let us pass to operators on infinite dimensional spaces. Let V be a Banach space — a linear vector space with a norm $\|\cdot\|$ and complete with respect to this norm. The dual space (also called the adjoint space) — the space of all continuous linear functionals on V — is denoted by V' and the value of a functional $f \in V'$ at a point $u \in V$ is denoted by $\langle f, u\rangle$. The space V' is also a Banach space with the norm $\|f\| = \sup\{|\langle f, u\rangle|,\ u \in V,\ \|u\| = 1\}$.

We will consider an operator $A: V \to V'$ and look for a solution to the equation

$$(3.1) \qquad Au = b$$

with the right-hand side $b \in V'$. The equality (3.1) is understood in the sense of equality of functionals Au and b in the dual space V', i.e.

$$(3.1') \qquad \langle Au, v \rangle = \langle b, v \rangle \quad \forall v \in V.$$

The equation (3.1) is an abstract formulation of many problems, e.g. boundary value problems for ordinary differential equations and stationary partial differential equations – see Section 8.

Definition 2.1 of monotony and coercivity can be simply rewritten for operators A on a Banach space V by replacing the scalar product $(f(x), x)$ in R^n by the duality $\langle Au, u \rangle$ on the space V. The strict monotony directly implies the uniqueness of the solution:

3.1. Theorem. *Let the operator $A: V \to V'$ be strictly monotone, i.e.*

$$(3.2) \qquad \langle Au_1 - Au_2, u_1 - u_2 \rangle > 0 \quad \forall u_1, u_2 \in V, \quad u_1 \neq u_2 .$$

Then equation (3.1) has at most one solution.

Proof. Supposing the equation has two solutions $u_1, u_2 \in V$, we have $Au_1 = Au_2$ and $\langle Au_1 - Au_2, u_1 - u_2 \rangle = 0$. Due to condition (3.2) we conclude $u_1 = u_2$. □

Infinite-dimensional spaces bring some difficulties. A closed bounded set, e.g. the ball $B_r = \{u \in V, \|u\| \leqq r\}$, is not compact in general, which implies e.g. that a bounded sequence need not contain a convergent subsequence. In addition, a continous mapping on B_r need not be bounded.

This is the reason why we introduce the following concept: Besides the strong convergence

$$(3.3) \qquad u_n \to u \quad \text{iff} \quad \|u_n - u\| \to 0$$

on a Banach space V we introduce the weak convergence on V, denoted by a half-arrow,

$$(3.4) \qquad u_n \rightharpoonup u \quad \text{iff} \quad \langle b, u_n - u \rangle \to 0 \quad \forall b \in V' .$$

Similarly, on the dual space V' we have the strong and weak convergences:

$$b_n \to b \quad \text{iff} \quad \|b_n - b\|_{V'} \to 0 ,$$

$$b_n \rightharpoonup b \quad \text{iff} \quad \langle \varphi, b_n - b \rangle \to 0 \quad \forall \varphi \in V'' ,$$

where V'' it the second dual space, i.e. the space of linear continuous functionals on V'. We can get some elements of V'' if we assign to each $u \in V$ a functional $\varphi \in V''$ by the relation $\langle \varphi, b \rangle = \langle b, u \rangle$, but in general we do not obtain the whole space V''.

The spaces in which V can be identified with V'' by this canonical imbedding are called reflexive. In these spaces the weak convergence on V' can be defined as

$$(3.5) \qquad b_n \rightharpoonup b \quad \text{iff} \quad \langle b_n - b, v \rangle \to 0 \quad \forall v \in V .$$

Moreover, the weak convergence makes bounded closed convex subsets of infinite-dimensional reflexive spaces compact:

3.2. Theorem. *In a reflexive Banach space the closed ball B_r is weakly sequentialy compact, i.e. each bounded sequence contains a weakly convergent subsequence.*

The theorem is a special case of the Eberlein-Schmulian theorem, which moreover asserts that if the ball B_r is weakly sequentially compact then the Banach space is reflexive, see e. g. [4], [9], [12]. In finite dimensional spaces both the strong and weak convergences coincide.

4. STRONGLY MONOTONE OPERATORS

The operators satisfying the monotony condition in a stronger form,

$$(4.1) \qquad \langle Au_1 - Au_2, u_1 - u_2 \rangle > \alpha \|u_1 - u_2\|^2 \quad \forall u_1, u_2 \in V \ (\alpha > 0) ,$$

are called strongly monotone operators. These operators forming a special subclass of monotone operators are in a certain sense close to linear elliptic operators. Existence and unicity of solutions can be easily proved by the Banach fixed point theorem.

In this section we restrict ourselves to the case that V is a Hilbert space. In this case we can identify the functionals from V' with the elements from V and replace the duality $\langle b, v \rangle$ by the scalar product (b, v).

4.1. Theorem on strongly monotone operators. *Let V be a Hilbert space and $A: V \to V$ an operator which is*

– strongly monotone, i.e. there exists $\alpha > 0$ such that

$$(4.1') \qquad (Au_1 - Au_2, u_1 - u_2) \geqq \alpha \|u_1 - u_2\|^2 \quad \forall u_1, u_2 \in V ,$$

– and Lipschitz continuous, i.e. there exists $M > 0$ such that

$$(4.2) \qquad \|Au_1 - Au_2\| \leqq M \|u_1 - u_2\| \quad \forall u_1, u_2 \in V .$$

Then the equation

$$(4.3) \qquad Au = b$$

has a unique solution for each $b \in V$.

Proof. Again we convert the problem of solving equation (4.3) into a fixed point problem. This equation has a solution u iff u is a fixed point of the mapping $T_\varepsilon(u) = u - \varepsilon(Au - b)$ for a constant $\varepsilon > 0$.

We shall find $\varepsilon > 0$ such that T_ε is a contractive mapping, i.e. $\|T_\varepsilon(u_1) - T_\varepsilon(u_2)\| \leqq \leqq c\|u_1 - u_2\|$ with a constant $c < 1$. In the estimate we use inequalities (4.1′), (4.2):

$$\begin{aligned}\|T_\varepsilon(u_1) - T_\varepsilon(u_2)\|^2 &= \|(u_1 - u_2) - \varepsilon(Au_1 - Au_2)\|^2 = \\ &= \|u_1 - u_2\|^2 - 2\varepsilon(Au_1 - Au_2, u_1 - u_2) + \varepsilon^2\|Au_1 - Au_2\|^2 \leqq \\ &\leqq \|u_1 - u_2\|^2 (1 - 2\varepsilon\alpha + \varepsilon^2 M^2)\,.\end{aligned}$$

For $\varepsilon = \alpha/M^2$ the constant $c = (1 - 2\varepsilon\alpha + \varepsilon^2 M^2)^{1/2} = (1 - \alpha^2/M^2)^{1/2}$ is less than 1. The mapping T_ε is a contractive mapping on the complete metric space, therefore, following the Banach fixed point theorem, T_ε has a unique fixed point u which is the unique solution of (4.3). □

4.2. Remarks.

(a) If A is a linear operator on a Hilbert space the condition (4.1′) is equivalent to the so-called ellipticity condition

$$(Au, u) \geqq \alpha\|u\|^2 \quad \forall u \in V\,.$$

Similarly, for linear operators the conditions of Lipschitz continuity, continuity, continuity at 0 and boundedness are equivalent and (4.2) can be replaced by

$$\|Au\| \leqq M\|u\| \quad \forall u \in V\,.$$

In this way we come to the well known Lax-Milgram lemma.

(b) The proof of Theorem 4.1 by means of the Banach fixed point theorem is constructive and yields an important approximate method. The sequence of approximate solutions $\{u_k\}$ defined by

$$(4.4) \qquad u_0 \in V - \text{arbitrary}\,, \quad u_{k+1} = T_\varepsilon(u_k)\,, \quad k = 0, 1, 2, \ldots$$

converges in the norm to the solution of equation (4.3). We can compute even the rate of convergence. The estimate $\|u_{k+1} - u_k\| \leqq c^k\|T_\varepsilon(u_0) - u_0\|$ (c is the constant of contractivity, $c < 1$) yields

$$(4.5) \qquad \|u - u_k\| \leqq \frac{c^k}{1 - c}\|T_\varepsilon(u_0) - u_0\|\,.$$

5. MAIN EXISTENCE THEOREM

We shall generalize Theorem 2.5 to operators on infinite-dimensional spaces. The assumptions introduced above are "fit to measure" for the proof of the theorem. In Section 7 we replace these assumptions by more natural ones.

We shall deal with the existence of a solution to the equation

$$(5.1) \qquad Au = b$$

with an operator $A: V \to V'$ and $b \in V'$. The problem can be written in the following form:

$$\text{Find } u \in V \text{ such that}$$

$$(5.1') \qquad \langle Au, v \rangle = \langle b, v \rangle \quad v \in V.$$

In order to be able to use Theorem 2.5 we define a restriction of the problem to a finite dimensional subspace V_n — the so-called Galerkin approximation.

5.1. Definition. *Let V_n be a subspace of V. The problem*

$$\textit{Find } u_n \in V_n \textit{ such that}$$

$$(5.2) \qquad \langle Au_n, v \rangle = \langle b, v \rangle \quad \forall v \in V_n$$

is called the Galerkin approximation of problem (5.1′) *on the subspace V_n.*

In other words, we have restricted the functionals Au_n, b to the subspace V_n.

In sequel we denote the strong or weak convergence by an arrow or half-arrow, respectively, see (3.3)–(3.5).

5.2. Main Theorem. *Let V be a reflexive separable Banach space and $A: V \to V'$ an operator which is*

— *coercive, i.e.*

$$(5.3) \qquad \lim_{\|u\| \to \infty} \frac{\langle Au, u \rangle}{\|u\|} = \infty\,,$$

— *continuous on finite-dimensional subspaces,*

— *bounded, i.e. there exists an increasing function $M: R^+ \to R^+$ such that*

$$(5.4) \qquad \|Au\|_{V'} \leqq M(\|u\|_V) \quad \forall u \in V\,,$$

— *and satisfying the so-called condition $(M)_0$, i.e.*

$$(5.5) \qquad \left.\begin{array}{l} u_n \rightharpoonup u\,, \quad Au_n \rightharpoonup b \\ \langle Au_n, u_n \rangle \to \langle b, u \rangle \end{array}\right\} \Rightarrow Au = b\,.$$

Then A is surjective, i.e. equation (5.1) *has a solution for each $b \in V'$. Moreover, A^{-1} as a multivalued mapping is bounded, i.e. there exists an increasing function $N: R^+ \to R^+$ such that*

$$(5.6) \qquad \|u\|_V \leqq N(\|Au\|_{V'}) \quad \forall u \in V\,.$$

Proof. We divide the proof into four steps:

(1) We construct a sequence of finite dimensional subspaces V_n. In this way we obtain a sequence of Galerkin approximations of problem (5.1).

(2) By means of Theorem 2.5 we prove the existence of a solution of problem (5.2). In this way we obtain a sequence of approximate solutions u_n.

(3) We prove that the sequences $\{u_n\}, \{Au_n\}$ contain weakly convergent subsequences: $u_{n_k} \rightharpoonup u$, $Au_{n_k} \rightharpoonup f$.

(4) We prove that u is a solution of equation (5.1).

In each step we indicate the assumptions being used.

1[st] *step.* By the definition, a separable space V contains a countable dense subset. We choose a linearly independent sequence

$$\{w_1, w_2, w_3, \ldots\}, \quad \|w_i\| = 1$$

which generates a sequence of finite dimensional subspaces V_n:

$$V_n = \operatorname{span}\{w_1, w_2, \ldots, w_n\}.$$

The sequence of subspaces V_n obviously has the following approximative property:

$$(5.7) \qquad \forall v \in V \quad \exists\{v_n\}, \quad v_n \in V_n \quad \text{such that} \quad v_n \to v.$$

By the Galerkin approximation (5.2) we restrict equation (5.1) to the finite dimensional subspaces V_n.

2[nd] *step.* We make use of the existence Theorem 2.5. The space V_n is isomorphic to R^n. The operator $A_n = A|_{V_n}$ induces a mapping f: $R^n \to R^n$:

$$\begin{array}{ccc} f\colon x \in R^n & \mapsto & \{\langle A(\sum x_i w_i), w_j\rangle\}_{j=1}^n \in R^n \\ \updownarrow & & \updownarrow \\ A_n\colon \sum x_i w_i \in V_n & \mapsto & A(\sum x_i w_i)|_{V_n} \in V_n'. \end{array}$$

We have transformed equation (5.2) into an equation of the form $f(x) = y$, where $y = \{\langle b, w_i\rangle\}_i \in R^n$. The continuity on the finite dimensional subspace V_n and the coercivity of the operator A yields the continuity and coercivity of the mapping f on R^n. Due to Theorem 2.5 there exists a solution x of the equation $f(x) = y$ and thus also a solution $u_n = \sum x_i w_i$ of the approximation (5.2).

3[rd] *step.* We prove that the sequence $\{u_n\}$ is bounded. The coercivity (5.3) implies the existence of an increasing function N: $R^+ \to R^+$ such that for all $u \in V$

$$\|u\| > N(r) \Rightarrow \frac{\langle Au, u\rangle}{\|u\|} > r.$$

Transposition of this implication yields

$$(5.8) \qquad \frac{\langle Au, u\rangle}{\|u\|} \leqq r \Rightarrow \|u\| \leqq N(r).$$

The approximate solutions u_n of equation (5.2) satisfy

$$\frac{\langle Au_n, u_n\rangle}{\|u_n\|} = \frac{\langle b, u_n\rangle}{\|u_n\|} \leqq \|b\|_{V'},$$

which due to (5.8) implies the boundedness of $\{u_n\}$,

$$\|u_n\|_V \leqq N(\|b\|_{V'}) . \tag{5.9}$$

Strengthening the left-hand side of implication (5.8) to $\|Au\|_{V'} \leqq r$ we obtain assertion (5.6).

We know that $\{u_n\}$ is bounded. Thanks to assumption (5.4) the sequence $\{Au_n\}$ is bounded as well:

$$\|Au_n\|_{V'} \leqq M(N(\|b\|_{V'})) . \tag{5.10}$$

The space V, and thus also its dual space V', are reflexive. Due to Theorem 3.2 we can extract weakly convergent subsequences

$$u_{n_k} \rightharpoonup u , \quad Au_{n_k} \rightharpoonup f ,$$

where the limits are some elements $u \in V, f \in V'$.

4[th] *step.* First, we show that $f = b$, where b is the right-hand side of (5.1). Let $v \in V$ be arbitrary. Due to the approximative property (5.7) there exists a sequence $\{v_n\}$, $v_n \in V_n$, $v_n \to v$. Using (5.2) we have

$$\langle Au_n, v_n \rangle = \langle b, v_n \rangle \to \langle b, v \rangle .$$

On the other hand,

$$\langle Au_{n_k}, v_{n_k} \rangle = \langle Au_{n_k}, v_{n_k} - v \rangle + \langle Au_{n_k}, v \rangle \to \langle f, v \rangle ,$$

since (5.10) implies

$$|\langle Au_{n_k}, v_{n_k} - v \rangle| \leqq \|Au_{n_k}\|_{V'} \|v_{n_k} - v\|_V \to 0 .$$

Thus we have obtained $\langle f, v \rangle = \langle b, v \rangle \ \forall v \in V$, which implies $f = b$.

To complete the proof we show that u is a solution. Besides $u_{n_k} \rightharpoonup u$, $Au_{n_k} \rightharpoonup b$ we have due to (5.2)

$$\langle Au_{n_k}, u_{n_k} \rangle = \langle b, u_{n_k} \rangle \to \langle b, u \rangle ,$$

which is the last assumption of condition $(M)_0$. The assertion of $(M)_0$ yields $Au = b$, thus u is a solution of equation (5.1), and the proof is complete. □

5.3. Remarks to the assertion of the theorem.

(a) The proof is constructive. Galerkin approximations represent the basis for many numerical methods.

(b) In general, the sequence of approximate solutions u_n converges neither strongly nor weakly,in contrast to Theorem 4.1. Since the solution need not be unique, only weak convergence of an extracted subsequence is ensured. On the other hand, if u_n converges to $u \in V$ weakly, then $Au_n \rightharpoonup b$ and u is a solution of (5.1). If the solution of (5.1) is unique, then we have $u_n \rightharpoonup u$. Moreover, if the operator A is monotone, solutions of (5.1) form a closed convex set as will be proved in 7.2.

5.4. Remarks to the assumptions of the theorem.

(a) The assumption of reflexivity is substantial. The assumption of separability can be omitted; however, this causes technical difficulties: the Galerkin approximation should then be defined for an uncountable system of finite dimensional subspaces and one must deal with nets – the generalized sequences. A proof of existence of a solution to an equation with a monotone operator, not requiring separability of the space V, is the proof of Theorem 7.5.

(b) Existence of a solution (not for all right-hand sides $b \in V'$) can be proved even for noncoercive operators, see e.g. Theorem 7.3.

(c) The assumptions of theorems appearing in literature usually require continuity or demicontinuity of the operator A instead of the weaker assumption of continuity on finite-dimensional subspaces.

(d) Assumption (5.4) is necessary for the proof of boundedness of $\{Au_n\}$. In the case of a monotone operator the boundedness can be omitted, see Theorem 7.5.

(e) In applications to boundary value problems for differential equations the assumptions of boundedness and continuity are usually not restrictive, since they follow from the assumptions ensuring that the differential equation can be formulated as an operator equation (5.1), see Theorem 8.9 (c).

(f) The special condition $(M)_0$ enables us to pass to the limit when pairing two weakly convergent sequences:

$$u_n \rightharpoonup u\,, \quad b_n = Au_n \rightharpoonup b \Rightarrow \langle b_n, u_n\rangle \to \langle b, u\rangle\,,$$

which in general need not be true. Let us introduce a simple counterexample. Sequences of functions $u_n(x) = \sin nx$, $b_n(x) = \sin nx$ [or $\cos nx$, or $-\sin nx$, resp.] in the space $V = V' = L_2((0, \pi))$ converge to zero: $u_n \rightharpoonup 0$, $b_n \rightharpoonup 0$. but $\langle b_n, u_n\rangle = \int_0^\pi b_n(x)\, u_n(x)\,\mathrm{d}x = \pi/2$ [or 0, or $-\pi/2$, resp.].

Usually a little stronger condition is used – the condition (M), see Definition 6.6, which is necessary in the existence theorems for variational inequalities.

(g) An analogous theorem holds for operators on a complex reflexive separable Banach space with the coercivity condition in the form

$$\lim_{\|u\|\to\infty} \frac{\operatorname{Re}\langle Au, u\rangle}{\|u\|} = \infty\,.$$

Similarly, in the complex case the monotony condition reads

$$\operatorname{Re}\langle Au_1 - Au_2, u_1 - u_2\rangle \geqq 0 \quad \forall u_1, u_2 \in V\,.$$

Other theorems on monotone operators on complex spaces can be found e.g. in [7].

In the literature one can meet various formulations of theorems on monotone operators. The theorems differ particularly by their assumptions on continuity and monotony. Moreover, in functional analysis there appear also various types of continuity. Therefore we present a survey of their definitions (terminology is not unified) and their mutual relations as well as some counterexamples. Some of these concepts correspond to physical, reality, the others are of rather theoretical character. Let us recall that monotony is an important property easy to verify, but many operators describing real physical processes are not monotone, see Section 8, Example III, 8.18. This is the reason why various generalizations of monotony have been introduced.

In what follows the strong and weak convergence will be denoted by an arrow and half-arrow, respectively, and the universal quantifiers as e.g. $\forall u \in V$ will be omitted.

6.1. Definition (various types of continuity and boundedness).

Let $A: V \to V'$ be an operator on a Banach space V. We say that the operator A is

– *continuous*

C *iff* $u_n \to u \Rightarrow Au_n \to Au$,

– *demicontinuous*

dC *iff* $u_n \to u \Rightarrow Au_n \rightharpoonup Au$,

– *strongly continuous*

sC *iff* $u_n \rightharpoonup u \Rightarrow Au_n \to Au$,

– *weakly continuous*

wC *iff* $u_n \rightharpoonup u \Rightarrow Au_n \rightharpoonup Au$,

– *completely continuous*

cC *iff* A *is continuous and maps closed bounded sets into compact ones, i.e.*

$M \subset V$, M – *closed bounded* $\Rightarrow A(M)$ – *compact*,

– *hemicontinuous* (*weakly continuous on lines*)

hC *iff* $\{t_n\} \subset R,\ t_n \to 0 \Rightarrow A(u + t_n v) \rightharpoonup A(u)$,

– *continuous on lines*

lC *iff* $\{t_n\} \subset R,\ t_n \to 0 \Rightarrow A(u + t_n v) \to A(u)$,

– *Lipschitz continuous*

LC *iff* $\exists L > 0,\ \|Au - Av\| \leqq L\|u - v\|$,

– *uniformly continuous*

uC *iff* $\exists M: R^+ \to R^+$, $\lim M(t) = 0$ for $t \to 0_+$,

$$\|Au - Av\| \leqq M(\|u - v\|),$$

– *continuous on finite dimensional subspaces*

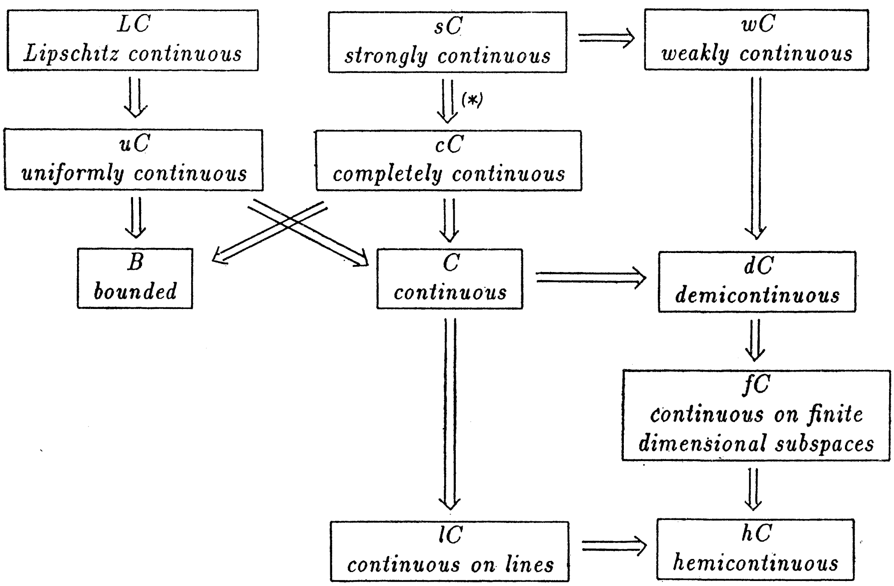
fC *iff* $V_n \subset V$, $\dim(V_n) < \infty \Rightarrow A|_{V_n}: V_n \to V_n'$ *is continuous*,

– *bounded*
B iff $\exists M: R^+ \to R^+$ *increasing*, $\|Au\|_{V'} \leqq M(\|u\|_V)$.

Remark. Terminology is not entirely unified, sometimes sC is called completely continuous and cC is called compact. In [8] there is another equivalent definition of fC:

fC' *iff* $V_n \subset V$, $\dim(V_n) < \infty$, $[\{u_i\} \subset V_n,\ u_i \to u \Rightarrow Au_i \rightharpoonup Au]$.

6.2. Lemma. (a) *The following implications holds*:



The implication $(*)$ *holds only if the space* V *is reflexive. In the non-reflexive case there is no relation between* sC *and* cC, *see* [6], *Chapter I part C, nevertheless* sC *implies* C *and* B. *On finite-dimensional spaces the following concepts coincide*: $C = cC = sC = wC = dC = fC$ *and* $lC = hC$. *For linear operators we have* $cC = sC$ *and* $LC = uC = C = B$.

(b) *The set of operators of each of the introduced continuities forms a linear space, i.e.*

$$A_1, A_2 \in xC \Rightarrow t_1 A_1 + t_2 A_2 \in xC \quad \forall t_1, t_2 \in R\ .$$

(c) *The sum of two operators of different but comparable continuities forms an operator of the "weaker" continuity.*

The proof follows from the definitions and properties of the strong and weak convergences.

Remark. In general, all definitions define mutually different sets of operators. For example, a cC operator need not be sC, e.g. $A: u \in \ell^2 \mapsto (\|u\|, 0, 0, \ldots) \in \ell^2$. Indeed, for $u_n = \{\delta_{in}\} = (0, \ldots, 0, 1, 0, \ldots)$ we have $u_n \rightharpoonup 0$, but $Au_n = (1, 0, 0, \ldots) \neq 0 = A(0)$ (the example is taken from [12]). Further, a continuous operator need not be bounded, e.g. the operator

$$A: u = (\xi_1, \xi_2, \xi_3, \ldots) \in \ell^2 \mapsto (\xi_1, \xi_2^2, \xi_3^3, \xi_4^4, \ldots) \in \ell^2$$

is continuous but not bounded since for $u_n = \{2\delta_{in}\}$ we have $\|u_n\| = 2$ and $\|Au_n\| = 2^n$.

6.3. Definition (types of monotony and coercivity).

Let $A: V \to V'$ be an operator on a Banach space. We say that the operator A is

– *strongly monotone*

sM *iff* $\exists \alpha > 0$, $\langle Au - Av, u - v \rangle \geqq \alpha \|u - v\|^2$ $\forall u, v \in V$,

– *uniformly monotone*

uM *iff* $\exists a: R^+ \to R^+$ *increasing*, $\lim a(t) = 0$ *for* $t \to 0_+$ *and* $\lim a(t) = \infty$ *for* $t \to \infty$,

$$\langle Au - Av\ u - v \rangle \geqq a(\|u - v\|)\, \|u - v\| \ \forall u, v \in V,$$

– *strictly monotone*

rM *iff* $\langle Au - Av, u - v \rangle > 0$ $\forall u, v \in V$, $u \neq v$,

monotone

M *iff* $\langle Au - Av, u - v \rangle \geqq 0$ $\forall u, v \in V$,

– *coercive*

$$K \text{ iff } \lim \frac{\langle Au, u \rangle}{\|u\|} = \infty \text{ for } \|u\| \to \infty,$$

– *weakly coercive*

wK *iff* $\lim \|Au\| = \infty$ *for* $\|u\| \to \infty$.

Moreover, we say that the operator A satisfies the condition

$(S)_+$ *iff* $[u_n \rightharpoonup u, \limsup \langle Au_n - Au, u_n - u \rangle \leqq 0] \Rightarrow u_n \to u$,

(S) *iff* $[u_n \rightharpoonup u, \langle Au_n - Au, u_n - u \rangle \to 0] \Rightarrow u_n \to u$,

$(S)_0$ *iff* $[u_n \rightharpoonup u, Au_n \rightharpoonup b, \langle Au_n, u_n \rangle \to \langle b, u \rangle] \Rightarrow u_n \to u$,

(P) *iff* $u_n \rightharpoonup u \Rightarrow \limsup \langle Au_n, u_n - u \rangle \geqq 0$.

6.4. Remarks.

(a) Monotony (sM, uM, rM, M) has local character in the following sense: if the inequality holds locally, i.e. for each $u, v \in U$, $U \in \mathcal{O}$, where $\mathcal{O}$ is an open covering of the space V, then the inequality holds for each $u, v \in V$.

(b) In the definition of uniform monotony we may assume that the function $a(t)/t$ is nondecreasing. Moreover, if there exists a positive one-sided derivative $a'(0_+) > 0$ then the operator is strongly monotone.

(c) Uniformly monotone operators satisfy the implication

$$\langle Au_n - Au, u_n - u\rangle \to 0 \Rightarrow u_n \to u .$$

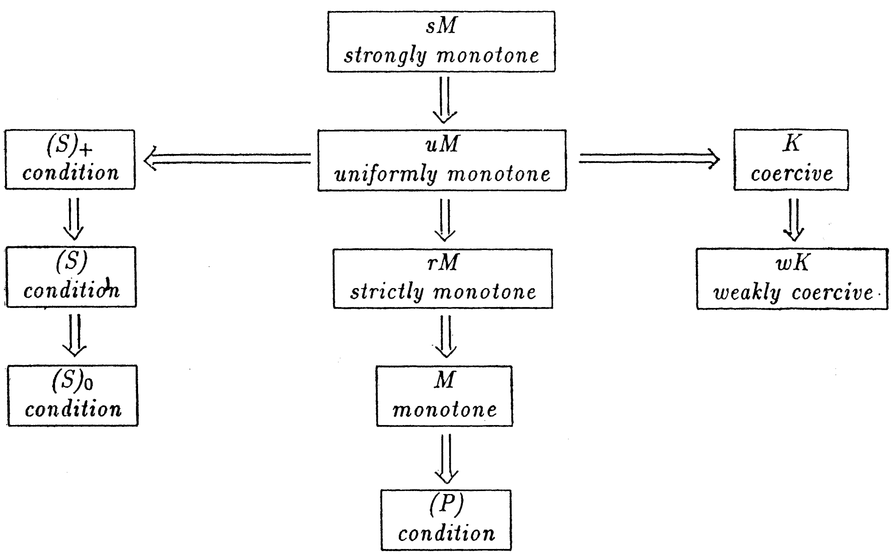
The S-conditions weaken this property to weakly convergent sequences. The S-conditions ensure the strong convergence of Galerkin approximations, see Theorem 7.2 (b).

(d) In the definitions of conditions $(S)_+$ and (S) we can replace $\langle Au_n - Au, u_n - u\rangle$ by $\langle Au_n, u_n - u\rangle$ since $u_n \rightharpoonup u$ implies $\langle Au, u_n - u\rangle \to 0$.

(e) In [16] one can meet a "generalized condition (S)" used for non-homogeneous boundary value problems of type (8.29):

$[u_n \rightharpoonup u$ in V, $v_n \rightharpoonup v$ in $W^{1,2}(\Omega)$, $\langle A(u_n + v_n) - A(u + v), u_n - u\rangle \to 0] \Rightarrow$ $\Rightarrow u_n + v_n \to u + v$, where V is a subspace satisfying $W_0^{1,2}(\Omega) \subset V \subset W^{1,2}(\Omega)$. This condition is satisfied if a bounded operator A satisfies condition (S) on $W^{1,2}(\Omega)$.

6.5. Lemma. (a) *The following implications hold*:



(b) *The sets of operators M, rM, uM, sM, K form cones, i.e.*

$$A_1, A_2 \in xM \Rightarrow A_1 + A_2 \in xM\,, \quad tA_1 \in xM \quad \textit{for} \quad t > 0\,.$$

(c) *The sum of two operators of various types of monotony* (M, rM, uM, sM) *forms an operator of the stronger monotony. Adding an operator M or rM does not violate coercivity.*

The proof follows from the definitions.

The following concepts contain both continuity and monotony.

6.6. Definition. *Let* $A: V \to V'$ *be an operator on a Banach space. We say that the operator is pseudomonotone iff the following implication holds*:

$$(PM) \quad \left[\begin{array}{l} \textit{if } u_n \rightharpoonup u \textit{ and } \limsup \langle Au_n, u_n - u\rangle \leqq 0 \\ \textit{then } \liminf \langle Au_n, u_n - v\rangle \geqq \langle Au, u - v\rangle \textit{ holds } \forall v \in V. \end{array}\right.$$

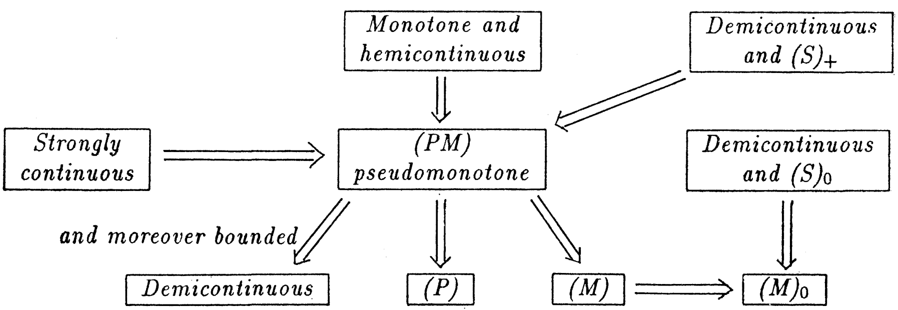
Further, the operator A satisfies condition

(M) *iff* $[u_n \rightharpoonup u,\ Au_n \rightharpoonup b,\ \limsup \langle Au_n, u_n\rangle \leqq \langle b, u\rangle] \Rightarrow Au = b,$

$(M)_0$ *iff* $[u_n \rightharpoonup u,\ Au_n \rightharpoonup b,\ \langle Au_n, u_n\rangle \to \langle b, u\rangle] \Rightarrow Au = b.$

Remark. In the definition of pseudomonotone operators some authors require, in addition to condition (PM), boundedness or demicontinuity. The condition $(M)_0$ – the weakened (M) – seems to be new. In the case of a finite-dimensional space a continuous operator is pseudomonotone and a locally bounded pseudomonotone operator is continuous.

6.7. Lemma. *Let* $A: V \to V'$ *be an operator on a reflexive Banach space V. Then the following implications hold*:



Proof. We give proofs of all implications. Most of them are taken from [11] and [14].

(a) *Strongly continuous* $\Rightarrow (PM)$

Let the assumptions of (PM) be satisfied. Due to strong continuity $u_n \rightharpoonup u$ implies $Au_n \to Au$. Therefore, we have $\langle Au_n, u_n - v\rangle \to \langle Au, u - v\rangle$, which yields the assertion of (PM). □

(b) *Demicontinuous and* $(S)_+ \Rightarrow (PM)$
Let the assumptions of (PM) be satisfied. Then the assumptions of $(S)_+$ are satisfied, too, and thus $(S)_+$ implies $u_n \to u$. Demicontinuity yields $Au_n \rightharpoonup Au$. Now we verify the assertion of (PM):

$\liminf \langle Au_n, u_n - v\rangle = \lim \langle Au_n, u_n - u\rangle + \lim \langle Au_n, u - v\rangle = \langle Au, u - v\rangle$

since $|\langle Au_n, u_n - u\rangle| \leqq \|Au_n\| \, \|u_n - u\| \to 0.$ □

(c) *Monotone and hemicontinuous* $\Rightarrow (PM)$
Let the assumptions of (PM) hold, i.e. $u_n \rightharpoonup u$ and

$$\limsup \langle Au_n, u_n - u\rangle \leqq 0 \,. \tag{6.1}$$

Let $v \in V$. We need to estimate lim inf of the term

$$\langle Au_n, u_n - v\rangle = \langle Au_n, u_n - u\rangle + \langle Au_n, u - v\rangle \,.$$

The first term on the right-hand side tends to zero since it is majorized by zero in (6.1) and the monotony yields the opposite bound

$$\langle Au_n, u_n - u\rangle \geqq \langle Au, u_n - u\rangle \to 0.$$

To estimate the second term $\langle Au_n, u - v\rangle$ let us put $w = u - t(u - v)$, $t > 0$ in the monotony inequality $\langle Au_n - Aw, \ u_n - w\rangle \geqq 0$. We obtain $\langle Au_n, u_n - u\rangle + t\langle Au_n, u - v\rangle - \langle Aw, u_n - u\rangle - t\langle Aw, u - v\rangle \geqq 0$. Let us pass to lim inf. By virtue of (6.1) and $u_n \rightharpoonup u$ the first and the third terms vanish. Dividing the remaining two terms by $t > 0$ we obtain $\langle Au_n, u - v\rangle \geqq \langle Aw, u - v\rangle$. Finally, due to hemicontinuity we have $Aw \rightharpoonup Au$ for $t \to 0$ and the desired inequality follows. □

(d) (PM) *and bounded* $\Rightarrow$ *demicontinuous*
Let $u_n \to u$. Then $\{u_n\}$ and $\{Au_n\}$ are bounded and we can extract a weakly converging subsequence $Au_{n_k} \rightharpoonup b$. Since the assumptions of (PM) are satisfied, we have $c = \liminf \langle Au_n, u_n - v\rangle \geqq \langle Au, u - v\rangle$. On the other hand,

$$c = \lim \langle Au_{n_k}, u_{n_k} - u\rangle + \lim \langle Au_{n_k}, u - v\rangle = \langle b, u - v\rangle \,.$$

We have obtained $\langle Au, u - v\rangle \leqq \langle b, u - v\rangle \; \forall u \in V$. Putting $v = 2u - v$ in the last inequality we get the opposite inequality. Thus the equality $b = Au$ follows. Since the limit Au of the extracted subsequence Au_{n_k} is unique, the whole sequence converges weakly to Au, i.e. $Au_n \rightharpoonup Au$, which proves demicontinuity. □

(e) $(PM) \Rightarrow (M)$

Let the assumptions of (M) be satisfied. Then the assmptions of (PM) are also satisfied, since

$$\limsup \langle Au_n, u_n - u\rangle = \limsup \langle Au_n, u_n\rangle - \lim \langle Au_n, u\rangle \leqq 0 \,.$$

In the chain of inequalities we use the assertion of (PM) and the assumption of (M):

$$\langle Au, u - v\rangle \leqq \liminf \langle Au_n, u_n - v\rangle \leqq \limsup \langle Au_n, u_n - v\rangle \leqq \langle b, u - v\rangle \,.$$

We have obtained $\langle Au, u - v \rangle \leqq \langle b, u - v \rangle \; \forall v \in V$. Analogously as in the previous implication we obtain $Au = b$, which proves condition (M). □

(f) *The implication* $(M) \Rightarrow (M)_0$ *is obvious.*

(g) *Demicontinuous and* $(S)_0 \Rightarrow (M)_0$.

Let the assumptions of $(M)_0$ be satisfied. They are identical with the assumptions of $(S)_0$, thus $u_n \to u$. Due to demicontinuity we have $Au_n \rightharpoonup Au$. Therefore $Au = b$ and the assertion of condition $(M)_0$ is proved. □

(h) *The implication* $(PM) \Rightarrow (P)$ *will be proved by contradiction*:

Let $u_n \rightharpoonup u$ and $\limsup \langle Au_n, u_n - u \rangle < 0$. Then (PM) yields $\liminf [Au_n, u_n - v \rangle \geqq \geqq \langle Au, u - v \rangle \; \forall v \in V$. However, putting $v = u$ we obtain $\liminf \langle Au_n, u_n - u \rangle \geqq \geqq 0$, which is a contradiction. □

In contrast to Lemmas 6.2, 6.5, Lemma 6.7 says nothing about the sum of two operators.

6.8. Lemma.

(a) *The sum of two pseudomonotone operators is a pseudomonotone operator, i.e. the pseudomonotone operators form a cone.*

(b) *The sum of two operators satisfying* $(S)_+$ *is an operator satisfying* $(S)_+$, *i.e. the operators satisfying condition* $(S)_+$ *form a cone.*

(c) *Adding a strongly continuous operator does not violate the property* $(S)_+$, (S), $(S)_0$, (P), (PM), (M) *or* $(M)_0$ *of the operator.*

Remark. Assertion (a) is taken from [12].

Proof. (a) Let A_1, A_2 be pseudomonotone, $u_n \rightharpoonup u$ and

$$\limsup \langle A_1 u_n + A_2 u_n, u_n - u \rangle \leqq 0 .$$

By contradiction we prove that

$$\limsup \langle A_i u_n, u_n - u \rangle \leqq 0 \quad \text{for} \quad i = 1, 2 .$$

Let e.g. $\limsup \langle A_2 u_n, u_n - u \rangle = \delta > 0$. We can extract a subsequence $\{u_{n_k}\}$ such that $\langle A_2 u_{n_k}, u_{n_k} - u \rangle = \delta$. Then $\limsup \langle A_1 u_{n_k}, u_{n_k} - u \rangle \leqq -\delta$ and we can apply (PM) to $u_{n_k} \to u$ and A_1 which yields $\liminf \langle A_1 u_{n_k}, u_{n_k} - v \rangle \geqq \langle A_1 u, u - v \rangle$. Putting $v = u$ we obtain $\liminf \langle A_1 u_{n_k}, u_{n_k} - u \rangle \geqq 0$ which contradicts $\limsup \langle A_1 u_{n_k}, u_{n_k} - u \rangle \leqq -\delta < 0$.

Thus the assumptions of (PM) for A_1, A_2 are satisfied and the sum of their assertions yields (PM) for $A_1 + A_2$. □

(b) Let A_1, A_2 satisfy $(S)_+$ and let $u_n \rightharpoonup u$ and $\limsup \langle A_1 u_n + A_2 u_n, u_n - u \rangle \leqq \leqq 0$ (we have used Remark 6.4 (d)). We prove by contradiction that $\limsup \langle A_1 u_n, u_n - u \rangle \leqq 0$. Let $\limsup \langle A_1 u_n, u_n - u \rangle = \delta > 0$. We can extract a subsequence $u_{n_k} \rightharpoonup u$ such that $\lim \langle A_1 u_{n_k}, u_{n_k} - u \rangle = \delta$. Then

$\limsup \langle A_2 u_{n_k}, u_{n_k} - u \rangle \leqq -\delta < 0$. Applying $(S)_+$ to $u_{n_k} \rightharpoonup u$ and A_2 we obtain $u_{n_k} \to u$. We arrive at a contradiction because the last convergence implies $\langle A_2 u_{n_k}, u_{n_k} - u \rangle \to 0$.

Thus $\limsup \langle A_1 u_n, u_n - u \rangle \leqq 0$ and $(S)_+$ applied to A_1 and $u_n \rightharpoonup u$ yields the desired convergence $u_n \to u$. □

(c) Let A_1 be strongly continuous and let A_2 satisfy condition (X) $[(X) = (S)_+, (S), (S)_0, (P), (PM), (M), (M)_0]$. Let the assumptions of (X) be satisfied for the operator $A_1 + A_2$. Since $u_n \rightharpoonup u$, the strong continuity of A_1 yields $A_1 u_n \to A_1 u$. Using this strong convergence it is easy to verify that in all cases the assumptions of (X) are satisfied also for A_2, and further that the assertion of (X) for A_2 remains valid also for $A_1 + A_2$, which proves the property (X) for $A_1 + A_2$. □

The last lemmas yield further properties for the sum of operators ,e.g.: If A_1 is monotone and continuous, A_2 strongly continuous then $A_1 + A_2$ is pseudomonotone.

6.9. Warnings.

(a) The sum of two operators satisfying condition (M) need not satisfy condition (M). A counter-example was given by Brézis (see [12], Chaper III, 5.2). Let V be a Hilbert space with an orthonormal base $\{e_1, e_2, e_3, \ldots\}$ and let us define operators $A_1 \colon u \mapsto -u$ (minus identity), A_2 the projection on the unit ball given by $A_2(u) = u/\|u\|$ for $\|u\| \geqq 1$ and $A_2(u) = u$ for $\|u\| \leqq 1$. Both operators satisfy condition (M) but their sum $A = A_1 + A_2$ does not. Indeed, for $u_n = e_1 + e_n$ we have $u_n \rightharpoonup u = e_1$, $Au_n = (e_1 + e_n)(2^{-1/2} - 1) \rightharpoonup e_1(2^{-1/2} - 1) = b$,
$\limsup \langle Au_n, u_n \rangle = 2^{1/2} - 2 \leqq 2^{-1/2} - 1 = \langle b, u \rangle$,
but $Au = 0 \neq e_1 (2^{-1/2} - 1) = b$.

(b) Complete continuity is not sufficient for condition $(M)_0$. A counter-example (Petryshyn, Fitzpatrick, see [12], Chaper III, 5.3) is a completely continuous operator (in the same notation) $A \colon u \mapsto e_1 \|u\|$ satisfying neither condition (M) nor $(M)_0$ for $u_n = e_n$, since $u_n \rightharpoonup 0 = u$, $Au_n = e_1 \rightharpoonup e_1 = b$, $\langle Au_n, u_n \rangle = \langle e_1, e_n \rangle \to 0 = \langle e_1, 0 \rangle = \langle b, u \rangle$, but $Au = 0 \neq e_1 = b$.

We conclude the section with a lemma often used in the applications to problems with differential operators with monotony in the principal part, see Section 8.

6.10. Lemma. *Let V be a reflexive Banach space and let the operator $A \colon V \to V'$ have the form*

$$Au = B(u, u)\,,$$

where $B \colon V \times V \to V'$ has the following properties:

(a) *$B(u, v)$ is hemicontinuous and bounded in u for each $v \in V$,*

(b) *$B(u, v)$ is hemicontinuous in v for each $u \in V$,*

(c) *$B(u, v)$ is monotone in v, i.e.*

$$\langle B(u, u) - B(u, v), u - v \rangle \geqq 0 \quad \forall u, v \in V,$$

(d) *if* $u_n \rightharpoonup u$ *and* $\langle B(u_n, u_n) - B(u_n, u), u_n - u \rangle \to 0$ *then* $B(u_n, v) \rightharpoonup B(u, v)$ $\forall v \in V$,

(e) *if* $u_n \rightharpoonup u$ *and* $B(u_n, v) \to b$ *in* V' *then* $\langle B(u_n, v), u_n \rangle \to \langle b, u \rangle$.

Then the operator A is pseudomonotone.

Remark. The operator with the above properties is called semi-monotone or of the variational calculus type. The lemma taken from [12] forms the essence of the proof of the Leray-Lions theorem, see [4], Theorem 29.6.

Proof. Let $u_n \rightharpoonup u$ and

$$\limsup \langle Au_n, u_n - u \rangle \leqq 0 . \tag{6.2}$$

First we prove the relations

$$B(u_n, v) \rightharpoonup B(u, v) \quad \forall v \in V, \tag{6.3}$$

$$c_n = \langle B(u_n, u_n) - B(u_n, u), u_n - u \rangle \to 0 . \tag{6.4}$$

Let us consider the sequence $\{B(u_n, u)\}$. Due to (a) it is a bounded sequence and thus it contains a weakly convergent subsequence $B(u_{n_k}, u) \rightharpoonup b$. Using (e) we obtain $\langle B(u_{n_k}, u), u_{n_k} \rangle \to \langle b, u \rangle$. Condition (c) yields the inequality

$$c_{n_k} = \langle B(u_{n_k}, u_{n_k}) - B(u_{n_k}, u), u_{n_k} - u \rangle \geqq 0 .$$

On the other hand, using (6.2) and the previous relations we obtain $\limsup c_{n_k} = \limsup \langle Au_{n_k}, u_{n_k} - u \rangle - \lim \langle B(u_{n_k}, u), u_{n_k} - u \rangle \leqq 0$. Thus $c_{n_k} \to 0$ and the assumptions of (d) are satisfied. The assertion yields $B(u_{n_k}, v) \rightharpoonup B(u, v)$ $\forall v \in V$. Putting $v = u$ we see that the sequence $\{B(u_{n_k}, u)\}$ has a unique limit $B(u, u)$ and thus the whole sequence convergences to $B(u, u)$, i.e. $B(u_n, u) \rightharpoonup B(u, u)$. Repeating the proof for the whole sequence we obtain (6.4) and (6.3).

In the second part of proof we derive two properties: First, since $B(u_n, w) \to B(u, w)$ for $w \in V$, condition (e) implies $\langle B(u_n, w), u_n \rangle \to \langle B(u, w), u \rangle$ and by virtue of (6.3) we obtain

$$\langle B(u_n, w), u_n - u \rangle \to 0 \quad \forall w \in V. \tag{6.5}$$

Further, due to (6.4) the sequences $\{\langle B(u_n, u_n), u_n - u \rangle\}$ and $\{\langle B(u_n, u), u_n - u \rangle\}$ have the same limits. However, the second sequence tends to zero due to (6.5) with $w = u$. Thus we have

$$\langle A(u_n), u_n - u \rangle \to 0 . \tag{6.6}$$

Finally, we prove the assertion of (PM). Let $v \in V$. We start with the inequality (c) which yields $B(u_n, u_n) - B(u_n, w), u_n - w \rangle \geqq 0$. Putting $w = u + t(v - u)$ with

$t > 0$ we obtain

$$\langle Au_n, u_n - u \rangle + t\langle Au_n, u - v \rangle \geqq$$
$$\geqq \langle B(u_n, w), u_n - u \rangle + t\langle B(u_n, w), u - v \rangle .$$

We pass to lim inf. The term $\langle Au_n, u_n - u \rangle$ can be omitted due to (6.2). Thanks to (6.5) we have $\langle B(u_n, w), u_n - u \rangle \to 0$. Dividing the remaining two terms by $t > 0$ and using (6.3) we obtain

$$\liminf \langle Au_n, u - v \rangle \geqq \langle B(u, w), u - v \rangle ,$$

and using (b) for $t \to 0$ we conclude that

$$\liminf \langle Au_n, u - v \rangle \geqq \langle Au, u - v \rangle . \tag{6.7}$$

Since $\langle Au_n, u - v \rangle = \langle Au_n, u_n - v \rangle - \langle Au_n, u_n - u \rangle$ we obtain using (6.6), $\liminf \langle Au_n, u - v \rangle = \liminf \langle Au_n, u_n - v \rangle$, which together with (6.7) yields the desired inequality (PM). The proof is complete. □

7. FURTHER THEOREMS

In this section we deduce some consequences of the main theorem and some additional results. Theorem 7.5 is presented for its elegant proof using the Minty lemma. Some remarks on variational inequalities and maximal monotone mappings conclude the section.

By virtue of Lemmas 6.2–6.8 the assumptions of Main Theorem 5.2 can be replaced by stronger ones, e.g.

7.1. Theorem. *Let an operator $A: V \to V'$ on a reflexive separable Banach space satisfy one of the following assumptions:*

(a) *A is coercive, demicontinuous, bounded and satisfies $(S)_+$,*

(b) *A is strongly monotone, continuous and bounded,*

(c) *A is continuous, bounded, coercive and $A = A_1 + A_2$, where A_1 is monotone and A_2 strongly continuous.*

Then the assertion of Theorem 5.2 holds.

Let us return to Theorem 5.2. Stronger assumptions yield stronger assertions:

7.2. Theorem. (Supplement to Theorem 5.2.)

Let us consider the problem (5.1) with a bounded operator A on a reflexive separable Banach space V and its Galerkin approximations (5.2). Let $\{u_n\}$ be a bounded sequence of solutions to the problem (5.2) and $\{u_{n_k}\}$ a weakly convergent subsequence, $u_{n_k} \rightharpoonup u$. Then the following assertions hold:

(a) *If A satisfies $(M)_0$ then u is a solution of (5.1).*

(b) *If A satisfies $(S)_0$ and is demicontinuous, then u is a solution of (5.1) and $u_{n_k} \to u$ strongly.*

(c) *If A is monotone and hemicontinuous, then the set of the solutions $A^{-1}b = \{u \in V, Au = b\}$ is nonempty, closed and convex.*

(d) *If A is strictly monotone and hemicontinuous then $A^{-1}b$ is a one-point set, i.e. the solution is unique.*

(e) *If A is uniformly monotone then A^{-1} is continuous.*

Proof. (a) Since A is bounded, $\{Au_{n_k}\}$ is bounded as well and we can extract a weakly convergent subsequence $Au_{n_k} \rightharpoonup b$. The final part of the proof is analogous to the proof of Theorem 5.2, step 4. □

(b) Demicontinuity and $(S)_0$ imply $(M)_0$, thus the limit u is a solution, the condition $(S)_0$ yields the strong convergence $u_{n_k} \to u$. □

(c) Monotony and hemicontinuity imply $(M)_0$, thus $A^{-1}b \neq \emptyset$. Let $u_1, u_2 \in A^{-1}b$. We prove that $u = u_1t_1 + u_2t_2 \in A^{-1}b$ for all $t_1, t_2 \in (0, 1)$, $t_1 + t_2 = 1$. Monotony and $Au_1 = b = Au_2$ yield $0 \leqq t_1\langle Au_1 - Av, u_1 - v\rangle + t_2\langle Au_2 - Av, u_2 - v\rangle = \langle b - Av, u - v\rangle$. Since the inequality $\langle b - Av, u - v\rangle \geqq 0$ holds for each $v \in V$ we have $Au = b$. Indeed, putting $v = u - sw$, $s > 0$, $w \in V$ in this inequality we obtain $s\langle b - A(u - sw), w\rangle \geqq 0$. Dividing it by $s > 0$ and passing to the limit $s \to 0$, we conclude by virtue of hemicontinuity that $\langle b - Au, w\rangle \geqq 0$. Then putting $-w$ instead of w we obtain the opposite inequality, thus $\langle b - Au, w\rangle = 0$ holds for each $w \in V$. Consequently, $Au = b$ and $A^{-1}b$ is convex.

It remains to prove that $A^{-1}b$ is closed. Let $Au_n = b$, $u_n \to u$. Then $\langle b - Av, u - v\rangle = \lim \langle Au_n - Av, u_n - v\rangle \geqq 0$ $\forall v \in V$. By a similar argument as above we again obtain $Au = b$, i.e. $A^{-1}b$ is closed. □

(d) See Theorem 3.1. □

(e) The continuity of A^{-1} follows directly from the definition of uniform monotony. □

The assumption of coercivity can be omitted if we guarantee boundedness of the solutions u_n to (5.2) – the finite-dimensional approximation of the problem (5.1). This can be ensured e.g. by means of Theorem 2.2:

7.3. Theorem. *Let V be a reflexive separable Banach space, $b \in V'$, and $A: V \to V'$ a demicontinuous bounded operator satisfying condition $(M)_0$ and the inequality*

$$\langle Au - b, u\rangle \geqq 0 \quad \forall u \in V, \quad \|u\| = R \quad (R > 0).$$

Then the equation $Au = b$ has a solution and we apply Theorem 7.2.

Although the theory of monotone operators was developed for nonvariational problems, the property of monotony is used also for potential operators, see e.g. [4], Theorem 26.11.

For monotone operators the main theorem can be proved without the assumptions of boundedness of the opetator and separability of the space by means of the following lemma:

7.4. Lemma. (Minty, see [8].) *Let A be a monotone hemicontinuous operator and $u \in V$. Then the following two conditions are equivalent*:

$$(7.1) \qquad \langle Au, v - u \rangle \geqq 0 \quad \forall v \in V,$$

$$(7.2) \qquad \langle Av, v - u \rangle \geqq 0 \quad \forall v \in V.$$

Remark. Lemma 7.4 holds even if we replace the space V by a closed convex subset $K \subset V$, see [8].

Proof. Monotony implies $\langle Av, v - u \rangle \geqq \langle Au, v - u \rangle$, which yields the implication (7.1) $\Rightarrow$ (7.2). Hemicontinuity yields the opposite implication. Let us assume (7.2). Let $w \in V$. Putting $v = u + t(w - u)$, $t > 0$ in (7.2) we obtain $\langle A(u + t(w - u), t(w - u) \rangle \geqq 0$. After dividing by $t > 0$ we pass to the limit $t \to 0_+$. Hemicontinuity implies $A(u + t(w - u)) \rightharpoonup Au$ and we obtain the inequality (7.1). □

7.5. Theorem. (Minty-Browder.) *Let V be a reflexive Banach space and A: $V \to V'$ a monotone coercive operator continuous on finite-dimensional subsapces.*

Then $A(V) = V'$, i.e. the equation $Au = b$ has a solution for each $b \in V'$. Moreover, A^{-1} as a multivalued mapping is bounded and $A^{-1}b$ is a closed convex set for each $b \in V'$.

Proof. Let $b \in V'$. We prove that $Au = b$ as a solution. The operator $A_1 u = Au - b$ is also monotone and hemicontinuous. Due to the Minty lemma 7.4 the following two conditions

$$(7.3) \qquad \langle A_1 u, v - u \rangle \geqq 0 \quad \forall v \in V,$$

$$(7.4) \qquad \langle A_1 v, v - u \rangle \geqq 0 \quad \forall v \in V$$

are equivalent for $u \in V$. Let us denote

$$U(v) = \{u \in V, \langle A_1 v, v - u \rangle \geqq 0\} .$$

Then the intersection $U_V = \bigcap\{U(v), v \in V\}$ is the set of $u \in V$ satisfying (7.4) and thus (7.3), which implies that u is a solution of $A_1 u = 0$.

We prove that the intersection U_V is nonempty by means of the following theorem on nonempty intersection:

Theorem. *Let $\{U_\iota, \iota \in I\}$ be an arbitrary system of closed subsets of a compact topological space such that the intersection of a finite number of U_ι is nonempty. Then also the intersection $\bigcap\{U_\iota, \iota \in I\}$ is nonempty.*

1st *step.* The coercivity of A implies (see the proof of Theorem 5.2, step 3) the existence of an increasing function N: $R^+ \to R^+$ such that

$$(7.5) \qquad \|u\|_V \leqq N(\|Au\|_{V'}) \quad \forall u \in V.$$

The constant $r = N(\|b\|_{V'})$ is an apriori estimate of the solution u, thus the ball B_r

contains all possible solutions. The closed ball B_r with the weak topology is a compact topological space due to the reflexivity of V.

2^{nd} *step.* The halfspaces $U(v)$ are convex and closed, thus they are closed in the weak topology. The same holds for $U_r(v) = U(v) \cap B_r$.

3^{rd} *step.* It remains to prove that the intersection of a finite number of $U_r(v)$ is nonempty. Let $v_1, v_2, \ldots, v_n \in V$ and let V_n be the finite-dimensional subspace generated by $\{v_1, v_2, \ldots, v_n\}$. In the same way as in the proof of Theorem 5.2, step 2 we obtain the existence of a solution u_n of the Galerkin approximation of the problem $Au = b$ on the subspace V_n. Due to (7.5) we have $\|u_n\| \leqq N(\|b|_{V_n}\|) \leqq r$. Therefore $u_n \in V_n \cap B_r$ satisfies

$$\langle A_1 u_n, v - u_n \rangle \geqq 0 \quad \forall v \in V_n .$$

Thanks to Lemma 7.4 it also satisfies

$$\langle A_1 v, v - u_n \rangle \geqq 0 \quad \forall v \in V_n .$$

Therefore the intersection $\bigcap\{U_r(v_i),\, i = 1, 2, \ldots, n\} = \bigcap\{U(v_i),\, i = 1, 2, \ldots, n\} \cap B_r$ is nonempty, since it contains at least the solution u_n.

Using the above theorem on nonempty intersection we conclude that the intersection U_V is nonempty, which yields the existence of a solution to the problem $Au = b$. Since each $U_r(v)$ is a closed convex set, their intersection $U_V = A^{-1}b$ is also a closed convex set in V. Due to (7.5) A^{-1} is a bounded mapping. □

The theory of monotone operators has applications also in the field of variational inequalities, see e.g. [8], [12]. As an example we introduce a theorem from [8], which can be proved similarly to Theorem 7.5 via the remark to Lemma 7.4 and an existence theorem for finite-dimensional approximations of variational inequalities.

7.6. Theorem. *Let K be a nonempty closed convex bounded subset of a reflexive Banach space V, and $A: K \to V'$ a monotone operator continuous on finite-dimensional subspaces. Then there exists a solution to the problem*

Find $u \in K$ such that

$$\langle Au, v - u \rangle \geqq 0 \quad \forall v \in K .$$

In the end of this section we introduce some remarks on maximal monotone mappings. The concept is defined for multivalued mappings $A: V \to \exp V'$ whose domain of definition need not be the whole space V. By $\exp X$ we denote the set of all subsets of X.

7.7. Definition. *The set $M \subset V \times V'$ is said to be monotone, iff*

$$\langle b_1 - b_2, u_1 - u_2 \rangle \geqq 0 \quad \forall (u_1, b_1), (u_2, b_2) \in M .$$

A mapping $A: V \to \exp V'$ is said to be monotone, if its graph is a monotone set.

A mapping is maximal monotone if its graph is a maximal monotone set in the sense of inclusion.

The following functions are examples of maximal monotone mappings on R: x^3, e^x, arctg $x, f(x) = \text{sign } x$ with $f(0) = [-1, 1]$ and their inverse functions. Let us present some assertions taken from [14]:

7.8. Assertions. *Let V be a reflexive Banach space.*

(a) *Let A be a monotone hemicontinuous mapping $A: V \to V'$. Then it is maximal monotone.*

(b) *Let A be a maximal monotone mapping $A: V \to \exp V'$. Then the inverse mapping $A^{-1}: V' \to \exp V = \exp V''$ is also maximal monotone.*

(c) *The subdifferential of a convex functional $\Phi: V \to R$ is a maximal monotone mapping.*

In [12] one can find definitions of pseudomonotone mappings, mappings of type (S) and (M) – i.e. mappings satisfying conditions similar to conditions (S), (M) discussed above – and applications to differential and integral equations. Let us present an existence theorem ([12], III, Theorem 2.10):

7.9. Theorem. *Let V be a strictly convex reflexive Banach space and $A: V \to \exp V'$ a maximal monotone coercive mapping. Then $A(V) = V'$.*

8. APPLICATION

In this section we give some examples of application of the theory developed in the previous sections to the solution of boundary value problems for differential equations. Due to the limited extent of the paper, the reformulation of a differential equation to an abstract operator equation will be only outlined. For details see e.g. [4], where also other examples can be found.

We adopt the following notation for function spaces:

$C^k(\Omega)$ – the space of k-times continuously differentiable functions,

$L_p(\Omega)$ – the Lebesgue space of functions integrable with the p-th power, with a.e. (almost everywhere) equality – i.e. two functions are equal if they differ at most on a subset of zero measure – and

$W^{k,p}(\Omega)$, $W_0^{k,p}(\Omega)$ – Sobolev spaces, see e.g. [4], [10], etc., where Ω is a domain in R^N. For an interval on the real line R we write I instead of Ω.

Example I. A simple ordinary differential equation

8.1. Classical formulation.

We shall deal with the boundary value problem

$$(8.1)\qquad -u'' + g(u) = f \quad \text{in} \quad I = (0, 1)$$
$$u(0) = u(1) = 0\,,$$

where f is a given function, $f \in L_2(I)$. We shall investigate four cases according to the type of the function g:

(a) $g(\xi) = c\xi$, $c \in R$, $c > 0$,

(b) $g(\xi) = \xi^3$,

(c) g is a continuous nondecreasing function,

(d) g is an arbitrary continuous function.

In order to be able to use the monotone operator theory we need to reformulate the problem in the form of an operator equation on a Banach space V.

8.2. Function space.

The suitable space is the Sobolev space $W_0^{1,2}(I) = W_0^1(I)$ often denoted by $H_0^1(I)$, which is a reflexive separable Banach space with the norm

$$\|u\| = [\textstyle\int_I (u'^2 + u^2)\,\mathrm{d}x]^{1/2}\,.$$

The space is defined as the completion of the linear set

$$\{u \in C^1([0, 1]),\ u(0) = u(1) = 0\}$$

in the above norm. It can be proved that $W_0^1(I)$ is the space of absolutely continuous functions with zero values on the boundary $\partial I = \{0, 1\}$ having a square-integrable first derivative (absolutely continuous functions have a derivative almost everywhere). Let us put $V = W_0^1(I)$.

8.3. Weak formulation.

We multiply equation (8.1) by a function $v \in V$ and integrate the equation with respect to x over I. Integrating the first term by parts and using the condition $v(0) = v(1) = 0$ we obtain

$$(8.2)\qquad \textstyle\int_I u'v'\,\mathrm{d}x + \int_I g(u)\,v\,\mathrm{d}x = \int_I fv\,\mathrm{d}x\,.$$

By the weak (often called generalized) formulation of the boundary value problem (8.1) we understand the problem

(8.3) Find $u \in V$ such that (8.2) holds for each $v \in V$.

The weak formulation is in fact the abstract operator equation $Au = b$, see (3.1). Indeed, we define the opetator A and the functional b by the relations

$$\langle Au, v\rangle = \textstyle\int_I u'v'\,\mathrm{d}x + \int_I g(u)\,v\,\mathrm{d}x\,, \quad u, v \in V\,,$$
$$\langle b, v\rangle = \textstyle\int_I fv\,\mathrm{d}x\,, \quad v \in V\,.$$

8.4. Justification of the weak formulation.

We have to prove that $A\colon u \in V \mapsto Au \in V'$ and $b \in V'$. Since $f \in L_2(I)$, the functional b can be estimated using the Schwarz inequality

$$|\langle b, v\rangle| = |\int_I fv \, dx| \leqq [\int_I f^2 \, dx]^{1/2} [\int_I v^2 \, dx]^{1/2} =$$
$$= \|f\|_{L_2} \|v\|_{L_2} \leqq \text{const.} \|v\|_V .$$

The functional b is bounded and linear, thus it is continuous, i.e. $b \in V'$.

Let us deal with the operator A. The form $\langle Au, v\rangle$ is linear in v. It remains to verify that $Au \in V'$. The operator A consists of two parts $A = A_1 + A_0$. The former part A_1 defined by

$$\langle A_1 u, v\rangle = \int_I u'v' \, dx$$

maps V into V' by virtue of the estimate

$$|\langle A_1 u, v\rangle| = |\int_I u'v' \, dx| \leqq [\int_I u'^2 \, dx]^{1/2} [\int_I v'^2 \, dx]^{1/2} \leqq \|u\|_V \|v\|_V .$$

Moreover, one can see that A_1 is a linear bounded operator and thus it is continuous. Since $[\langle A_1 u, u\rangle]^{1/2} = [\int_I u'^2 \, dx]^{1/2}$ is an equivalent norm on $V = W_0^1(I)$, the linear operator A_1 is strongly monotone.

The latter part A_0 of the operator A is defined by

$$\langle A_0 u, v\rangle = \int_I g(u(x)) \, v(x) \, dx .$$

We need to prove that $A_0 u \in V'$ for $u \in V$, i.e.

$$\langle A_0 u, v\rangle \leqq \text{const}\,(u) \, \|v\|_V , \quad u, v \in V.$$

In the linear case (a) the estimate is clear. In the other cases we shall use the imbedding of Sobolev spaces, see e.g. [4], [10]:

$$V = W_0^{1,2}(I) \hookrightarrow C^0(I) .$$

Indeed, the functions of V are absolutely continuous with $u(0) = 0$ and thus $u(x) = \int_0^x u'(s) \, ds$, which implies

$$|u(x)| \leqq \int_0^x |u'(s)| \, ds \leqq \int_I |u'| \, ds \leqq [\int_I u'^2 \, ds]^{1/2} [\text{meas}\,(I)]^{1/2}$$

for all $x \in I$. Consequently

$$\max_I |u| \leqq \|u\|_V . \tag{8.4}$$

Since in all cases the function g is continuous, $g(u(x))$ is also bounded. Thus we obtain the desired estimate

$$|\langle A_0 u, v\rangle| = |\int_I g(u) \, v \, dx| \leqq \max_I |g(u)| \max_I |v| \leqq \text{const}\,(u) \, \|v\|_V \tag{8.5}$$

and the weak formulation (8.3) is justified. Moreover, we have proved that in all cases the operator A_0 is bounded and continuous. Indeed, due to continuity of u, g

and (8.4) the constant in (8.5) depends on the norm $\|u\|_V$, thus A_0 is bounded. Similarly, by the same argument, $u_n \to u$ in V implies $\|A_0 u_n - A_0 u\| \to 0$, i.e. A_0 is continuous. Thus A is bounded and continuous.

Let us remark that the mapping $u \mapsto g(u)$ is a special case of the so-called Nemyckij operator, see 8.9.

8.5. Application of the monotone operator theory.

Now we shall investigate the individual cases using the lemmas of Section 6.

(a) The operator A is linear and continuous, thus it is Lipschitz continuous. For a nonnegative constant c the operator A_0 is monotone and thus A is strongly monotone. Theorem 4.1 yields the existence of a unique solution u. Moreover, we have the strong convergence (4.5) of the approximative solutions given by (4.4) or the strong convergence of the Galerkin approximations (5.2) due to Theorem 7.2 (b). Let us remark that if the constant c is negative, the solution need not exist.

(b) Again the operator A is continuous, bounded and strongly monotone, since A_0 is monotone

$$\langle A_0 u - A_0 v, u - v \rangle = \int_I (u - v)^2 (u^2 + uv + v^2) \, \mathrm{d}x \geqq 0 .$$

Theorems 7.1 (b), 7.2 (b) yield the existence of a unique solution and the strong convergence of the Galerkin approximations.

(c) Since g is a nondecreasing function, the operator A_0 is again monotone and we have the same result as in the case (b).

(d) Since g may have a decreasing segment, A_0 need not be monotone. Therefore we make use of the fact that A_0 is strongly continuous. Indeed, let $u_n \rightharpoonup u$ in V. The compact imbedding $W_0^{1,2}(I) \subset\subset C^0(I)$ (see e.g. [4], [10]) yields $u_n \to u$ in $C^0(I)$. Since g is continuous, we have $g(u_n) \to g(u)$ in $C^0(I)$. Thus $A_0 u_n \to A_0 u$ in V' which follows from the estimate $\|A_0 u_n - A_0 u\|_{V'} = \sup_{\|v\|=1} \int_I [g(u_n) - g(u)] \, v \, \mathrm{d}x \leqq$ $\leqq \|g(u_n) - g(u)\|_{C^0} \to 0$.

To obtain the coercivity of A we have to add another assumption for g:

$$\liminf_{|\xi| \to \infty} g(\xi) \operatorname{sign} \xi > -\infty . \tag{8.6}$$

This condition ensures that A_0 is not "too negative", i.e. it does not violate the coercivity of A.

If the assumption (8.6) is satisfied then Theorem 7.1 (c) yields the existence of a solution – which need not be unique.

If (8.6) is not satisfied, the operator A need not be coercive and the problem need not have a solution for some right-hand sides f, see [4], Chapter VI.

Let us remark that the operator A is potential and the problem can be studied by means of variational methods with similar results, see [4], Theorem 26.13.

Example II. General ordinary differential equation

8.6. Classical formulation.

We shall consider a nonlinear second order ordinary differential equation in divergent form, with Dirichlet boundary conditions:

$$(8.7) \qquad -\frac{\mathrm{d}}{\mathrm{d}x}\left[a_1(x, u(x), u'(x))\right] + a_0(x, u(x), u'(x)) = f(x) \quad \text{on} \quad I = (0, 1)$$

$$u(0) = u(1) = 0 .$$

8.7. Weak formulation.

We rewrite the problem in the form of an operator equation on a Banach space. The suitable space is the Sobolev space $W_0^1(I)$ described in 8.2. Putting $V = W_0^1(I)$ we have a reflexive separable Banach space. The equation is in the divergent form, hence multiplying it by v and integrating the first term by parts we obtain the integral identity

$$(8.8) \qquad \int_I \left[a_1(\cdot, u, u')\, v' + a_0(\cdot, u, u')\, v\right] \mathrm{d}x = \int_I fv \,\mathrm{d}x .$$

We define the operator A: $V \to V'$ by the relation

$$(8.9) \qquad \langle Au, v\rangle = \int_I \left[a_1(\cdot, u, u')\, v' + a_0(\cdot, u, u')\, v\right] \mathrm{d}x , \quad u, v \in V .$$

We can consider a more general right-hand side $f = f_0 - f_1'$, $f_0, f_1 \in L_2(I)$. Since we admit also discontinuous f_1, this case includes the Dirac distribution in f. We define $b \in V'$ by the relation

$$(8.10) \qquad \langle b, v\rangle = \int_I (f_0 v + f_1 v')\, \mathrm{d}x , \quad v \in V .$$

By the weak formulation of problem (8.7) we understand the problem

(8.11) Find $u \in V$ such that

$\langle Au, v\rangle = \langle b, v\rangle$ holds for all $v \in V$.

8.8. Justification of the weak formulation.

We have to specify the coefficients in such a way that the integrals in the formulation exist and are finite, in other words that the operator A defined by (8.9) really acts from V into V', and $b \in V'$.

We assume $f_0, f_1 \in L_2(I)$. Due to the estimate

$$|\langle b, v\rangle| \leqq \|f_0\|_{L_2} \|v\|_{L_2} + \|f_1\|_{L_2} \|v'\|_{L_2} \leqq \text{const.}\ \|v\|_V$$

the functional b: $V \to R$ is continuous, i.e. $b \in V'$.

Let us turn to the operator A. We have to find conditions general enough and such that the composed functions $a_i(\cdot, u(\cdot), u'(\cdot))$ are measurable, integrable (have a finite integral) so that A acts from V into V'. Let us remark that superposition of measurable functions need not be measurable. The problem is solved by the theorem on Nemyckij operators, see e.g. [4], which gives sufffcient conditions.

8.9. Theorem on the Nemyckij operators.

Let Ω be a domain in R^N and $h(x, \xi)$ a function

$$h: \Omega \times R^m \to R .$$

(a) *Let h satisfy the Carathéodory conditions, i.e.*

$$h(x, \xi) \text{ is measurable in } x \text{ for all fixed } \xi \in R^m , \qquad (8.12)$$

$$h(x, \xi) \text{ is continuous in } \xi \text{ for almost all } x \in \Omega .$$

Then the composed function $h(x, u_1(x), u_2(x), \ldots, u_m(x))$ is measurable for all measurable $u_1, u_2, \ldots, u_m$.

(b) *Let the function h satisfy the Carathéodory conditions* (8.12) *and let constants $p_1, p_2, \ldots, p_m, r \in [1, \infty)$ be given. Then the Nemyckij operator*

$$H: u_1, u_2, \ldots, u_m \mapsto h(\cdot, u_1(\cdot), u_2(\cdot), \ldots, u_m(\cdot)) \qquad (8.13)$$

acts between the spaces

$$H: L_{p_1}(\Omega) \times L_{p_2}(\Omega) \times \ldots \times L_{p_m}(\Omega) \to L_r(\Omega) \qquad (8.14)$$

if and only if h satisfies the growth condition

$$|h(x, \xi_1, \xi_2, \ldots, \xi_m)| \leqq g(x) + c \sum_{i=1}^{m} |\xi_i|^{p_i/r} , \qquad (8.15)$$

where $g \in L_r(\Omega)$ and c is a positive constant.

(b') *The preceding assertion* (b) *holds even if $p_i = \infty$ for some i. Let $p_i = \infty$ for $i = 1, 2, \ldots, s$; $s \leqq m$ and $p_i, r \in [1, \infty)$ for $i = s + 1, \ldots, m$.*

Then the assertion (b) *holds if we replace* (8.15) *by the condition*

$$|h(x, \xi_1, \ldots, \xi_m)| \leqq c\Big(\sum_{i=1}^{s} |\xi_i|\Big) \Big[g(x) + \sum_{i=s+1}^{m} |\xi_i|^{p_i/r}\Big] , \qquad (8.15')$$

where $g \in L_r(\Omega)$ and $c(t)$ is a continuous function.

(c) *If condition* (8.15) *or* (8.15') *is satisfied, then the operator H is continuous and bounded as the mapping* (8.14).

8.10. Justification of the weak formulation — continuation.

We adopt a natural assumption that the coefficients a_0, a_1 satisfy the Carathéodory conditions, i.e.

(8.16) $a_i(x, \xi_0, \xi_1)$, $i = 0, 1$ are measurable in x for all $\xi \in R^2$ and continuous in $\xi_1, \ldots, \xi_m$ for almost all $x \in I$.

In this case we have in (8.9) integrals $\int a_0(\cdot, u, u')\, v \,\mathrm{d}x$ and $\int a_1(\cdot, u, u')\, v' \,\mathrm{d}x$. Since $v, v' \in L_2(I)$, we need $a_i(\cdot, u, u') \in L_2(I)$, i.e. we need

$$(u, u') \in L_2(I) \times L_2(I) \mapsto a_i(\cdot, u, u') \in L_2(I) .$$

Condition (8.15) yields the growth condition

$$(8.17) \qquad |a_i(x, \xi_0, \xi_1)| \leqq g(x) + c(|\xi_0| + |\xi_1|) , \quad i = 0, 1 .$$

where $g \in L_2(I)$, $c > 0$. Indeed, the estimate (8.17) gives

$$|a_i(x, u(x), u'(x))| \leqq g(x) + c(|u(x)| + |u'(x)|) ,$$

which implies the estimate

$$\int |a_i(x, u(x), u'(x))|^2 \,\mathrm{d}x \leqq 3[\int g^2 \,\mathrm{d}x + c^2 \int u^2 \,\mathrm{d}x + c^2 \int u'^2 \,\mathrm{d}x] .$$

Condition (8.17) can be weakened if we take into account that a function from V is in a better space than $L_2(I)$, and use the imbeddings of Sobolev spaces. In our case $V \hookrightarrow C^0(I) \hookrightarrow L_\infty(I)$, see (8.4). Since $v \in L_\infty(I)$, it is sufficient to require $a_0(\cdot, u, u') \in L_1(I)$. Thus we need Nemyckij operators

$$(u, u') \in L_\infty(I) \times L_2(I) \mapsto a_1(\cdot, u, u') \in L_2(I) ,$$

$$(u, u') \in L_\infty(I) \times L_2(I) \mapsto a_0(\cdot, u, u') \in L_1(I) .$$

Condition (8.15') yields the growth conditions

$$(8.18) \qquad |a_1(x, \xi_0, \xi_1)| \leqq c_1(|\xi_0|)\,(g_1(x) + |\xi_1|) ,$$

$$|a_0(x, \xi_0, \xi_1)| \leqq c_0(|\xi_0|)\,(g_0(x) + |\xi_1|^2) ,$$

where $c_i(t)$ are continuous functions and $g_0 \in L_1(I)$, $g_1 \in L_2(I)$.

We can conclude: *If the coefficients c_i satisfy* (8.16) *and* (8.17) or (8.18) *then the operator A maps V into V' and the variational formulation of the problem is justified.*

8.11. Application of the monotone operator theory.

Due to the theorem on Nemyckij operators – assertion (c), the operator A is continuous and bounded. It remains to deal with the problem of coercivity and monotony.

Let us assume the coercivity condition in the form

$$(8.19) \qquad a_1(x, \xi_0, \xi_1)\, \xi_1 + a_0(x, \xi_0, \xi_1)\, \xi_0 \geqq c|\xi_1|^2 - K \quad \forall \xi_0, \xi_1 \in R$$

a.e. in I, where $c > 0$, $K \in R$. Then we have

$$\langle Au, u \rangle = \int_I [a_1(\cdot, u, u')\, u' + a_0(\cdot, u, u')\, u] \,\mathrm{d}x \geqq c \int_I u'^2 \,\mathrm{d}x - K .$$

Since $[\int u'^2 \,\mathrm{d}x]^{1/2}$ is an equivalent norm on V, the operator A is coercive. Concerning monotony we distinguish three cases:

(a) *Monotone case.* The monotony condition

$$(8.20) \qquad [a_1(x, \xi_0, \xi_1) - a_1(x, \eta_0, \eta_1)] (\xi_1 - \eta_1) + \\ + [a_0(x, \xi_0, \xi_1) - a_0(x, \eta_0, \eta_1)] (\xi_0 - \eta_0) \geqq 0 \\ \forall \xi_0, \xi_1, \eta_0, \eta_1 \in R$$

a.e. in I implies monotony of the operator A:

$$\langle Au - Av, u - v \rangle = \int_I \{[a_1(\cdot, u, u') - a_1(\cdot, v, v')] (u' - v') + \\ + [a_0(\cdot, u, u') - a_0(\cdot, v, v')] (u - v)\} \, dx \geqq 0 \quad \forall u, v \in V.$$

Using Theorem 7.5 we obtain the conclusion:

Let the assumptions (8.16), (1.18), (1.19), (8.20) *be satisfied and* $f_0, f_1 \in L_2(I)$. *Then the problem* (8.11) *has a solution. The solutions form a nonempty closed convex subset of* V.

(b) *The case of the* $(S)_+$ *condition.* In many important problems monotony is not satisfied. Instead of it we can assume only strong monotony in the principal part of the operator, i.e.

$$(8.21) \qquad [a_1(x, \xi_0, \xi_1) - a_1(x, \xi_0, \eta_1)] (\xi_1 - \eta_1) \geqq \alpha |\xi_1 - \eta_1|^2 \ (\alpha > 0) .$$

Then the operator A_1, given by $\langle A_1 u, v \rangle = \int_I a_1(\cdot, u, u') v' \, dx$, satisfies condition $(S)_+$. Indeed, let $u_n \rightharpoonup u$ in V and let $\limsup \langle A_1 u_n - A_1 u, u_n - u \rangle \leqq 0$. Due to (8.21) we obtain

$$\alpha \|u_n' - u'\|_{L_2}^2 \leqq \int_I [a_1(\cdot, u_n, u_n') - a_1(\cdot, u_n, u')] (u_n' - u') \, dx = \\ = \langle A_1 u_n - A_1 u, u_n - u \rangle + \\ + \int_I [a_1(\cdot, u, u') - a_1(\cdot, u_n, u')] (u_n' - u') \, dx .$$

Let us pass to lim sup. Due to the assumption, lim sup of the first term is nonpositive. Since $u_n \rightharpoonup u$ in V, the compact imbedding $V \subset\subset C^0(I)$ yields strong convergence $u_n \to u$ in $C^0(I)$ and due to the continuity of $a_1(x, \xi_0, \xi_1)$ in ξ_0 the second integral tends to zero. Since $\|u'\|_{L_2}$ is an equivalent norm on V, we obtain $\|u_n - u\|_V^2 \to 0$ which proves condition $(S)_+$.

Further, we suppose that a_0 is independent of ξ_1, i.e.

$$(8.22) \qquad a_0 = a_0(x, \xi_0) .$$

Then the remaining part of A, the operator A_0 (defined by $\langle A_0 u, v \rangle = \int_I a_0(\cdot, u) v \, dx$, is strongly continuous. Indeed, $u_n \rightharpoonup u$ in V implies $\|u_n - u\|_{C^0} \to 0$ and the continuity of a_0 in ξ_0 yields $A_0 u_n \to A_0 u$ in V', thus A_0 is strongly continuous.

Due to Lemma 6.8 the sum A also satisfies $(S)_+$. Thus using Theorem 7.1 (a) we reach the following conclusion:

Let (8.16), (8.18), (8.19), (8.21), (8.22) *be satisfied and* $f_0, f_1 \in L_2(I)$. *Then problem* (8.11) *has a solution. Moreover, due to* 7.2 (b) *the sequence of Galerkin approximate solutions has a strongly converging subsequence.*

(c) *Pseudomonotone case.* We assume only monotony in the principal part of the operator A, i.e.

$$(8.23) \qquad [a_1(x, \xi_0, \xi_1) - a_1(x, \xi_0, \eta_1)](\xi_1 - \eta_1) \geqq 0 \quad \forall \xi_0, \xi_1, \eta_1 \in R .$$

Defining the form $B: V \times V \to V'$ by the relation

$$(8.24) \qquad \langle B(u, v), w \rangle = \int_I a_1(\cdot, u, v')\, w' \,\mathrm{d}x \quad \forall u, v, w \in V ,$$

it is not difficult to verify that the principal part A_1 satisfies the assumptions of Lemma 6.10 and that A_1 is pseudomonotone. Adding assumption (8.22) we see that the operator A_0 is strongly continuous, and using Theorems 6.8 (c), 7.1 (c) we obtain the existence of a solution:

Let (8.16), (8.18), (8.19), (8.22), (8.23) *be satisfied and* $f_0, f_1 \in L_2(I)$. *Then the problem* (8.11) *has a solution.*

Let us consider the case without (8.22), i.e. a_0 does depend on ξ_1. Since superposition with a continuous nonlinear function does not preserve weak convergence in $L_2(I)$, i.e.

$$f_n \rightharpoonup f \quad \text{does not imply} \quad g(f_n) \rightharpoonup g(f) ,$$

the operator A_0 need not be pseudomonotone. Nevertheless, if the inequality (8.23) is strict for $\xi_1 \neq \eta_1$ one can prove pseudomonotony of the whole operator A. We only outline the proof. Again, we define a form $B: V \times V \to V'$ by

$$(8.24') \qquad \langle B(u, v), w \rangle = \int_I [a_1(\cdot, u, v')\, w' + a_0(\cdot, u, u')\, w] \,\mathrm{d}x , \quad u, v, w \in V$$

and verify the assumptions of Lemma 6.10. The main difficulty appears in property (d). The crucial step consists in the proof that the assumptions of (d) imply the pointwise convergence a.e. $u_n'(x) \to u'(x)$; for details see [11], [15].

8.12. Remarks.

(a) If the coefficients $a_i(x, \xi_0, \xi_1)$ are differentiable in ξ_0, ξ_1, then the monotony condition (8.20) can be rewritten in the form

$$(8.20') \qquad \frac{\partial a_1}{\partial \xi_1}(x, \xi_0, \xi_1)\, \eta_1 \eta_1 + \left[\frac{\partial a_1}{\partial \xi_0}(x, \xi_0, \xi_1) + \frac{\partial a_0}{\partial \xi_1}(x, \xi_0, \xi_1)\right] \eta_0 \eta_1 + \\ + \frac{\partial a_0}{\partial \xi_0}(x, \xi_0, \xi_1)\, \eta_0 \eta_0 \geqq 0 \quad \forall \xi_0, \xi_1, \eta_0, \eta_1 \in R .$$

Indeed, using the mean value theorem we can write

$$[a_1(x, \xi_0, \xi_1) - a_1(x, \eta_0, \eta_1)](\xi_1 - \eta_1) +$$

$$+ [a_0(x, \xi_0, \xi_1) - a_0(x, \eta_0, \eta_1)] (\xi_0 - \eta_0) =$$

$$= \int_0^1 \left[\frac{\partial a_1}{\partial \xi_1} (\vartheta) (\xi_1 - \eta_1) + \frac{\partial a_1}{\partial \xi_0} (\vartheta) (\xi_0 - \eta_0)\right] (\xi_1 - \eta_1) \, dt +$$

$$+ \int_0^1 \left[\frac{\partial a_0}{\partial \xi_1} (\vartheta) (\xi_1 - \eta_1) + \frac{\partial a_0}{\partial \xi_0} (\vartheta) (\xi_0 - \eta_0)\right] (\xi_0 - \eta_0) \, dt ,$$

where ϑ stands for $(x, \eta_0 + t(\xi_0 - \eta_0), \eta_1 + t(\xi_1 - \eta_1))$. Thus (8.20′) yields (8.20) Similarly we can rewrite (8.21) or (8.23).

(b) The fact that only homogeneous Dirichlet boundary conditions were considered is not substantial. Other boundary conditions bring only technical difficulties.

(c) The problems with coefficients growing more rapidly than in (8.18) can be investigated using Sobolev spaces $W^{1,p}(I)$ with $p > 2$ or Orlicz spaces, see e.g. [4], [10].

(d) The above introduced procedure can be applied also to boundary value problems for differential equations of order $2m$, for partial differential equations and even for systems of equations, see [2], [4], [11], [14].

Example III. Stationary nonlinear heat-conduction equation

8.13. Classical formulation.

Let Ω be a bounded domain in R^N with the boundary $\partial\Omega$ divided into two parts Γ_0, Γ_1. We shall consider the equation

$$(8.25) \qquad -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left[a(x, u) \frac{\partial u}{\partial x_i}\right] = f \quad \text{in} \quad \Omega$$

with mixed boundary conditions

$$(8.26) \qquad u = U_0 \quad \text{on} \quad \Gamma_0 ,$$

$$(8.27) \qquad a(x, u) \frac{\partial u}{\partial n} = g \quad \text{on} \quad \Gamma_1 ,$$

where $\partial u/\partial n$ is the outward normal derivative, $\partial u/\partial n = \sum \partial u/\partial x_i \, n_i$.

The equation describes the steady state of heat conduction – $u(x, t)$ represents the temperature at a point x at a time t – in a body occupying the volume Ω, with internal heat sources f. On the boundary the temperature (8.26) or the heat flow (8.27) is prescribed. The formulation covers both the space and the plane cases for $N = 3$ and $N = 2$.

The heat conduction properties of the material are described by the function $a(x, \xi)$ which corresponds to a nonhomogeneous isotropic material. If the material is homogeneous then the coefficients do not depend on x. In the case of an anisotropic material the properties are characterized by a matrix function $a_{ij}(x, \xi)$ and the operator in (8.25) is written in the form

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left[a_{ij}(x, u)\frac{\partial u}{\partial x_j}\right].$$

This case causes only technical difficulties. If the function a is independent of ξ then we have a linear conduction equation, i.e. a linear problem. This case occurs if the conductivity, the specific mass and the specific heat do not depend on the temperature u.

8.14. Domain and function space.

First, we have to exclude domains with a "bad" boundary. It is sufficient to consider domains with a Lipschitz boundary, i.e. with a boundary which can be locally expressed as the graph of a Lipschitz continuous function in convenient local coordinates. Moreover, each part of the boundary separates the domain Ω from its complement $R^N - \bar{\Omega}$, see e.g. [4]. Such a boundary has the normal vector almost everywhere, which is important for the condition (8.27). Further we assume that the part Γ_0 of the boundary is nonempty and relatively open in $\partial\Omega$, and that $\Gamma_1 = \partial\Omega - \Gamma_0$.

The space of functions convenient for our purposes is a subspace of the Sobolev space $W^{1,2}(\Omega)$. Taking into account the boundary condition (8.26), we define V as the closure of the set $\{u \in C^1(\bar{\Omega}), u = 0 \text{ on } \Gamma_0\}$ in the norm of the space $W^{1,2}(\Omega)$. The space V is a reflexive separable Banach space, see [4], [10]. The functions of $W^{1,2}(\Omega)$ can be characterized as functions absolutely continuous on almost all lines parallel to the coordinate axes and having square-integrable derivatives. The subspace V consists of functions with zeros (in the sense of traces) on Γ_0.

8.15. Weak formulation.

We multiply (8.25) by $v \in V$ and integrate over Ω. Applying the Green theorem to integrals on the left hand-side and using the boundary condition (8.27) we obtain

$$\int_\Omega \sum_{i=1}^{N} a(x, u)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x + \int_{\Gamma_1} gv\,\mathrm{d}S\,. \tag{8.28}$$

The boundary condition (8.27) is implicitly contained in this equality, condition (8.26) must be added explicitly. Let $u_0 \in W^{1,2}(\Omega)$ be a function with the prescribed values u_0 on Γ_0 in the sense of traces. Then we can formulate the problem as follows:

(8.29) Find $u \in W^{1,2}(\Omega)$ such that $u - u_0 \in V$ and the equality (8.28) holds for each $v \in V$.

We define the operator $A: W^{1,2}(\Omega) \to [W^{1,2}(\Omega)]'$ by the relation

$$\langle Au, v\rangle = \int_\Omega \sum_{i=1}^{N} a(x, u)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x \quad u, v \in W^{1,2}(\Omega) \tag{8.30}$$

and the functional b: $W^{1,2}(\Omega) \to R$ by

$$(8.31) \qquad \langle b, v\rangle = \int_\Omega fv \, dx + \int_{\Gamma_1} gv \, dS .$$

In accordance with the formulation (8.29) we look for the solution in the set $u_0 + V = \{u = u^* + u_0,\ u^* \in V\}$. Substituting $u = u_0 + u^*$ we reformulate the problem in the following way:

Find $u^* \in V$ such that

$$(8.29') \qquad \langle A^* u^*, v\rangle = \langle b, v\rangle \quad \forall v \in V ,$$

where A^* is defined by $A^* u^* = A(u^* + u_0)$.

8.16. Justification of the weak formulation.

Let us assume

$$(8.32) \qquad u_0 \in W^{1,2}(\Omega) , \quad f \in L_2(\Omega) , \quad g \in L_2(\Gamma_1) .$$

Then due to the inequality $\|v\|_{L_2(\Gamma_1)} \leqq \text{const.}\ \|v\|_{W^{1,2}(\Omega)}$ (the theorem on traces, see e.g. [4], [10]), b is a continuous linear functional on V:

$$|\langle b, v\rangle| \leqq \|f\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)} + \|g\|_{L_2(\Gamma_1)} \|v\|_{L_2(\Gamma_1)} \leqq \text{const.}\ \|v\|_{W^{1,2}(\Omega)} .$$

Let us deal with the operator A. Since $\partial u/\partial x_i,\ \partial v/\partial x_i \in L_2(\Omega)$ the integral in (8.30) is finite if $a(\cdot, u(\cdot)) \in L_\infty(\Omega)$. Thus we assume that the coefficient a: $\Omega \times R \to R$ satisfies the Carathéodory condition (see Theorem 8.9 (a)):

$$(8.33) \qquad a(x, \xi) \text{ is measurable in } x \text{ for each } \xi \in R ,$$

$$a(x, \xi) \text{ is continuous in } \xi \text{ for almost each } x \in \Omega ,$$

which ensures that $a(\cdot, u(\cdot))$ is measurable for measurable u. Further, we assume that

$$(8.34) \qquad |a(x, \xi)| \leqq c < \infty \quad \text{a.e. in } \Omega \quad \forall \xi \in R ,$$

which implies $a(\cdot, u(\cdot)) \in L_\infty(\Omega)$ for each $u \in L_2(\Omega)$. In fact, due to the imbedding theorems the values of $u \in W^{1,2}(\Omega)$ are in a better space than $L_2(\Omega)$, but this does not enable us to weaken the restriction (8.34).

Let us conclude. If the assumptions of 8.14 and (8.32)–(8.34) are satisfied then the problems (8.29), (8.29′) are well defined.

8.17. Application of the monotone operator theory.

Due to Theorem 8.9 (c) the above introduced assumptions yield, in addition, boundedness and continuity of the operators A and A^*. Further, we assume

$$(8.35) \qquad a(x, \xi) \geqq \alpha \quad (\alpha > 0) ,$$

which yields the coercivity of the operators A and A^*. Indeed, $\langle Au, u\rangle \geqq \alpha \|\nabla u\|^2_{L_2(\Omega)}$

and $\|\nabla u\|_{L_2(\Omega)}$ is an equivalent norm on V since the functions from V have zero traces on Γ_0, where Γ_0 is the set of positive $(N-1)$-dimensional measure due to the assumptions in 8.14.

Let us deal with the monotony of the operator. If the coefficient $a(x, \xi)$ is not constant in ξ, then the operator is not monotone. A counterexample can be constructed using the following simple one-dimensional example:

8.18. Example.

Let $a(\xi)$ assume at least two different values

$$a(\xi) = a_i \quad \text{for} \quad \xi \in J_i\,, \quad i = 1, 2\,,$$

where J_1, J_2 are disjoint intervals and $0 < a_1 < a_2$. Outside J_1, J_2 the function $a(\xi)$ may be arbitrary.

We construct two „saw" functions $u_1, u_2 \in W^{1,2}((0, 1))$ such that u_i has "teeth" with slopes $\pm b_i$ and values in J_i ($u_i' = \pm b_i$ a.e. in $(0, 1)$, $u_i(x) \in J_i$) $i = 1, 2$, see Fig. 3.

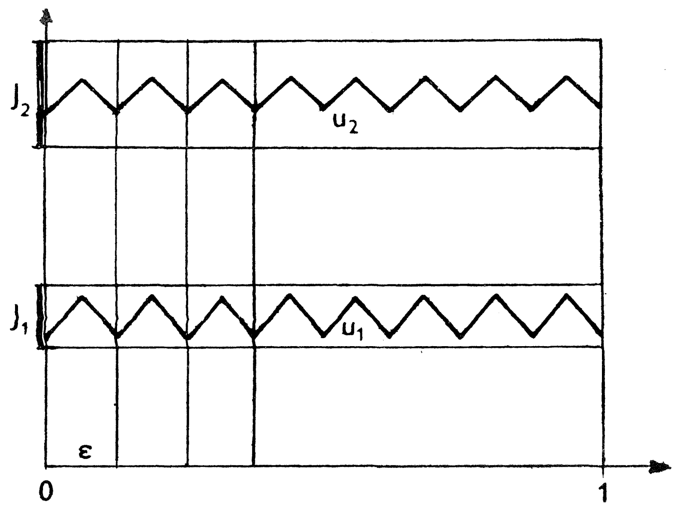


Fig. 3

The constants b_1, b_2 are chosen such that $b_1 > b_2$ but $a_1 b_1 - a_2 b_2 < 0$. If the period ε is small enough, such functions exist.

The above introduced functions violate the monotony condition. Indeed, $\langle Au_1 - Au_2, u_1 - u_2 \rangle = \int_0^1 [a(u_1) u_1' - a(u_2) u_2'] (u_1' - u_2') \,\mathrm{d}t = (a_1 b_1 - a_2 b_2) \,.$ $. (b_1 - b_2) < 0.$

8.19. Application of the monotone operator theory — continuation.

Thus the operator is not monotone. However, the assumption (8.35) yields monotony in the principal part, which makes it possible to prove pseudomonotony of the operator by using Lemma 6.10. Since we are interested in the operator $A^*u = A(u + u_0)$, we define the form $B: V \times V \to V'$ by the relation

$$\langle B(u, v), w\rangle = \int_\Omega a(\cdot, u + u_0)\, \nabla(v + u_0)\, \nabla w \,\mathrm{d}x\,, \quad u, v, w \in V \tag{8.36}$$

and verify the assumptions of Lemma 6.10.

Clearly $B(u, u) = A(u + u_0) = A^*(u)$. Due to Theorem 8.9 (c) the assumptions (a), (b) are satisfied, (8.35) implies (c):

$$\langle B(u, u) - B(u, v), u - v\rangle \geqq \alpha \|\nabla(u - v)\|^2_{L_2(\Omega)} \geqq 0\,.$$

In the proof of the implication (d) we assume that the continuous coefficient $a(x, \xi)$ is Lipschitz continuous, i.e.

$$|a(x, \xi) - a(x, \eta)| \leqq L|\xi - \eta|\,. \tag{8.37}$$

The assumption is not necessary but it simplifies the proof. Let $u_n \rightharpoonup u$ in V. Then $u_n \to u$ in $L_2(\Omega)$ strongly due to the compact imbedding $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow L_2(\Omega)$. Let $v \in V$ and $w \in C^1(\bar{\Omega})$. Then thanks to (8.37) we obtain

$$\begin{aligned}&\langle B(u_n, v) - B(u, v), w\rangle = \\ &= \int_\Omega [a(\cdot, u_n + u_0) - a(\cdot, u + u_0)]\, \nabla(v + u_0)\, \nabla w \,\mathrm{d}x \leqq \\ &\leqq \text{const. } L\|u_n - u\|_{L_2}\, \|\nabla(v + u_0)\|_{L_2}\, \|\nabla w\|_{L_\infty} \to 0\,.\end{aligned}$$

Since $C^1(\bar{\Omega})$ is dense in $W^{1,2}(\Omega)$ and B is continuous, the desired convergence $B(u_n, v) \rightharpoonup B(u, v)$ follows.

Finally, we prove the implication (e). Let $u_n \rightharpoonup u$ in V, which implies $u_n \to u$ in $L_2(\Omega)$. In the previous step we have proved $B(u_n, v) \rightharpoonup B(u, v)$. Hence we have to estimate

$$\begin{aligned}&\langle B(u_n, v), u_n\rangle - \langle B(u, v), u\rangle = \\ &= \langle B(u_n, v) - B(u, v), u_n\rangle + \langle B(u, v), u_n - u\rangle = \\ &= \int_\Omega [a(\cdot, u_n + u_0) - a(\cdot, u + u_0)]\, \nabla(v + u_0)\, \nabla u_n \,\mathrm{d}x + \\ &+ \int_\Omega a(\cdot, u + u_0)\, \nabla(v + u_0)\, \nabla(u_n - u)\,\mathrm{d}x\,.\end{aligned}$$

Clearly, the second integral tends to zero. In the first integral we replace $v + u_0$ by its approximation v^* in $C^1(\bar{\Omega})$ using the fact that $C^1(\bar{\Omega})$ is dense in $W^{1,2}(\Omega)$ and B is continuous. Then due to (3.37) and the boundedness of ∇u_n in $L_2(\Omega)$ we obtain convergence to zero of the first integral and (e) is proved. Thus A^* is pseudomonotone and using Theorem 7.1 (c) we reach the following conclusion:

Let the assumptions of 8.14 *and* (8.32)–(8.34), (8.35), (8.37) *be satisfied. Then the problem* (8.29) *has a solution.*

We conclude the section with a general nonlinear second-order partial differential equation and indicate how to proceed in the case of operators of order $2m$. We introduce only the results for problems with simple boundary conditions. For more general cases we refer to [4], [11], [15].

Example IV. Partial differential equation — general case

8.20. Formulation of the problem.

Let Ω be a bounded domain in R^N with a Lipschitz boundary $\partial\Omega$ divided into two parts Γ_0, Γ_1 and let us consider the equation

$$(8.38)\qquad -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} [a_i(x, u, \nabla u)] + a_0(x, u, \nabla u) = f \quad \text{in} \quad \Omega$$

with mixed boundary conditions

$$(8.39)\qquad u = u_0 \quad \text{on} \quad \Gamma_0$$

$$(8.40)\qquad \sum_{i=1}^{N} a_i(x, u, \nabla u)\, n_i = g \quad \text{on} \quad \Gamma_1 .$$

8.21. Weak formulation and its justification.

Taking into account the stable boundary condition (8.39) we define the Banach space V as the closure of the set $\{u \in C^1(\overline{\Omega}),\ u = 0 \text{ on } \Gamma_0\}$ in the Sobolev space $W^{1,2}(\Omega)$. We define the operator A: $W^{1,2}(\Omega) \to V'$ by

$$(8.41)\qquad \langle Au, v\rangle = \sum_{i=1}^{N} \left[a_i(x, u, \nabla u) \frac{\partial v}{\partial x_i} + a_0(x, u, \nabla u)\, v \right] \mathrm{d}x$$

and the functional $b \in V'$ by

$$(8.42)\qquad \langle b, v\rangle = \int_\Omega fv \,\mathrm{d}x + \int_{\Gamma_1} gv \,\mathrm{d}S .$$

Thus we obtain the weak formulation of the problem (8.38)–(8.40):

(8.43) Find $u \in W^{1,2}(\Omega)$ such that $u - u_0 \in V$ and $\langle Au, v\rangle = \langle b, v\rangle$ holds for each $v \in V$.

To justify this formulation we adopt the assumptions

$$(8.44)\qquad u_0 \in W^{1,2}(\Omega)\,, \quad f \in L_2(\Omega)\,, \quad g \in L_2(\Gamma_1)\,.$$

According to Theorem 8.9 on Nemyckij operators it is sufficient to suppose that the coefficients a_i: $\Omega \times R \times R^N \to R$, $i = 0, 1, \ldots, N$ satisfy the Carathéodory conditions (8.33) and the growth conditions

$$(8.45)\qquad |a_i(x, \xi_0, \xi_1, \ldots, \xi_N)| \leqq g_i(x) + c_i \sum_{j=0}^{N} \xi_j\,,$$

where $g_i \in L_2(\Omega)$ and $c_i > 0$. The restriction (8.45) can be weakened by using imbeddings of Sobolev spaces, see [4], Theorem 16.14.

8.22. Application of the theory of monotone operators.

The above introduced assumptions yield also boundedness and continuity of the operator A. The condition

$$\sum_{i=0}^{N} a_i(x, \xi_0, \xi_1, \ldots, \xi_N)\, \xi_i \geqq d_1 \sum_{i=1}^{N} \xi_i^2 + d_0 \xi_0^2 - h(x) \tag{8.46}$$

with $d_0, d_1 > 0$, $h \in L_1(\Omega)$ implies the coercivity of the operator. If $\|\nabla u\|_{L_2}$ is an equivalent norm on V, then we can admit $d_0 = 0$. Further, if the monotony condition is satisfied, i.e.

$$\sum_{i=0}^{N} \left(a_i(x, \xi_0, \xi_1, \ldots, \xi_N) - a_i(x, \eta_0, \eta_1, \ldots, \eta_N)\right)(\xi_i - \eta_i) \geqq 0\,, \tag{8.47}$$

then the operator is monotone and using Theorem 7.5 we obtain the following result: Let (8.33), (8.44)–(8.47) be satisfied. Then the problem (8.43) has a solution. The solutions form a closed convex subset in $W^{1,2}(\Omega)$.

If (8.47) is not satisfied, we can assume only strict monotony in the principal part of the operator, i.e.

$$\begin{gathered}\sum_{i=1}^{N} \left[a_i(x, \xi_0, \xi_1, \ldots, \xi_N) - a_i(x, \xi_0, \eta_1, \ldots, \eta_N)\right](\xi_i - \eta_i) > 0 \\ \text{for} \quad (\xi_1, \ldots, \xi_N) \neq (\eta_1, \ldots, \eta_N)\,.\end{gathered} \tag{8.48}$$

Then using Lemma 6.10 we can prove the pseudomonotony. For the proof we refer to [11], [15]. Thus using Theorem 7.1 (c) we obtain the following result

Let (8.33), (8.44)–(8.46) *and* (8.48) *be satisfied. Then the problem* (8.43) *has a solution.*

8.23. Remarks.

(a) The same procedure can be applied to systems of equations, see e.g. [11]; one obtains the same results, only the formulae have more indices.

(b) If the coefficients are differentiable, the conditions (8.46)–(8.48) are often expressed in terms of derivatives, see Remark 8.12 (a).

8.24. The case of the $2m^{\text{th}}$-order equation — an outline.

We shall briefly mention the case of the equation

$$\sum_{|\alpha| \leqq m} (-1)^{|\alpha|} D^\alpha[a_\alpha(x, u, D_1 u, \ldots, D_m u)] = f \quad \text{on} \quad \Omega \tag{8.49}$$

with suitable boundary conditions, see e.g. [4]. The simplest case is

$$D_\beta u = 0 \quad \text{on} \quad \partial\Omega\,, \quad |\beta| \leqq m - 1\,. \tag{8.50}$$

We use the notation with multiindices denoted by Greek characters $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \{0, 1, \ldots, m\}$, $|\alpha| = \sum \alpha_i$, D^α means $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}$ and $D_k u = (D^\alpha u, |\alpha| = k)$.

The suitable Banach space V is a subspace of the Sobolev space $W^{m,2}(\Omega)$, $W_0^{m,2}(\Omega) \subset V \subset W^{m,2}(\Omega)$ chosen in accordance with the given boundary conditions; in the case (8.50) we choose $V = W_0^{m,2}(\Omega)$. The corresponding operator A: $W^{m,2}(\Omega) \to V'$ is defined by

$$\langle Au, v\rangle = \int_\Omega \Big(\sum_{|\alpha| \leqq m} a_\alpha(x, u, D_1 u, \ldots, D_m u)\, D^\alpha v\Big)\, \mathrm{d}x\,. \tag{8.51}$$

In order that A be "well" defined, the coefficients a_α are supposed to satisfy the Carathéodory conditions and the growth conditions

$$|a_\alpha(x, \xi)| \leqq c \sum_{|\beta| \leqq m} |\xi_\beta| + g(x) \quad \forall \xi = (\xi_\beta, |\beta| \leqq m)\,. \tag{8.52}$$

Again, using imbedding theorems, the restriction (8.52) can be weakened, see [4] Theorem 16.14. The condition

$$\sum_{|\alpha| \leqq m} a_\alpha(x, \xi)\, \xi_\alpha \geqq d_1 \sum_{|\alpha| = m} \xi_\alpha^2 + d_0 \xi_0^2 - h(x) \tag{8.53}$$

implies coercivity, the condition

$$\sum_{|\alpha| \leqq m} [a_\alpha(x, \xi) - a_\alpha(x, \eta)]\, (\xi_\alpha - \eta_\alpha) \geqq 0 \tag{8.54}$$

ensures monotony of the operator and the existence result follows. It is sufficient to assume only strict monotony in the principal part, i.e.

$$\sum_{|\alpha| = m} [a_\alpha(x, \hat{\xi}, \xi_m) - a_\alpha(x, \hat{\xi}, \eta_m)]\, (\xi_\alpha - \eta_\alpha) > 0 \quad \forall \xi_m \neq \eta_m\,, \tag{8.55}$$

where $\hat{\xi} = (\xi_\beta, |\beta| \leqq m - 1)$, $\xi_m = (\xi_\beta, |\beta| = m)$, which implies pseudomonotony. However, the proof is rather complicated, see [15].

HISTORICAL REMARKS

The assumption of monotony for operators in a Hilbert space was used by M. Golomb already in 1935. The term "monotone mapping" was invented by R. I. Kačurovskij (1960), who also noticed that the differential of a convex functional is a monotone mapping. The surjectivity of a continuous coercive monotone operator was proved by G. J. Minty (1962) and F. E. Browder (1963). Pseudomonotone operators were introduced by H. Brézis (1968) and F. E. Browder (1968), the operators satisfying condition (M) also by H. Brézis. Hundreds of papers have been devoted to the theory of monotone operators and its applications, more then 300 items are quoted in the monogoraph [12] from 1978 (which has been also the main source for these remarks), further references can be found in [3], [4], [7], [8], [11], [14].

References

[1] *K. Deimling:* Nonlinear functional analysis, Springer 1985.

[2] *P. Doktor:* Modern methods of solving partial differential equations (Czech), Lecture Notes, SPN, Prague, 1976.

[3] *S. Fučík:* Solvability of nonlinear equations and boundary value problems, D. Reidel Publ. Comp., Dordrecht; JČSMF, Prague, 1980.

[4] *S. Fučík, A. Kufner:* Nonlinear differential equations; Czech edition — SNTL, Prague 1978; English translation — Elsevier, Amsterdam 1980.

[5] *S. Fučík, J. Milota:* Mathematical analysis II (Czech), Lecture Notes, SPN, Prague 1980.

[6] *S. Fučík, J. Nečas, J. Souček, V. Souček:* Spectral analysis of nonlinear operators, Lecture Notes in Math. 346, Springer, Berlin 1973; JČSMF, Prague 1973.

[7] *R. I. Kačurovskij:* Nonlinear monotone operators in Banach spaces (Russian), Uspechi Mat. Nauk 23 (1968), 2, 121—168.

[8] *D. Kinderlehrer, G. Stampacchia:* An introduction to variational inequalities and their applications, Academic Press, New York 1980; Russian translation — Mir, Moscow 1983.

[9] *A. N. Kolmogorov, S. V. Fomin:* Introductory real analysis (Russian), Moscow 1954, English translation — Prentice Hall, New York 1970, Czech translation — SNTL, Prague 1975.

[10] *A. Kufner, O. John, S. Fučík:* Function spaces, Academia, Prague 1977.

[11] *J. Nečas:* Introduction to the theory of nonlinear elliptic equations, Teubner-Texte zur Math. 52, Leipzig, 1983.

[12] *D. Pascali, S. Sburlan:* Nonlinear mappings of monotone type, Editura Academiei, Bucuresti 1978.

[13] *A. Pultr:* Subspaces of Euclidean spaces (Czech), Matematický seminář — 22, SNTL, Prague 1987.

[14] *E. Zeidler:* Lectures on nonlinear functional analysis II — Monotone operators (German), Teubner-Texte zur Math. 9, Leipzig 1977; Revised extended English translation: Nonlinear functional analysis and its application II, Springer, New York (to appear).

[15] *J. Nečas:* Nonlinear elliptic equations (French), Czech. Math. J. 19 (1969), 252—274.

[16] *M. Feistauer, A. Ženíšek:* Compactness method in the finite element theory of nonlinear elliptic problems, Numer. Math. 52 (1988), 147—163.

Souhrn

MONOTÓNNÍ OPERÁTORY

Přehled zaměřený na aplikace v diferenciálních rovnicích

JAN FRANCŮ

Článek se zabývá existencí řešení rovnic tvaru $Au = b$ s operátorem monotónním v širším smyslu, včetně pseudomonotónních operátorů a operátorů splňujících podmínky S a M. První část práce má metodický charakter a vrcholí důkazem existence řešení rovnice na reflexivním separabilním prostoru s ohraničeným demispojitým koercivním operátorem splňujícím podmínku $(M)_0$. Druhá část má přehledový charakter, srovnává různé druhy spojitosti a monotonie a uvádí řadu dalších výsledků. Použití této teorie pro důkaz existence řešení okrajových úloh pro obyčejné a parciální diferenciální rovnice je ilustrováno na příkladech.

Резюме

МОНОТОННЫЕ ОПЕРАТОРЫ

Обзор результатов применяющихся в теории дифференциальных уравнений

JAN FRANCŮ

Работа посвящена проблеме существования решений уравнений вида $Au = b$ с оператором монотонным в широком смысле, включая псевдомонотонные операторы и операторы удовлетворяющие условиям S и M. Первая часть работы имеет методический характер и заканчивается доказательством существования решения для урвнения в рефлексивном сепарабельном банаховом пространстве с ограниченным полунепрерывным коэрцитивным оператором удовлетворяющим условию $(M)_0$. Во второй части обзорного характера сравниваются различные типы непрерывности и монотонности а также приводится ряд последующих результатов. Применение этой теории к доказательству существования решений краевых задач для обыкновенных дифференциальных уравнений и уравнений в частных производных иллюстрируется примерами.

Author's address: RNDr. *Jan Franců,* CSc., katedra matematiky, strojní fakulta VUT, Technická 2, 616 69 Brno.