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ON EXISTENCE OF THE WEAK SOLUTION FOR NONLINEAR DIFFUSION EQUATION

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Summary. The paper concerns the existence of bounded weak solutions of a nonlinear diffusion equation with nonhomogeneous mixed boundary conditions.

Keywords: Nonlinear diffusion, method of lines.

AMS Classification: 45K65.

1. INTRODUCTION

The main of this paper is to find a bounded weak solution of the initial-boundary value problem for the nonlinear diffusion equation of the form

$$\begin{aligned} &\frac{\partial}{\partial t} b(u) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u) = f(x, u) \quad \text{in} \quad I \times D\,, \\ &u = u^D \quad \text{on} \quad I \times \Gamma_1\,, \\ &\sum_{i=1}^{N} a_i(x, u, \nabla u) \cos(\nu, x_i) = 0 \quad \text{on} \quad I \times \Gamma_2\,, \\ &u(x, 0) = u_0(x) \quad \text{in} \quad D\,, \end{aligned}$$

where $b(z) = |z|^m \,.\, \operatorname{sgn}(z)$, $m > 0$, D is a bounded domain in R^N with smooth boundary ∂D, Γ_1, Γ_2 are open subdomains of ∂D such that $\Gamma_1 \cap \Gamma_2 = 0$, meas Γ_1 + + meas Γ_2 = meas ∂D, meas $\Gamma_1 > 0$, a_i: $D \times R \times R^N \to R$ $(i = 1, \ldots, N)$, f: $D \times R \to R$ satisfy Carathéodory's conditions.

The equation in (1.1) appears in various physical, chemical and biological models. For $0 < m < 1$ it is known as the slow diffusion equation, for $m = 1$ as the classical heat equation and for $m > 1$ as the fast diffusion equation.

A similar equation is solved in [1], but one of the main assumptions of that paper is ellipticity of the operator $A(=\sum a_i)$ while we assume only monotonicity. We also prove boundedness of the solution. In the case of linear operator A it is possible to get a smoother solution (see [6]). We solve Problem (1.1) using the method of lines which has been intensively studied in [2], and we apply it to the slow and fast diffusions simultaneously.

In the sequel we shall adopt the following notation: Let $I = (0, T)$, $T < \infty$, $Q = D \times I$, $V = \{v \in W_p^1,\ v = 0 \text{ in } \Gamma_1\}$, $B(z) = m/(m+1)\,|z|^{m+1}$,

$$a(u; v, w) = \sum_{i=1}^{N} \int_D a_i(x, u, \nabla v) \frac{\partial w}{\partial x_i}\, dx\,,$$

$$(f(v), w) = \int_D f(x, v)\, w\, dx\,,$$

$$\partial_t^h u(t) = \frac{u(t) - u(t-h)}{h}\,.$$

2. EXISTENCE OF THE WEAK SOLUTION

We will assume that the elliptic part in (1.1) is continuous in all variables and monotone in ∇u, i.e.

$$\text{(2.1)} \qquad \sum_{i=1}^{N} (a_i(x, \eta, \xi) - a_i(x, \eta, \zeta))(\xi_i - \zeta_i) \geqq 0$$

for $x \in D$, $\eta \in R$, $\xi, \zeta \in R^N$, $a_i(x, \eta, 0) = 0$ for $i = 1, \ldots, N$, and satisfies

$$\text{(2.2)} \qquad \frac{\partial a_i(x, \eta, \xi)}{\partial \xi_j} = \frac{\partial a_j(x, \eta, \xi)}{\partial \xi_i}$$

is the sense of distribution,

$$\text{(2.3)} \qquad \sum_{i=1}^{N} a_i(x, \eta, \xi)\, \xi_i \geqq C_1 \sum_{i=1}^{N} |\xi_i|^p - C_2\,, \quad 1 < p < \infty\,.$$

The growth conditions are of the form

$$\text{(2.4)}_1 \qquad \sum_{i=1}^{N} |a_i(x, \eta, \xi)| \leqq C_3 + B(\eta)^{1/q} + |\xi|^{p-1}\ (p^{-1} + q^{-1} = 1)\,,$$

$$\text{(2.4)}_2 \qquad |f(x, \eta)| \leqq C_4(|d(x)| + |\eta|^\alpha)\,, \quad \alpha = \min\left(m, \frac{m+1}{q}\right), \quad d(x) \in L_\infty(D)\,.$$

Boundary and Initial Data satisfy

$$\text{(2.5)} \qquad u_0 \in L_\infty(D) \cap V, \quad u^D \in L_\infty(I, W_p^1(D)) \cap L_\infty(Q),\ \frac{d}{dt} u^D \in L_\infty(I \times D)\,,$$

$$u^D(t) \to u_0 \quad \text{for} \quad t \to 0 \quad \text{in} \quad L_\infty(D)\,.$$

Definition 2.6. *We call $u \in u^D + L_p(I, V)$ a weak solution of the initial boundary value problem* (1.1) *if the following two conditions are fulfilled*:

i) $b(u) \in L_\infty(Q)$, $(d/dt)\, b(u) \in L_q(I, V^*)$ *satisfy*

$$\int_I \left(\frac{d}{dt} b(u), v\right) dt = -\int_I \int_D (b(u) - b(u_0)) \frac{dv}{dt}\, dx\, dt$$

for every $v \in L_p(I, V) \cap L_\infty(Q)$, $dv/dt \in L_\infty(Q)$ *and* $v(T) = 0$;

ii) $f(u) \in L_q(I. V^*)$ *and the identity*

$$\int_I \left(\frac{\mathrm{d}}{\mathrm{d}t} b(u), v\right) \mathrm{d}t + \int_I a(u; u, v) \,\mathrm{d}t = \int_I (f(u), v) \,\mathrm{d}t$$

holds for every $v \in L_p(I, V)$.

The main result of this paper is

Theorem 2.7. *Suppose* (2.1), (2.2), (2.3), (2.4) *and* (2.5). *Then there exists a weak solution of Problem* (1.1) *in the sense of Definition* 2.6.

We now prove a series of assertions, which contain most of the essential elements for the proof of Theorem 2.7.

Suppose that an integer n is specified and set $h = T/n$. Applying a time discretization formula we use the approximation scheme

$$\frac{b(u_i) - b(u_{i-1})}{h} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u_{i-1}, \nabla u_i) = f(x, u_{i-1}), \tag{2.8}$$

$u_i = u_i^D + V$, where $u_i^D = (1/h) \int_{t_{i-1}}^{t_i} u^D(s) \,\mathrm{d}s$, $i = 1, \ldots, n$, $u_{i-1} = u_0$ for $i = 1$.

Definition 2.9. *We call* u_i, $i = 1, \ldots, n$ *a solution of Problem* (2.8) *in* V, *if* $u_i \in u_i^D + V$, $B(u_i) \in L_1(D)$, *the functional* $F_i(v) = \int_D b(u_i) v \,\mathrm{d}x$ *can be uniquely extended to* V *and the following identity holds for all* $v \in V$ $(i = 1, \ldots, n)$:

$$(\partial_i^h b(u_i), v) + a(u_{i-1}; u_i, v) = (f(u_{i-1}), v). \tag{2.10}$$

Lemma 2.11. *There exists a unique solution* u_i *of Problem* (2.8) *in the sense of Definition* 2.9 *for any positive integers* $i, n \geqq n_0$. *Moreover, each* $u_i \in L_\infty(D)$.

Proof. By induction with respect to i.

Let us suppose the assertion is true for $j = 1, \ldots, i - 1$. Let us now prove it for $j = i$.

Existence. Let

$$\Phi(v) = \int_0^1 \mathrm{d}t \int_D \sum_{i=1}^{N} a_i(x, u_{i-1}, t \nabla v) \frac{\partial v}{\partial x_i} \,\mathrm{d}x + \int_0^1 \mathrm{d}t \int_D \frac{1}{h} b(tv) \, v \,\mathrm{d}x - $$
$$- \int_D \left(\frac{1}{h} b(u_{i-1}) - f(u_{i-1})\right) v \,\mathrm{d}x .$$

Due to (2.1), (2.2), (2.3) Φ is continuous, strictly convex and coercive over V and has a G-differential

$$D\Phi(u, v) = \left(\frac{b(u) - b(u_{i-1})}{h}, v\right) + a(u_{i-1}; u, v) - (f(u_{i-1}), v).$$

The classical results concerning the minimization of Φ imply existence of a solution u_i^0 of Problem 2.8 for $j = i$ (with the homogeneous boundary condition (see [3])). Then the function $u_i = u_i^D + u_i^0$ is the solution of Problem 2.8.

Boundedness. Suppose the contrary (see [4]), that is, there exists $\{c_j\}_{j=1}^{\infty}$, $c_j \leqq c_{j+1}$, $c_j \to \infty$ for $j \to \infty$ and $K_j = \{x \in D, |u_i^0(x)| > c_j\}$ with meas $(K_j) > 0$. Putting

$$u_i^0|^{c_j} = u_i^0(x)|^{c_j} = \begin{cases} u_i^0(x) & x \in D - K_j\,, \\ c_j \operatorname{sgn}(u(x)) & x \in K_j \end{cases}$$

we easily obtain $\Phi(u_i^0|^{c_j}) < \Phi(u_i^0)$ for sufficiently large j, which contradicts the minimum property of u_i^0.

Let us construct sequences of step functions $\{u_n^D\}$, $\{u_n\}$ defined by

$$(2.12) \qquad \begin{aligned} u_n^D(t) &= u_i^D \quad t_{i-1} \leqq t < t_i\,, \quad i = 1, \ldots, n\,, \\ u_n(t) &= u_i \quad t_{i-1} \leqq t < t_i\,, \quad i = 1, \ldots, n\,. \end{aligned}$$

We can write (2.10) in the form

$$(2.13) \qquad (\partial_t^h b(u_n), v) + a(u_{nh}; u_n, v) = (f(u_{nh}), v)$$

where $n \geqq n_0$, $v \in V$, $u_{nh} = u_n(t - h)$ and $u_n(t) = u_0$ for $t \in (-h, 0)$. The main role in the proof of Theorem 2.7 is played by the uniform boundedness of the sequence $\{u_i\}$ in $L_\infty(D)$. First we have to prove a priori estimates.

Lemma 2.14. *The estimates*

i) $\int_I \|u_n\|_V^p \, \mathrm{d}t \leqq C$,

ii) $\int_D B(u_n(t)) \, \mathrm{d}x \leqq C$

hold for all $n \geqq n_0$.

Proof. Putting in (2.13) $v = u_n - u_n^D$ and integrating it over $(0, \tau)$ we have

$$(2.15) \qquad \int_0^\tau \int_D \partial_t^h b(u_n) (u_n - u_n^D) \, \mathrm{d}x \, \mathrm{d}t + \int_0^\tau a(u_{nh}; u_n, u_n - u_n^D) \, \mathrm{d}t = \\ = \int_0^\tau \int_D f(u_{nh}) (u_n - u_n^D) \, \mathrm{d}x \, \mathrm{d}t\,.$$

First let us estimate the first term in (2.15). Using the inequality

$$(2.16) \qquad B(z) - B(z_0) \leqq (b(z) - b(z_0))\, z\,,$$

which follows from the fact that the function $f(x) = m/(m + 1)\, x^{m+1} - x^m + 1/(m + 1) \geqq 0$ for $x > 0$, we can write

$$(2.17) \qquad B(u_n) - B(u_{nh}) \leqq (b(u_n) - b(u_{nh}))\, u_n \quad \text{for a.e.} \quad t \in (h, T)\,.$$

We can integrate it over $(0, \tau) \times D$:

$$(2.18) \qquad 1/h \int_0^\tau \int_D (B(u_n) - B(u_{nh})) \, \mathrm{d}x \, \mathrm{d}t \leqq \int_0^\tau \int_D \partial_t^h b(u_n)\, u_n \, \mathrm{d}x \, \mathrm{d}t\,.$$

In virtue of the equality

$$\int_0^\tau \int_D \partial_t^h b(u_n)\, u_n \, \mathrm{d}x \, \mathrm{d}t = \int_0^\tau (\partial_t^h b(u_n), u_n - u_n^D) \, \mathrm{d}t + \int_0^\tau \int_D \partial_t^h b(u_n)\, u_n^D \, \mathrm{d}x \, \mathrm{d}t$$

in (2.18) we have

$$1/h \int_0^\tau \int_D (B(u_n) - B(u_{nh})) \, \mathrm{d}x \, \mathrm{d}t \leqq \int_0^\tau (\partial_t^h b(u_n), u_n - u_n^D) \, \mathrm{d}t + \int_0^\tau \int_D \partial_t^h b(u_n)\, u_n^D \, \mathrm{d}x \, \mathrm{d}t\,.$$

Using integration by parts in the above inequality, we obtain

(2.19) $$1/h \int_{\tau-h}^{\tau} \int_D B(u_n)\, dx\, dt - \int_D B(u_0)\, dx \leqq \int_0^{\tau} (\partial_t^h b(u_n), u_n - u_n^D)\, dt -$$
$$- \int_0^{\tau-h} \int_D (b(u_n) - b(u_0))\, \partial_t^{-h} u_n^D\, dx\, dt + 1/h \int_{\tau-h}^{\tau} \int_D (b(u_n) - b(u_0))\, u_n^D\, dx\, dt\,,$$

which yields

(2.20) $$\int_0^{\tau} \int_D \partial_t^h b(u_n)\,(u_n - u_n^D)\, dx\, dt \geqq 1/h \int_{\tau-h}^{\tau} \int_D B(u_n)\, dx\, dt - \int_D B(u_0)\, dx +$$
$$+ \int_0^{\tau-h} \int_D (b(u_n) - b(u_0))\, \partial_t^{-h} u_n^D\, dx\, dt - 1/h \int_{\tau-h}^{\tau} \int_D (b(u_n) - b(u_0))\, u_n^D\, dx\, dt\,.$$

In virtue of (2.3), (2.4) and Young's inequality we estimate

(2.21) $$\int_0^{\tau} a(u_{nh}; u_n, u_n - u_n^D)\, dt \geqq C_1 \int_0^{\tau} \|u_n\|_V^p\, dt + C_2 \int_0^{\tau} \|u_n^D\|_V^p\, dt - C_3 \int_0^{\tau} \int_D B(u_{nh})\, dx\, dt\,,$$

(2.22) $$\left|\int_0^{\tau} \int_D f(u_{nh})(u_n - u_n^D)\, dx\, dt\right| \leqq \delta \int_0^{\tau} \|u_n\|_V^p\, dt + C_\delta \int_0^{\tau} \int_D B(u_{nh})\, dx\, dt + C \int_0^{\tau} \|u_n^D\|_V^p\, dt\,,$$

which together with (2.20) yields (if δ is sufficiently small)

(2.23) $$1/h \int_{\tau-h}^{\tau} \int_D B(u_{nh})\, dx\, dt + \int_0^{\tau} \|u_n\|_V^p\, dt \leqq C_1 \int_0^{\tau} \int_D B(u_{nh})\, dx\, dt + C_2$$

in view of

$$\int_D B(u_0)\, dx \leqq C\,,$$
$$1/h \int_{\tau-h}^{\tau} \int_D |b(u_n)|\, dx\, dt \leqq \delta/h \int_{\tau-h}^{\tau} \int_D B(u_n)\, dx\, dt + C_\delta\,,$$
$$\int_0^{\tau-h} \int_D b(u_n)\, \partial_t^{-h} u_n^D\, dx\, dt \leqq C \int_0^{\tau-h} \int_D |b(u_n)|\, dx\, dt \leqq$$
$$\leqq C_1 \int_0^{\tau} \int_D B(u_n)\, dx\, dt + C_2\,.$$

Using Gronwall's lemma in the discrete form (letting $a_i = \int_D B(u_n(t))\, dx$ for $t_{i-1} \leqq \leqq t < t_i$) we obtain the required estimates.

Lemma 2.24. *For any $n \geqq n_0$ and $i = 1, \ldots, n$ the solution u_i satisfies*

$$\|u_i\|_{L_\infty(D)} \leqq C\,.$$

Proof. Putting $v = b^s(u_i) - b^s(u_i^D)$ (see [6]), where s is odd, $s > s_0(m)$ we obtain

(2.25) $$\int_D (b(u_i) - b(u_{i-1}))\, b^s(u_i)\, dx - \int_D (b(u_i) - b(u_{i-1}))\, b^s(u_i^D)\, dx +$$
$$+ h \int_D a(u_{i-1}, \nabla u_i)\, \nabla b^s(u_i)\, dx - h \int_D a(u_{i-1}, \nabla u_i)\, \nabla b^s(u_i^D)\, dx =$$
$$= \int_D f(u_{i-1})\,(b^s(u_i) - b^s(u_i^D))\, dx\,.$$

In virtue of (2.1) and $(2.4)_2$ we can estimate the third term on the left-hand side and the term on the right-side obtaining

(2.26) $$\int_D b^{s+1}(u_i) \leqq \int_D (1 + C_4 h)\, |b(u_{i-1})|\, |b^s(u_i)| + C_4 h \int_D |d|\, |b^s(u_i)| +$$
$$+ \int_D |b(u_i) - b(u_{i-1})|\, |b^s(u_i^D)| + h \int_D |a(u_{i-1}, \nabla u_i)|\, |\nabla b^s(u_i^D)| +$$
$$+ C_4 h \int_D |d|\, |b^s(u_i^D)| + C_4 h \int_D |b(u_{i-1})|\, |b^s(u_i^D)|\,.$$

Now we can successively estimate the third, the fourth, the fifth and the sixth terms on the right-side in (2.26):

$$\int_D |b(u_i) - b(u_{i-1})|\, |b^s(u_i^D)| \leqq |b^s(c_M)| \int_D (|b(u_i)| + |b(u_{i-1})|) \leqq$$
$$\leqq C|b^s(c_M)|$$

where $c_M = \sup |u_i^D|$, in view of (2.5), $(2.14)_{ii}$;

$$h \int_D |a(u_{i-1}, \nabla u_i)|\, |\nabla b^s(u_i^D)| = h \int_D |a(u_{i-1}, \nabla u_i)\, s b^{s-1}(u_i^D)|\, |\nabla u_i^D| \leqq$$
$$\leqq hs|b^{s-1}(c_M)| \left(\int_D B(u_{i-1}) + \int_D |\nabla u_i|^p + \int_D |\nabla u_i^D|^p \leqq Chs|b^{s-1}(c_M)|\right.$$

holds in view of (2.5) and Lemma 2.14;

$$C_4 h \int_D |d|\, |b^s(u_i^D)| \leqq C_4 h |b^s(c_M)| \int_D |d| \leqq Ch|b^s(c_M)|\,;$$
$$C_4 h \int_D |b(u_{i-1})|\, |b^s(u_i^D)| \leqq C_4 h |b^s(c_M)| \int_D |b(u_{i-1})| \leqq Ch|b^s(c_M)|\,.$$

All these estimates together with (2.26) imply

$$\int_D b^{s+1}(u_i) \leqq \int_D (1 + C_4 h)\, |b(u_{i-1})|\, |b^s(u_i)| + C_4 h \int_D |d|\, |b^s(u_i)| +$$
$$+ sC|b^s(c_M)|\,.$$

Applying twice Young's inequality we obtain

$$\int_D b^{s+1}(u_i) \leqq (1 + \eta h)^{s+1} \int_D b^{s+1}(u_{i-1}) + \frac{(s + 1)\, C_4 h C(\varepsilon)}{1 - \varepsilon C_4 h} \int_D |d|^{s+1} +$$
$$+ \frac{sC}{1 - \varepsilon C_4 h} |b^s(c_M)|\,, \quad \text{where} \quad 0 < \varepsilon < \frac{1}{C_4 h} \quad \text{and} \quad \eta = \frac{C_4(\varepsilon + 1)}{1 - \varepsilon C_4 h}\,.$$

This inequality may be formally rewritten as $y_i \leqq a y_{i-1} + b$, from which we recurrently obtain

$$y_i \leqq a y_{i-1} + b \leqq a^i y_0 + b \frac{a^i - 1}{a - 1} \leqq a^i \left(y_0 + \frac{b}{a - 1} \right).$$

So we have

$$\int_D b^{s+1}(u_i) \leqq (1 + \eta h)^{i(s+1)} \left(\int_D b^{s+1}(u_0) + C_1' \int_D |d|^{s+1} + \right.$$
$$\left. + C_2' |b^s(c_M)|\right) \tag{2.27}$$

where the constants C_i' $(i = 1, 2)$ are such that their $(s + 1)$-st root tends to 1 if $s \to \infty$.

Now taking the $(s + 1)$-st root of (2.27) and letting $s \to \infty$ we obtain

$$\|b(u_i)\|_{L_\infty(D)} \leqq (1 + \eta h)^i \left(\|b(u_0)\|_{L_\infty(D)} + \|d(x)\|_{L_\infty(D)} + |b(c_M)|\right),$$

where we can estimate $(1 + \eta h)^i \leqq \exp(\eta T)$. This completes the proof.

Lemma 2.28. *The sequence of functions $\{u_n\}$ defined by* (2.12) *is uniformly bounded in $L_\infty(Q)$.*

Proof. This lemma immediately follows from Lemma 2.24.

Lemma 2.29. *The estimate $\int_0^{T-\tau} (b(u_n(t + \tau)) - b(u_n(t)), u_n(t + \tau) - u_n(t))\, dt \leqq \leqq C\tau$ holds for $n \geqq n_0$ and $0 < \tau \leqq \tau_0 < T$.*

Proof. Sum up (2.10) for $i = j + 1, \ldots, j + k$ and then put $v = u_{j+k} - u_j - (u^D_{j+k} - u^D_j)$. We estimate

$$\begin{aligned}
&\int_D (b(u_{j+k} - b(u_j))(u_{j+k} - u_j) \leqq \int_D |b(u_{j+k}) - b(u_j)|\, |u^D_{j+k} - u^D_j| + \\
&+ \sum_{i=j+1}^{j+k} \{|a(u_{i-1}; u_{j+k} - u_j - (u^D_{j+k} - u^D_j))| + \\
&+ \int_D |f(u_{i-1})|\, |u_{j+k} - u_j - (u^D_{j+k} - u^D_j)|\}\, h \leqq \\
&\leqq C\{(\int_D (B(u_{j+k}) + B(u_j)) + 1)\, \|u^D_{j+k} - u^D_j\|_{L_\infty(D)} + \\
&+ h \sum_{i=j}^{j+k} \int_D B(u_i) + h \sum_{i=j+1}^{j+k} \|u_i\|^p + kh(\|u_{j+k}\|^p + \|u_j\|^p + 1)\} \leqq \\
&\leqq C\{kh(1 + \|u_{j+k}\|^p + \|u_j\|^p) + h \sum_{i=j+1}^{j+k} \|u_i\|^p\} \leqq Ckh
\end{aligned}$$

because of $(2.14)_{ii}$, (2.4) and (2.5).

Using our notation, we conclude that

$$\int_0^{T-kh} \int_D (b(u_n(t + kh) - b(u_n(t))(u_n(t + kh) - u_n(t))\, dx\, dt \leqq Ckh$$

and Lemma 2.29 is proved.

From Lemma 2.29 it follows that the sequence $\{b(u_n)\}$ is compact in $L_1(Q)$ (see [1], Lemma 1.8, 1.9); thus there exists a subsequence (in the sequel, we denote a subsequence of $\{u_n\}$ again by $\{u_n\}$) and a function u such that

(2.30) $$b(u_n) \to b(u) \quad \text{in} \quad L_1(Q)\,,$$
$$b(u_{nh}) \to b(u) \quad \text{in} \quad L_1(Q)\,.$$

From the fact that the operator $b(u) = |u|^m \operatorname{sgn}(u)$ is strictly monotone and from (2.30) it follows (see [7]) that

(2.31) $$u_n \to u \quad \text{a.e. in} \quad Q$$

and $$u_n \to u \quad \text{in} \quad L_r(Q) \quad \text{for} \quad r > 1\,,$$

because u_n is bounded in $L_\infty(Q)$.

Lemma 2.32. *The sequence $\{u_n\}$ satisfies*

i) $\partial^h_t b(u_n) \rightharpoonup (d/dt)\, b(u)$ *in* $L_q(I, V^*)$,

ii) $f(u_{nh}) \rightharpoonup f(u)$ *in* $L_q(Q)$,

iii) $a_i(u_{nh}, \nabla u_n) \rightharpoonup a_i(u, \nabla u)$ *in* $L_q(Q)$, $i = 1, \ldots, N$.

Proof. i) In virtue of (2.14), (2.28) and (2.13) we have

$$(2.33) \qquad \sup_{\|v\|_{L_p(I,V)} \leqq 1,\, v \in L_\infty(D)} \left| \int_I \int_D \partial_t^h b(u_n)\, v \, \mathrm{d}x \, \mathrm{d}t \right| \leqq C .$$

So the sequence $\{\partial_t^h b(u_n)\}$ is uniformly bounded in $L_q(I, V^*)$. Then there exists a subsequence and $\chi \in L_q(I, V^*)$ such that

$$(2.34) \qquad \partial_t^h b(u_n) \rightharpoonup \chi \quad \text{in} \quad L_q(I, V^*) .$$

From the fact that $\|\partial_t^h b(u)\|_{L_q(I,V^*)} \leqq C$ it follows that there exsits a subsequence such that

$$(2.35) \qquad \partial_t^h b(u) \rightharpoonup \psi := \frac{\mathrm{d}}{\mathrm{d}t} b(u) \quad \text{in} \quad L_q(I, V^*) .$$

In virtue of (2.30) the identity

$$\int_I (\chi, v) \, \mathrm{d}t = -\int_I \int_D (b(u) - b(u_0)) \frac{\mathrm{d}v}{\mathrm{d}t} \, \mathrm{d}t$$

holds for $v \in L_p(I, V) \cap L_\infty(Q)$, $\mathrm{d}v/\mathrm{d}t \in L_\infty(Q)$, $v(T) = 0$.

Putting $v = \varphi_h = (1/h) \int_{t-h}^t \varphi \, \mathrm{d}s$, $\varphi \in L_p(I, V)$ and realizing that $\varphi_h \rightharpoonup \varphi$ in $L_p(I, V)$ we obtain

$$\int_I (\chi, \varphi) = \int_I \left(\frac{\mathrm{d}}{\mathrm{d}t} b(u), \varphi \right), \quad \text{which yeidls} \quad \chi = \frac{\mathrm{d}}{\mathrm{d}t} b(u) .$$

ii) In virtue of (2.31) we have

$$(2.36) \qquad f(u_{nh}) \to f(u) \quad \text{a.e. in } Q .$$

From $(2.4)_2$ and from Lemma 2.28 we obtain that

$$\|f(u_{nh})\|_{L_q(Q)}^q \leqq C_1 + C_2 \int_I \int_D B(u_{nh}) \, \mathrm{d}x \, \mathrm{d}t \leqq C .$$

Hence there exists $\chi \in L_q(Q)$ such that

$$(2.37) \qquad f(u_{nh}) \rightharpoonup \chi \quad \text{in} \quad L_q(Q) .$$

(2.36), (2.37) together give the assertion ii) of Lemma 2.32.

iii) In virtue of $(2.4)_1$, Lemma 2.28 and (2.14) we obtain

$$\|a_i(u_{nh}, \nabla u_n)\|_{L_q(Q)} \leqq C , \quad i = 1, \ldots, N ,$$

which implies that there exists $\chi_i \in L_q(Q)$ $(i = 1, \ldots, N)$ such that

$$a_i(u_{nh}, \nabla u_n) \rightharpoonup \chi_i \quad \text{in} \quad L_q(Q) , \quad i = 1, \ldots, N .$$

To show that $\chi_i = a_i(u, \nabla u)$ we use Minty's trick (see [3]), which is based on the relation

$$(2.38) \qquad \limsup \int_I a(u_{nh}; u_n, (u_n - u)) \leqq 0 .$$

(We prove this inequality later on.)

From the monotonicity we have

$$\int_I \int_D (a_i(u_{nh}, \nabla u_n) - a_i(u_{nh}, w))\left(\frac{\partial}{\partial x_i} u_n - w\right) \geqq 0\,,$$

where $w \in [L_p(Q)]^N$,

and letting $n \to \infty$ we obtain

$$\int_I \int_D \sum_{i=1}^{N} (\chi_i - a_i(u, w))\left(\frac{\partial}{\partial x_i} u - w\right) \geqq 0\,.$$

Putting $w = \nabla u + \varepsilon v$, $\varepsilon > 0$, $v \in L_\infty(Q)$, the above inequality (after $\varepsilon \to 0_+$) yields

$$\int_I \int_D \sum_{i=1}^{N} (\chi_i - a_i(u, \nabla u))\, v \,\mathrm{d}x \,\mathrm{d}t \geqq 0\,. \tag{2.39}$$

Now putting $w = \nabla u - \varepsilon v$, $\varepsilon > 0$ we obtain

$$\int_I \int_D \sum_{i=1}^{N} (\chi_i - a_i(u, \nabla u))\, v \,\mathrm{d}x \,\mathrm{d}t \leqq 0\,. \tag{2.40}$$

(2.39), (2.40) together yield

$$\int_I \int_D \sum_{i=1}^{N} (\chi_i - a_i(u, \nabla u))\, v \,\mathrm{d}x \,\mathrm{d}t = 0\,,$$

which holds for all $v \in L_\infty(Q)$. This implies that

$$\chi_i = a_i(u, \nabla u) \quad \text{a.e. in } Q\,, \quad i = 1, \ldots, N\,.$$

Now we prove (2.38).

Putting $v = u_n - u_n^D$ in (2.13) and integrating it over $(0, t)$ we obtain

$$\int_0^t a(u_{nh}; u_n, (u_n - u_n^D)) \,\mathrm{d}t =$$
$$= \int_0^t \int_D f(u_{nh}) (u_n - u_n^D) \,\mathrm{d}x \,\mathrm{d}t - \int_0^t (\partial_t^h b(u_n), (u_n - u_n^D))\,.$$

Using (2.19) in the above equality we have

$$\int_0^t a(u_{nh}; u_n, u_n - u_n^D) \leqq \int_0^t f(u_{nh}) (u_n - u_n^D) - 1/h \int_{t-h}^t B(u_n(t)) +$$
$$+ \int_D B(u_0) - \int_0^{t-h} \int_D (b(u_n(t)) - b(u_0))\, \partial_t^{-h} u_n^D +$$
$$+ 1/h \int_{t-h}^t \int_D (b(u_n(t)) - b(u_0))\, u_n^D\,.$$

Letting $n \to \infty$ and using (2.28), (2.30), (2.32) and Lemma 1.5 from [1] we obtain the estimate

$$\limsup \int_0^t a(u_{nh}; u_n, u_n - u_n^D) \leqq \int_0^t \int_D f(u) (u - u^D) - \tag{2.41}$$
$$- \int_0^t \left(\frac{\mathrm{d}}{\mathrm{d}t} b(u), u - u^D\right),$$

which holds for a.e. $t \in (0, T)$.

Now let $\varphi \in L_p(I, V) \cap L_\infty(Q)$, $d\varphi/dt \in L_\infty(Q)$, $\varphi(s) = 0$ for $s \in (T - \delta, T)$, $\varphi(0) = 0$; then

$$(2.42) \qquad \int_I a(u_{nh}; u_n, \varphi) = \int_I \int_D f(u_{nh}) \varphi - \int_I \int_D \partial_t^h b(u_n) \varphi =$$
$$= \int_I \int_D f(u_{nh}) \varphi + \int_I \int_D b(u_n) \partial_t^{-h} \varphi \to \int_I \int_D f(u) \varphi +$$
$$+ \int_I \int_D b(u) \frac{d\varphi}{dt} = \int_I \int_D f(u) \varphi - \int_I \left(\frac{d}{dt} b(u), \varphi \right).$$

Since the set $\Phi = \{\varphi \in L_p(I, V) \cap L_\infty(Q),\ d\varphi/dt \in L_\infty(Q),\ \varphi(s) = 0,\ s \in (0, \delta), (T - \delta, T),\ \delta > 0\}$ is dense in $L_p(I, V)$ (see [8]) we can put $\varphi = u - u^D \in L_p(I, V)$ in (2.42) and obtain

$$(2.43) \qquad \int_I a(u_{nh}; u_n, (u - u^D)) \to \int_I \int_D f(u)(u - u^D) - \int_I \left(\frac{d}{dt} b(u), u - u^D \right).$$

(2.41) and (2.42) together yield (2.38).

Proof of Theorem 2.7: Let us put $v \in L_p(I, V)$ in (2.13) and then integrate it over $(0, T)$. Taking the limit as $n \to \infty$ we obtain (in virtue of Lemma 2.32) that u is a weak solution of (1.1) in the sense of Definition 2.6.

3. GENERALIZATIONS

a) The growth of the coefficients a_i: Instead of the condition $(2.4)_1$ we can consider

$$(3.1) \qquad \sum_{i=1}^N |a_i(x, \eta, \xi)| \leqq \mu(|\eta|)(C + |\xi|^{p-1}),$$

where $\mu(z) \in C((0, \infty))$ is increasing

Theorem 3.2. *If* (2.1), (2.2), (2.3), (3.1), $(2.4)_2$ *and* (2.5) *are satisfied, then there exists a weak solution of Problem* (1.1).

Proof. We replace the coefficients a_i in (1.1) by

$$(3.3) \qquad a_i^R = a_i(x, \lambda_R \eta, \xi), \quad \text{where} \quad \lambda_R = \min\left(1, \frac{R}{|\eta|}\right), \quad R > 0.$$

In virtue of (3.1) we have

$$\sum_{i=1}^N |a_i^R(x, \eta, \xi)| = \sum_{i=1}^N |a_i(x, \lambda_R \eta, \xi)| \leqq \mu(|\lambda_R \eta|)(C + |\xi|^{p-1}) \leqq$$
$$\leqq \mu(R)(C + |\xi|^{p-1}),$$

where $\mu(R)$ is a constant, because of $|\lambda_R \eta| \leqq R$. This growth condition is a special case of $(2.4)_1$.

Now we consider the problem

$$\frac{\partial}{\partial t} b(u) - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i^R(x, u, \nabla u) = f(x, u), \tag{3.4}$$

$$u = u^D \quad \text{on} \quad I \times \Gamma_1,$$

$$\sum_{i=1}^{N} a_i(x, u, \nabla u) \cos(\nu, x_i) = 0 \quad \text{on} \quad I \times \Gamma_2,$$

$$u(x, 0) = u_0(x) \quad \text{in} \quad D.$$

In virtue of Theorem 2.7 there exists a solution of (3.4). Let us denote it by u_R. Lemma 2.38 yields

$$\|u_R\|_{L_\infty(Q)} \leqq P, \quad \text{where} \quad P = C_1(\|u_0\|_{L_\infty(D)} + \|d(x)\|_{L_\infty(D)} + b(c_M)) \exp(C_2 T) \tag{3.5}$$

and P does not depend on R.

Putting $R > P$ in (3.3) and considering (3.5) we obtain $a_i^R \equiv a_i$ for $i = 1, \ldots, N$, because $\lambda_R = 1$. Now Problem (3.4) is identical with Problem (1.1) and the proof is complete.

b) Time dependent coefficients: All arguments remain the same, if we assume that the coefficients a_i and f depend on t. In this case we have to assume that $a_i(x, t, \eta, \xi)$ $(i = 1, \ldots, N)$ and $f(x, t, \eta)$ are smooth in t, η, ξ and we have to use approximations a_h, f_h in (2.8) which are piecewise constant in time, for example

$$a_h(x, t, u_{i-1}, \nabla u_i) = \frac{1}{h} \int_{t_{i-1}}^{t_i} a(x, s, u_{i-1}, \nabla u_i)\, ds,$$

$$f_h(x, t, u_{i-1}) = \frac{1}{h} \int_{t_{i-1}}^{t_i} f(x, s, u_{i-1})\, ds, \quad \text{for} \quad t_{i-1} \leqq s < t_i.$$

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Súhrn

O EXISTENCII SLABÉHO RIEŠENIA NELINEÁRNEJ DIFÚZNEJ ROVNICE

JURAJ ZEMAN

Práca je venovaná otázkam existencie ohraničeného slabého riešenia nelineárnej difúznej rovnice s nehomogénnymi zmiešanými okrajovými podmienkami.

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