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GLOBAL SOLUTION TO THE ISOTHERMAL COMPRESSIBLE BIPOLAR FLUID IN A FINITE CHANNEL WITH NONZERO INPUT AND OUTPUT

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Summary. The paper contains the proof of global existence of weak solutions viscous compressible isothermal bipolar fluid of initial boundary value in a finite channel.

Keywords: Viscous compressible bipolar fluid, initial boundary value problem, global existence of weak solutions.

AMS classification: 35Q20, 76N10.

1. INTRODUCTION

This article is inspired by the paper by J. Nečas, A. Novotný, M. Šilhavý [7] concerning the global solution to the isothermal compressible bipolar fluid where Orlicz spaces were used for describing finite entropy and theorems of the compensated compactness type. I follow in this paper the ideas of M. Feistauer, J. Nečas, V. Šverák [1], J. Nečas, A. Novotný, M. Šilhavý [7], M. Padula [9].

The main step in this work is the study of a bipolar fluid in a finite channel. Higher stress tensor implies the use of higher derivations of the velocity field. The existence of a global Hopf solution, under general initial data \((0, t_0) \times \Omega\) with \(t_0\) arbitrary and \(\Omega \subseteq \mathbb{R}^N, N = 2\) or 3 is proved.

In the present case, only one new stress tensor is needed, such that the momentum equations are of the 4th order. So we come to a bipolar fluid. The corresponding stress strain relations are supposed to be linear.

We suppose that density \(\rho\) on input is \(\rho_0 > 0\), the velocity \(v = v^0\) on input and output, where \(v_0\) is extended to the entire \(Q_t\).

2. FORMULATION OF THE PROBLEM

We consider the classical state equation

\[
(2.1) \quad p = R \rho \, T
\]
where $p$, $\varrho$, $T$ are pressure, density and temperature, respectively and $R$ is the universal gas constant.

The isothermal process means that

$$
(2.2) \quad p = \varrho \lambda, \quad \lambda = \text{const} > 0. 
$$

As usual we denote by $v$ the velocity vector. The continuity equation assumes its standard form

$$
(2.3) \quad \frac{\partial \varrho}{\partial t} + \frac{\partial (\varrho v_i)}{\partial x_i} = 0. 
$$

A standard symmetric stress tensor $\tau_{ij}$ is considered such that

$$
(2.4) \quad \tau_{ij} = -p\delta_{ij} + \tau^d_{ij}. 
$$

The general linear form for $\tau^d_{ij}$, with coefficients depending on the temperature $T$ only and therefore constant in our case, provided $\tau^d_{ij}$ are symmetric, is

$$
(2.5) \quad \tau^d_{ij} = \gamma \frac{\partial v_i}{\partial x_1} \delta_{ij} + 2\mu e_{ij} - \gamma_1 A \frac{\partial v_i}{\partial x_1} \delta_{ij} - 2\mu_1 A e_{ij} + \gamma_2 \frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{\partial v_i}{\partial x_1} \right), 
$$

see [3].

We shall suppose that $\gamma \geq -\frac{3}{2} \mu$, $\mu > 0$, $\gamma_1 > -\frac{3}{2} \mu$, $\mu_1 > 0$, $\gamma_2 = 0$, $2\mu_1 = -\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$. We consider further a 3rd order stress tensor $\tau^d_{ijk}$. For it we require symmetry in $i, j$, then its general form according to [3] is

$$
(2.6) \quad \tau_{ijk} = 2\mu \frac{\partial e_{ij}}{\partial x_k} + \gamma_1 \delta_{ij} \frac{\partial e_{lk}}{\partial x_k} + \gamma_3 \delta_{ij} A v_k + \gamma_4 \delta_{ik} A v_j + \gamma_4 \delta_{jk} A v_i + \\
+ \gamma_3 \delta_{ik} \frac{\partial e_{lj}}{\partial x_j} + \gamma_5 \delta_{jk} \frac{\partial e_{li}}{\partial x_j} + \gamma_6 \frac{\partial^2 v_k}{\partial x_i \partial x_j} + \gamma_7 \frac{\partial^2 v_i}{\partial x_j \partial x_k} + \gamma_7 \frac{\partial^2 v_j}{\partial x_i \partial x_k}. 
$$

We shall restrict ourselves to the case $\gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = 0$. The Clausius-Duhem inequality (see [3])

$$
(2.7) \quad \tau^d_{ijk} e_{ij} + \tau^d_{ijk} \frac{\partial^2 v_i}{\partial x_j \partial x_i} + \frac{\partial^2 v_i}{\partial x_k \partial x_k} \geq 0
$$

is satisfied for (2.6), (2.5).

Let $\Omega \subseteq R^N$, $N = 2$ or $3$ be a bounded domain with a smooth, infinitely differentiable boundary and let $Q_{t_0} = (0, t_0) \times \Omega$ be the time-space cylinder.

The momentum equations combined with (2.3) yield

$$
(2.8) \quad \frac{\partial (\varrho v_i)}{\partial t} + \frac{\partial}{\partial x_j} \left( \varrho v_i v_j + \delta_{ij} p - \tau^d_{ij} \right) = 0. 
$$

In addition to initial conditions for $v$ and $\varrho$, we suppose that $\Omega$ is a finite channel, where we have the following conditions: for the input, output and on the upper and lower sides.
We suppose that the velocity \( v \) need not be equal to zero in two parts of \( \partial \Omega \):

\[
\Gamma_{\text{inp}} = \{ x \in \partial \Omega ; \; v \nu < 0 \},
\]

\[
\Gamma_{\text{out}} = \{ x \in \partial \Omega ; \; v \nu > 0 \}; \text{ and we denote}
\]

\[
\Gamma_{\text{upp+low}} = \{ x \in \partial \Omega ; \; \partial \Omega \setminus [\Gamma_{\text{inp}} \cup \Gamma_{\text{out}}] \},
\]

where \( \nu \) is the outer normal.

Conditions for the velocity are:

\[
v = v^0 \quad \text{on} \quad \Gamma_{\text{inp}} \cup \Gamma_{\text{out}},
\]

\[
v = 0 \quad \text{on} \quad \Gamma_{\text{upp+low}}.
\]

Conditions for density:

we suppose that

\[
\rho = \rho_0 \quad \text{on} \quad \Gamma_{\text{inp}},
\]

\[
\rho = \rho_0 \quad \text{in} \quad \Omega \text{ for } \quad t = 0.
\]

Let us suppose that we are already given a solution \( \rho, v (\rho \geq 0) \) which is sufficiently smooth. Assume that \( v_0 \) is such a function that there exists its extension onto the whole cylinder \( C_{t_0} \) so that this extension is an element of \( L^2((0, T), W^{2,2}(\Omega)) \).

We shall denote the extension by \( v_0 \) again.

Then we can write

\[
v = v^0 + w,
\]

where

\[
w = 0 \quad \text{on} \quad (0, t_0) \times \partial \Omega.
\]

We assume another boundary condition:

\[
\tau_{ij,k} v_j v_k = 0 \quad \text{on} \quad (0, t_0) \times \partial \Omega.
\]

Now we shall need apriori estimates.

**Theorem 2.1.** Let \( \rho, v, v^0 \) be smooth enough. Then

\[
\int_{\Omega} \rho \; dx \leq \int_{\Omega_0} \rho \; dx + \int_0^t \int_{\Gamma_{\text{inp}}} \rho_0 v^0_i v_i \; ds \; dt,
\]

\[
\int_{\Omega} \frac{1}{2} \rho |w|^2 \; dx - \frac{1}{2} \int_{\Omega_0} \rho |w|^2 \; dx + \int_{\Omega} \left( \partial_t v^0_i + \rho (v^0_j + w_j) \partial_x v^0_i \right) dx \; dt + \lambda \int_{\Omega_0} (\rho \ln \rho - \rho) \; dx - \lambda \int_{\Omega_0} (\rho \ln \rho - \rho) \; dx +
\]

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\[ + \lambda \int_0^t \int_{\Gamma_{inp}} \varrho \ln \varrho v_i^0 v_i \, ds \, dt + \lambda \int_0^t \int_{\Gamma_{out}} \varrho \ln \varrho v_i^0 v_i \, ds \, dt + \]

\[ + \lambda \int_{\Omega_{t}} \varrho \, dx - \lambda \int_{\Omega_{0}} \varrho \, dx + \lambda \int_0^t \int_{\Omega} \frac{\partial v_i^0}{\partial x_i} \, dx \, dt + \]

\[ + \int_{\Omega_{t}} \left\{ v e^2(w_{tt}) + 2\mu e_{ij}(w) e_{ij}(w) + \gamma_1 \frac{\partial e_{ii}(w)}{\partial x_k} \frac{\partial e_{pp}(w)}{\partial x_k} + \right. \]

\[ + 2\mu_1 \frac{\partial e_{ij}(w)}{\partial x_k} \frac{\partial e_{ij}(w)}{\partial x_k} + \gamma e_{ii}(v^0) e_{kk}(w) + \]

\[ + 2\mu e_{ij}(v^0) e_{ij}(w) + \gamma_1 \frac{\partial e_{ii}(v^0)}{\partial x_k} \frac{\partial e_{pp}(w)}{\partial x_k} + \]

\[ + 2\mu_1 \frac{\partial e_{ij}(v^0)}{\partial x_k} \frac{\partial e_{ij}(w)}{\partial x_k} \right\} = 0, \]

where \( \Omega_{t} = \{(x, t), x \in \Omega\}, \Omega_{0} = \{(x, 0), x \in \Omega\} \).

**Proof.** Let us prove (2.18) (we denote \( Q_{0} \) by \( Q_{t} \)). The proof of (2.18) is based on the integration of equations (2.3) over \( Q_{t} \), the use of Green’s theorem and the boundary condition

\[ \int_{Q_{t}} \left[ \frac{\partial \varrho}{\partial t} + \frac{\partial (\varrho v_i)}{\partial x_i} \right] \, dx \, dt = 0, \]

hence

\[ \int_{\Omega} \left[ \frac{\partial \varrho}{\partial t} + \frac{\partial (\varrho v_i)}{\partial x_i} \right] \, dx \, dt = \int_0^t \int_{\Omega_{t}} \left\{ \int_{\Omega} \left[ \frac{\partial \varrho}{\partial t} \right] \, dx \right\} \, dt + \]

\[ + \int_0^t \int_{\Omega_{t}} \varrho v_i v_i \, ds \, dt = \int_{\Omega_{t}} \varrho \, dx - \int_{\Omega_{0}} \varrho \, dx + \int_0^t \int_{\Omega_{t}} \varrho v_i v_i \, ds \, dt. \]

Now we write the last integral

\[ \int_0^t \int_{\Omega_{t}} \varrho v_i v_i \, ds \, dt = \int_0^t \int_{\Omega_{t}} \varrho_0 v_i^0 v_i \, ds \, dt + \]

\[ + \int_0^t \int_{\Gamma_{inp}} \varrho v_i v_i \, ds \, dt + \int_0^t \int_{\Gamma_{out} + \Gamma_{upp} + \Gamma_{low}} \varrho v_i v_i \, ds \, dt. \]

We know that the last integral in (2.20) is equal to zero.

Because

\[ \int_0^t \int_{\Gamma_{inp}} \varrho_0 v_i^0 v_i \, ds \, dt \leq 0, \]

(2.21)

\[ \int_0^t \int_{\Gamma_{out}} \varrho_0 v_i^0 v_i \, ds \, dt \geq 0, \]

(2.22)
it follows that

$$\int_{\Omega_t} q \, dx \leq \int_{\Omega_0} q \, dx - \int_0^t \int_{\Gamma_{in,p}} q v^0_i v_i \, ds \, dt.$$

Now we prove (2.19).

Let us multiply equations (2.8) by \( w \), where \( v = v^0 + w \), and integrate over \( \Omega_t \).

We have

$$0 = \int_{\Omega_t} \left[ \frac{\partial}{\partial t} (qv_i) w_i + \frac{\partial}{\partial x_j} (qv_i v_j + \delta_{ij} p - \tau^0_{ij}) w_i \right] \, dx \, dt.$$

Let us divide the right hand side into three parts.

The first part:

$$\int_{\Omega_t} \left\{ \frac{\partial}{\partial t} (qv_i) w_i + \frac{\partial}{\partial x_j} (qv_i v_j) w_i \right\} \, dx \, dt =$$

$$= \int_{\Omega_t} \left\{ \frac{\partial v_i}{\partial t} q w_i + \frac{\partial q}{\partial t} v_i w_i + \frac{\partial (qv_j) v_j}{\partial x_j} w_i \right\} \, dx \, dt.$$

We use the continuity equation and obtain

$$\int_{\Omega_t} \left( \frac{\partial v_i}{\partial x_j} q v_j w_i + \frac{\partial v_i}{\partial t} q w_i \right) \, dx \, dt = \int_{\Omega_t} \left( \frac{\partial (v^0_i + w_i)}{\partial t} q w_i +$$

$$+ \frac{\partial v_i}{\partial x_j} q v_j w_i \right) \, dx \, dt = \int_{\Omega_t} \frac{\partial w_i}{\partial t} q w_i \, dx \, dt + \int_{\Omega_t} \frac{\partial v^0_i}{\partial t} q w_i \, dx \, dt +$$

$$+ \int_{\Omega_t} \frac{\partial (v^0_i + w_i)}{\partial x_j} q (v^0_j + w_j) w_i \, dx \, dt = \frac{1}{2} \int_{\Omega_t} \frac{\partial}{\partial t} (q|w|^2) \, dx \, dt -$$

$$- \frac{1}{2} \int_{\Omega_t} \frac{\partial q}{\partial t} |w|^2 \, dx \, dt + \int_{\Omega_t} \frac{\partial v^0_i}{\partial t} q w_i \, dx \, dt +$$

$$+ \int_{\Omega_t} \left( q w_j \frac{\partial v^0_j}{\partial x_j} w_i + q v_j \frac{\partial w_i}{\partial x_j} \right) \, dx \, dt +$$

$$+ \int_{\Omega_t} \left( q w_i \frac{\partial w_j}{\partial x_j} v_j + q v_i \frac{\partial v_j}{\partial x_j} w_i \right) \, dx \, dt =$$

$$= \frac{1}{2} \int_{\Omega_t} q |w|^2 \, dx - \frac{1}{2} \int_{\Omega_0} q |w|^2 \, dx - \frac{1}{2} \int_{\Omega_t} \frac{\partial q}{\partial t} |w|^2 \, dx \, dt +$$

$$+ \int_{\Omega_t} \frac{\partial v^0_i}{\partial t} q w_i \, dx \, dt + \frac{1}{2} \int_{\Omega_t} q v_j \frac{\partial |w|^2}{\partial x_j} \, dx \, dt +$$

$$+ \frac{1}{2} \int_{\Omega_t} q v_j \frac{\partial |w|^2}{\partial x_j} \, dx \, dt + \int_{\Omega_t} q v_i \frac{\partial v^0_i}{\partial x_j} \, dx \, dt + q w_j \frac{\partial v^0_i}{\partial x_j} \, dx \, dt =$$

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\[
\begin{align*}
&= \frac{1}{2} \int_{\Omega_t} q |\omega|^2 \, dx - \frac{1}{2} \int_{\Omega_0} q |\omega|^2 \, dx + \int_{\Omega_t} \frac{\partial v_i^0}{\partial t} q \omega_i \, dx \, dt - \\
&- \frac{1}{2} \int_{\Omega_t} \left( \frac{\partial q}{\partial t} |\omega|^2 + \frac{\partial (q v_i)}{\partial x_j} |\omega|^2 \right) \, dx \, dt + \int_{\Omega_t} q v_j \omega_i \frac{\partial v_i^0}{\partial x_j} \, dx \, dt \quad (2.3) \\
&= (2.3) \frac{1}{2} \int_{\Omega_t} q |\omega|^2 \, dx - \frac{1}{2} \int_{\Omega_0} q |\omega|^2 \, dx + \int_{\Omega_t} \frac{\partial v_i^0}{\partial t} q \omega_i \, dx \, dt + \\
&+ \int_{\Omega_t} q v_j \omega_i \frac{\partial v_i^0}{\partial x_j} \, dx \, dt.
\end{align*}
\]

The second part:

\[
\begin{align*}
&\int_{\Omega_t} \frac{\partial}{\partial x_j} (\partial_{ij} p) \omega_i \, dx \, dt = \int_{\Omega_t} \frac{\partial p}{\partial x_i} \omega_i \, dx \, dt = (2.1) \lambda \int_{\Omega_t} \frac{\partial q}{\partial x_i} \omega_i \, dx \, dt = \\
&= \lambda \int_{\Omega_t} \frac{\partial q}{\partial x_i} \frac{1}{q} q \omega_i \, dx \, dt = \lambda \int_{\Omega_t} \frac{\partial}{\partial x_i} \left( \ln q \right) q \omega_i \, dx \, dt = \\
&= -\lambda \int_{\Omega_t} \ln q \frac{\partial (q \omega_i)}{\partial x_i} \, dx \, dt = -\lambda \int_{\Omega_t} \ln q \frac{\partial (q (v_i - v_i^0))}{\partial x_i} \, dx \, dt = \\
&= -\lambda \int_{\Omega_t} \ln q \frac{\partial}{\partial x_i} \left( q v_i - v_i^0 \right) \, dx \, dt = \\
&= \lambda \int_{\Omega_t} \ln q \frac{\partial}{\partial t} (q - q) \, dx \, dt + \lambda \int_{\Omega_t} \ln q \frac{\partial}{\partial x_i} (q v_i^0) \, dx \, dt = \\
&= \lambda \int_{\Omega_t} (q \ln q - q) \, dx - \lambda \int_{\Omega_0} (q \ln q - q) \, dx + \\
&+ \lambda \int_{\Omega_t} \ln q \frac{\partial}{\partial x_i} (q v_i^0) \, dx \, dt = \lambda \int_{\Omega_t} (q \ln q - q) \, dx - \\
&- \lambda \int_{\Omega_0} (q \ln q - q) \, dx + \lambda \int_0^t \int_{\Omega_t} (q v_i^0) \ln q \, dS \, dt - \\
&- \lambda \int_0^t \int_{\Omega_t} \frac{\partial q}{\partial x_i} v_i^0 \, dx \, dt = \lambda \int_{\Omega_t} (q \ln q - q) \, dx - \\
&- \lambda \int_0^t \int_{\Omega_t} (q \ln q - q) \, dx + \lambda \int_0^t \int_{\Gamma_{in}} q v_i^0 v_i \ln q \, dS \, dt + \\
&+ \lambda \int_0^t \int_{\Gamma_{out}} q v_i^0 v_i \ln q \, dS \, dt - \lambda \int_0^t \int_{\Gamma_{in}} q v_i^0 v_i \, dS \, dt -
\end{align*}
\]
\[-\lambda \int_0^t \int_{\Omega_r} qv_i^0 v_i \, dS \, dt + \lambda \int_{\Omega_r} q \frac{\partial v_i^0}{\partial x_i} \, dx \, dt = \cdot \]

\[= (2.19)(2.20) \lambda \int_{\Omega_r} (q \ln q - q) \, dx - \lambda \int_{\Omega_0} (q \ln q - q) \, dx + \]

\[+ \lambda \int_0^t \int_{\Gamma_{in}} qv_i^0 v_i \ln q \, dS \, dt + \lambda \int_0^t \int_{\Gamma_{out}} qv_i^0 v_i \ln q \, dS \, dt - \]

\[- \lambda \int_0^t \int_{\Gamma_{in}} qv_i^0 v_i \, dS \, dt + \lambda \int_{\Omega_r} q \, dx - \lambda \int_{\Omega_0} q \, dx + \]

\[+ \lambda \int_0^t \int_{\Gamma_{in}} qv_i^0 v_i \, dS \, dt + \lambda \int_0^t \int_{\Gamma_{out}} qv_i^0 v_i \, dS \, dt + \]

\[= \lambda \int_{\Omega_r} (q \ln q - q) \, dx - \lambda \int_{\Omega_0} (q \ln q - q) \, dx + \]

\[+ \lambda \int_0^t \int_{\Gamma_{in}} qv_i^0 \ln qv_i \, dS \, dt + \lambda \int_0^t \int_{\Gamma_{out}} qv_i^0 \ln qv_i \, dS \, dt + \]

\[+ \lambda \int_{\Omega_r} q \, dx - \lambda \int_{\Omega_0} q \, dx + \lambda \int_0^t \int_{\Omega_i} q \frac{\partial v_i^0}{\partial x_i} \, dx \, dt . \]

The third part:

\[- \int_{\Omega_t} \frac{\partial}{\partial x_j} (t_{ij}) w_i \, dx \, dt = - \int_0^t \int_{\Omega_i} \tau_{ij} w_i \, dS \, dt + \]

\[+ \int_{\Omega_t} \tau_{ij} \frac{\partial w_i}{\partial x_j} \, dx \, dt = \int_{\Omega_t} \tau_{ij} \frac{\partial w_i}{\partial x_j} \, dx \, dt = \int_0^t \int_{\Omega_i} \left\{ \gamma \frac{\partial v_i}{\partial x_i} + \frac{\partial w_i}{\partial x_i} \right\} \, dx \, dt = \]

\[= \int_0^t \int_{\Omega_i} \left\{ \gamma \frac{\partial v_i}{\partial x_i} \frac{\partial w_i}{\partial x_i} + 2 \mu_1 \frac{\partial w_i}{\partial x_i} \left( \frac{1}{2} \frac{\partial w_i}{\partial x_i} + \frac{1}{2} \frac{\partial w_i}{\partial x_i} \right) \right\} \, dx \, dt = \]

\[= \int_0^t \int_{\Omega_i} \left\{ \gamma \frac{\partial v_i}{\partial x_i} \frac{\partial w_i}{\partial x_i} + 2 \mu_1 \frac{\partial w_i}{\partial x_i} \left( \frac{1}{2} \frac{\partial w_i}{\partial x_i} + \frac{1}{2} \frac{\partial w_i}{\partial x_i} \right) \right\} \, dx \, dt = \]

\[- \lambda \int_0^t \int_{\Omega_i} \left\{ \gamma \frac{\partial v_i}{\partial x_i} \frac{\partial w_i}{\partial x_i} + 2 \mu_1 \frac{\partial w_i}{\partial x_i} \left( \frac{1}{2} \frac{\partial w_i}{\partial x_i} + \frac{1}{2} \frac{\partial w_i}{\partial x_i} \right) \right\} \, dx \, dt - \]

\[- \lambda \int_0^t \int_{\Omega_i} \left\{ \gamma \frac{\partial v_i}{\partial x_i} \frac{\partial w_i}{\partial x_i} + 2 \mu_1 \frac{\partial w_i}{\partial x_i} \left( \frac{1}{2} \frac{\partial w_i}{\partial x_i} + \frac{1}{2} \frac{\partial w_i}{\partial x_i} \right) \right\} \, dx \, dt + \]

\[+ \int_0^t \int_{\Omega_i} \gamma_1 \frac{\partial v_i}{\partial x_i} \frac{\partial w_i}{\partial x_i} \, dx \, dt + \]

\[+ \int_0^t \int_{\Omega_i} \gamma_1 \frac{\partial v_i}{\partial x_i} \frac{\partial w_i}{\partial x_i} \, dx \, dt + \]

\[= 52 \]
\[ + 2\mu_1 \frac{\partial}{\partial x_i} \left( e_{ij}(v) \right) \frac{\partial}{\partial x_k} \left( \frac{\partial w_i}{\partial x_j} \right) \, dx \, dt = \]
\[ = \int_0^T \int_{\Omega} \left\{ \gamma e_{ii}(w) + 2\mu e_{ij}(w) e_{ij}(w) + \gamma_1 \frac{\partial}{\partial x_k} e_{ii}(w) \frac{\partial}{\partial x_k} e_{pp}(w) + \right. \]
\[ + 2\mu_1 \frac{\partial}{\partial x_k} e_{ij}(w) \frac{\partial}{\partial x_k} e_{ij}(w) + \gamma e_{ii}(v^0) e_{kk}(w) + \]
\[ + 2\mu e_{ij}(v^0) e_{ij}(w) + \frac{\partial}{\partial x_k} e_{ii}(v^0) \frac{\partial}{\partial x_k} e_{pp}(w) + \]
\[ + 2\mu_1 \frac{\partial}{\partial x_k} e_{ij}(v^0) \frac{\partial}{\partial x_k} e_{ij}(w) \right\} \, dx \, dt - \int_0^T \int_{\Gamma_{\text{in}}} \tau_{ijk} \frac{\partial w_i}{\partial x_j} v_k \, dS \, dt \]

Let us denote the last term by \( B \).
\[ B = \int_0^T \int_{\Gamma_{\text{in}}} \tau_{ijk} \frac{\partial w_i}{\partial x_j} v_k \, dS \, dt . \]

We know that \( \frac{\partial w_i}{\partial x_j} = \frac{\partial w_i}{\partial v} v_j \).

This means that
\[ B = \int_0^T \int_{\Gamma_{\text{in}}} \tau_{ijk} \frac{\partial w_i}{\partial v} v_j v_k \, dS \, dt \]

and we use condition (2.17). It implies that \( B = 0 \).

Thus
\[ \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial t} |w|^2 \, dx - \frac{1}{2} \int_{\Omega} \frac{\partial}{\partial t} \left( q e_{ii}(w) + 2 \mu e_{ij}(w) e_{ij}(w) + \gamma_1 \frac{\partial}{\partial x_k} e_{ii}(w) \frac{\partial}{\partial x_k} e_{pp}(w) + \right. \]
\[ + 2\mu_1 \frac{\partial}{\partial x_k} e_{ij}(w) \frac{\partial}{\partial x_k} e_{ij}(w) + \gamma e_{ii}(v^0) e_{kk}(w) + \]
\[ + 2\mu e_{ij}(v^0) e_{ij}(w) + \frac{\partial}{\partial x_k} e_{ii}(v^0) \frac{\partial}{\partial x_k} e_{pp}(w) + \]
\[ + 2\mu_1 \frac{\partial}{\partial x_k} e_{ij}(v^0) \frac{\partial}{\partial x_k} e_{ij}(w) \right\} \, dx \, dt = 0 . \]
The last term in denoted by $A_1 + B_1$, where $A_1 = ((w, w))$, $B_1 = ((v^0, w))$ are scalar products in $W^{2,2}(\Omega, R^n) \cap W^{1,2}_0(\Omega, R^n)$, see below and suppose that the following condition (C) is satisfied:

(C): $\varrho_0 \in C^1(\overline{\Omega})$ (\(\varrho = \varrho_0\) on input and \(\varrho = \varrho_0\) for \(t = 0\)) ; \(\varrho_0 > 0\).

\(v_0 \in C^1(\Omega),\)

\(\varrho \in L^p(I, L^1(\Omega)),\)

\(w \in L^2(I, W^{2,2}(\Omega, R^n))\),

\(v^0 \in L^2(I, W^{2,2}(\Omega, R^n))\).

**Theorem 2.2.** Let us suppose (C), then

\[
\frac{1}{2} \int_{\Omega_t} \varrho |w|^2 \, dx + \int_{\Omega_t} \varrho \ln \varrho \, dx + \frac{1}{4} \int_0^t \|w\|^2 \, dt \leq h \int_{\Omega_t} \varrho |w|^2 \, dx \, dt + k \leq l,
\]

where \(h, k, l \geq 0\), \(h, k, l\) are constants.

**Proof.** From (2.19) we know that (we denote \(Q^*\) by \(Q\))

\[
\frac{1}{2} \int_{\Omega_t} \varrho |w|^2 \, dx - \frac{1}{2} \int_{\Omega_0} \varrho |w|^2 \, dx + \int_{\Omega_t} \varrho w_i \frac{\partial v^0}{\partial t} \, dx \, dt + \\
+ \int_{\Omega_t} \varrho v^0 \frac{\partial v^0}{\partial x_j} \, dx \, dt + \lambda \int_{\Omega_t} \varrho \ln \varrho - \varrho \, dx - \\
- \lambda \int_{\Omega_0} \varrho \ln \varrho - \varrho \, dx + \lambda \int_0^t \int_{\Gamma_{in}} \varrho_0 v^0 \ln \varrho_0 v_i \, dS \, dt + \\
+ \lambda \int_0^t \int_{\Gamma_{out}} \varrho v^0 \ln \varrho v_i \, dS \, dt + \lambda \int_{\Omega_t} \varrho \, dx - \\
- \lambda \int_{\Omega_0} \varrho \, dx + \lambda \int_0^t \int_{\Omega} \varrho \frac{\partial v^0}{\partial x_i} \, dx \, dt + \int_0^t [((w, w)) + ((v^0, w))] \, dt = 0.
\]

First we move the known terms to the right hand side

\[
\frac{1}{2} \int_{\Omega_t} \varrho |w|^2 \, dx + \int_{\Omega_t} \varrho w_i \frac{\partial v^0}{\partial t} \, dx \, dt + \int_{\Omega_t} \varrho v^0 \frac{\partial v^0}{\partial x_j} \, dx \, dt + \\
+ \lambda \int_{\Omega_t} \varrho \ln \varrho - \varrho \, dx + \lambda \int_0^t \int_{\Omega} \varrho \frac{\partial v^0}{\partial x_i} \, dx \, dt + \\
+ \int_0^t [((w, w)) + ((v^0, w))] \, dt + \int_{\Omega_t} \varrho \, dx =
\]

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= \lambda \int_{\Omega_0} (q \ln q - q) \, dx + \frac{1}{2} \int_{\Omega_0} q |w|^2 \, dx + \lambda \int_{\Omega_0} q \, dx -

- \lambda \int_0^t \int_{\Gamma_{inp}} q_0 v_i^0 \ln q_0 v_i \, dS \, dt - \lambda \int_0^t \int_{\Gamma_{out}} q v_i^0 \ln q v_i \, dS \, dt.

Now let us estimate. From (2.18) we know that

\begin{align}
(2.25) & \quad \| \varphi \|_{L^\infty(I, L^1(\Omega))} \leq k ; \\
(2.26) & \quad \left| \int_{Q_t} q w_i \frac{\partial v_i^0}{\partial t} \right| \leq c_1 \int_0^t \| w \|^{W^{2,1}(\Omega)} \| \varphi \|_{L^\infty(I, W^{2,1}(\Omega))} \leq \hat{c}_1 \sqrt{(t)} \| w \|_{L^2(I, W^{2,1}(\Omega))} . \\
(2.27) & \quad \left| \int_{Q_t} q v_i^0 w_i \frac{\partial v_i^0}{\partial x_j} \right| \leq c_2 \int_0^t \| w \|^{W^{2,1}(\Omega)} \| \varphi \|_{L^\infty(I, W^{2,1}(\Omega))} \leq \hat{c}_2 \sqrt{(t)} \| w \|_{L^2(I, W^{2,1}(\Omega))} ,
\end{align}

\int_0^t \left[ (w, w) + ((v^0, w)) \right] \, dt = \int_0^t \| w \|^2 + ((v^0, w)) \, dt \leq

\leq \int_0^t \left( \| w \|^2 + \frac{1}{\varepsilon} \| v^0 \|^2 + \varepsilon \| w \|^2 \right) \, dt ,

\begin{align}
(2.29) & \quad \lambda \int_0^t \int_{\Omega} q \frac{\partial v_i^0}{\partial x_i} \leq \hat{c}_4 .
\end{align}

Thus

\begin{align}
(2.30) & \quad \frac{1}{2} \int_{\Omega_t} q |w|^2 \, dx + \lambda \int_{\Omega_t} q \ln q \, dx + \int_0^t \| w \|^2 \, dt \leq \\
& \leq \frac{1}{2} \int_{\Omega_0} q |w|^2 \, dx + \lambda \int_{\Omega_0} q \ln q \, dx - \lambda \int_0^t \int_{\Gamma_{inp}} q_0 v_i^0 \ln q_0 v_i \, dS \, dt -

- \lambda \int_0^t \int_{\Gamma_{out}} q v_i^0 \ln q v_i \, dS \, dt + \hat{c}_1 \sqrt{(t)} \| w \|_{L^2(I, W^{2,1})} +

+ \hat{c}_2 \sqrt{(t)} \| w \|_{L^2(I, W^{2,1})} + c_3 \int_0^t \int_{\Omega} q w_i w_i \, dx \, dt +

+ \int_0^t \left( \frac{1}{\varepsilon} \| v^0 \|^2 + \varepsilon \| w \|^2 \right) \, dt + c .
\end{align}
Now we use the Gronwall lemma:

\[ f'(t) \leq K_1 f(t) + K_2 , \]

where

\[ f'(t) = \frac{1}{2} \int_{\Omega_t} q |w|^2 \, dt , \]
\[ f(t) = \int_{\Omega_t} q |w|^2 \, dx \, dt , \]
\[ f'(t) \leq f'(t) + \lambda \int_{\Omega_t} q \ln q \, dx + \int_0^t \|w\|^2 \, dt \leq \varepsilon = 1/2 \]
\[ \leq \varepsilon c_3 \int_{\Omega_t} q |w|^2 \, dx \, dt + K + c_5 \|w\|_{L^2(I,W^{2,2}(\Omega))} + \int_0^t \frac{1}{2} \|w\|^2 \, dt , \]

where

\[ K = \frac{1}{2} \int_{\Omega_0} q |w|^2 \, dx + \lambda \int_{\Omega_t} q \ln q \, dx - \lambda \int_{\Gamma_{\text{in}}} q_0 v_i^0 \ln q_0 v_i \, dS \, dt - \lambda \int_0^t \int_{\Gamma_{\text{out}}} q v_i^0 \ln q v_i \, dS \, dt + \int_0^t 2 \|v_0\|^2 \, dt + c . \]

For \( q \leq 1 \) we have \( |\lambda \int_{\Omega_t} q \ln q \, dx| \leq c_6 \). Thus

\[ f'(t) + \lambda \int_{\Omega_t} q \ln q \, dx + \frac{1}{2} \int_0^t \|w\|^2 \, dt \leq c_3 \int_{\Omega_t} q |w|^2 \, dx \, dt + 
+ K_1 + c_5 \|w\|_{L^2(I,W^{2,2}(\Omega))} , \]
\[ K_1 = K + c_6 , \]
\[ c_5 \|w\|_{L^2(I,W^{2,2}(\Omega))} \leq \frac{1}{4} \int_0^t \|w\|^2 \, dt + c_6 . \]

Thus

\[ \frac{1}{2} \int_{\Omega_t} q |w|^2 \, dx \leq \frac{1}{2} \int_{\Omega_t} q |w|^2 \, dx + \frac{1}{2} \int_0^t \|w\|^2 \, dt + 
+ \lambda \int_{\Omega_t} q \ln q \, dx \leq c_3 \int_0^t \int_{\Omega_t} q |w|^2 \, dx \, dt + c_7 . \]

This implies (use Gronwall lemma) that

\[ \frac{1}{2} \int_{\Omega_t} q |w|^2 \, dx - 2c_3 \frac{1}{2} \int_0^t \int_{\Omega_t} q |w|^2 \, dx \, dt \leq c_7 . \]
Multiply this inequality by $e^{-2c_{3}t}$:

\[
\left\{ \frac{1}{2} \int_{0}^{t} \int_{Q_{t}} q |w|^{2} \, dx \, dt \, e^{-2c_{3}t} \right\} \leq c_{7} e^{-2c_{3}t},
\]

\[
\int_{0}^{t} \int_{Q_{t}} q |w|^{2} \, dx \, dt \, e^{-2c_{3}t} \bigg|_{0}^{t} \leq \frac{c_{7} e^{-2c_{3}t}}{2c_{3}} \bigg|_{0}^{t} = \frac{c_{7} - c_{7} e^{-2c_{3}t}}{2c_{3}}.
\]

Now we multiply by $e^{2c_{3}t}$:

(2.32) \quad \frac{1}{2} \int_{0}^{t} \int_{Q_{t}} q |w|^{2} \, dx \, dt \leq \frac{c_{7}}{2c_{3}} \, e^{2c_{3}t} - \frac{c_{7}}{2c_{3}},

(2.33) \quad f'(t) = \frac{1}{2} \int_{Q_{t}} q |w|^{2} \, dx \leq 2c_{3} \left( \frac{1}{2} \int_{Q_{t}} q |w|^{2} \, dx \, dt \right) + c_{7} \leq

\leq 2c_{3} \left[ \frac{c_{7}}{2c_{3}} e^{2c_{3}t} - \frac{c_{7}}{2c_{3}} \right] + c_{7} = c_{7} e^{2c_{3}t},

f'(t) + \frac{1}{4} \int \|w\|^{2} \, dt + \lambda \int_{Q_{t}} q \ln q \, dx \leq 2c_{3} f(t) + c_{7},

(2.34) \quad \frac{1}{4} \int \|w\|^{2} \, dt + \lambda \int_{Q_{t}} q \ln q \, dx \leq 2c_{3} c_{7} e^{2c_{3}t} + c_{7}.

Let $\varphi(t)$ and $\psi(t)$ be two right continuous $(s, t > 0)$ nondecreasing functions such that

(2.35) \quad \varphi(t) = \sup_{s \leq t} \varphi(s), \quad \psi(t) = \sup_{s \leq t} \psi(s)

which satisfy the conditions

(2.36) \quad \varphi(0) = \psi(0) = 0,

\quad \varphi(\infty) = \psi(\infty) = \infty.

The convex functions $\Phi(u)$ and $\Psi(v)$ defined by the relations

(2.37) \quad \Phi(t) = \int_{0}^{[w]} \varphi(t) \, dt

\quad \Psi(t) = \int_{0}^{[w]} \psi(s) \, ds

are called mutually complementary Young (or $\Psi -$) functions [2].

A convex function $\Phi_{1}(t)$ will be called the principal part of the $\Psi$-function $\Phi_{2}(t)$ if

$\Phi_{1}(t) = \Phi_{2}(t)$ for sufficiently large $t$.  

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By $\tilde{L}_\phi(\Omega)$ we denote the Orlicz class corresponding to $\phi$, i.e. the set of all Lebesgue measurable functions $u$ in $\Omega$ such that

$$(2.38) \quad \int_\Omega \phi(u) \, dx < \infty.$$ 

The Orlicz space $L_\phi(\Omega)$ is defined to be the linear hull of the Orlicz class $\tilde{L}_\phi(\Omega)$ with the Luxemburg norm

$$(2.39) \quad \|u\|_\phi = \inf \left\{ h > 0 : \int_\Omega \phi(u(x))/h \, dx \leq 1 \right\},$$

$$(2.40) \quad \|u\|_\psi = \inf \left\{ h > 0 : \int_\Omega \psi(u(x))/h \, dx \leq 1 \right\}.$$ 

For $\|u\|_\phi > 1$ we have

$$(2.41) \quad \int_\Omega \phi(u(x))/\|u\|_\phi \, dx \leq 1.$$ 

Let $C(\Omega)$ be the set of all functions $u$ continuous on $\Omega$ up to the boundary. The space $C_\phi$ and $C_\psi$ are defined as the closures of the set $C(\Omega)$ with respect to the Luxemburg norms $\|\cdot\|_\phi$ and $\|\cdot\|_\psi$, respectively.

The following inclusions hold:

$$(2.42) \quad C_\phi(\Omega) \subset \tilde{L}_\phi(\Omega) \subset L_\phi(\Omega).$$

**Definition.** We say that $\phi$ satisfies the $A_2$-condition if for large values of $t$ we have

$$(2.43) \quad \exists a > 0 : \phi(2t) \leq a \phi(t).$$

We use

$$(2.44) \quad \psi(i) = (1 + i) \ln(1 + i) - i,$$

$\phi(i) = e^i - i - 1.$

**Remark.**

$$(2.45) \quad L_\phi$$ is a Banach space [2].

$$(2.46) \quad L_\psi = (C_\phi)^* [2].$$

$$(2.47) \quad If \phi satisfies the A_2-condition, then L_\phi is separable [2].$$

**Theorem 2.3.** If $\psi$ satisfies the $A_2$-condition, then

$$(2.48) \quad C_\psi(\Omega) = \tilde{L}_\psi(\Omega) = L_\psi(\Omega).$$

**Proof.** See [2].
Theorem 2.4. If $\Psi$ satisfies the $A_2$-condition, then

\begin{equation}
L_p(\Omega) = (L_p(\Omega))^*. \tag{2.49}
\end{equation}

Proof. See [2].

Of course $C_0(\Omega), C_p(\Omega)$ are separable Banach spaces. We realize that $\Psi$ satisfies the $A_2$-condition, hence

\begin{equation}
C_p(\Omega) = L_p(\Omega). \tag{2.50}
\end{equation}

Definition. If $X$ is any Banach space, we set $X = [X]^d$ while $X^*$ will denote the dual space. Moreover, the symbols $(\ldots)$ and $\|\cdot\|_2$ will denote, as customary, the scalar product and the norm in $L_2(\Omega)$, respectively. For $1 \leq p < \infty$ and $X$ a Banach space will the norm $\|\cdot\|_X$, we denote by $L_p(I, X)$ the set of all mappings $f: I = (0, t) \to X$ which are strongly measurable and such that

\begin{equation}
\int_0^t \|f\|^p_X dt < \infty. \tag{2.51}
\end{equation}

Theorem 2.5. If $X$ is a separable space and $1 < p < \infty$, then

\begin{equation}
(L^p(I, X))^* = L^q(I, X^*), \quad \frac{1}{p} + \frac{1}{q} = 1. \tag{2.52}
\end{equation}

Proof. See [10].

Remark. This theorem implies that

\begin{equation}
L^2(I, C_0(\Omega))^* = L^2(I, L_p(\Omega)). \tag{2.53}
\end{equation}

Definition. Let $1 \leq p \leq \infty$. The space $W^{k,p}(\Omega)$ is the subspace of $L^p(\Omega)$ of functions $u$ for which there exists $\omega_\xi \in L^p(\Omega)$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $\xi_i \geq 0$, $|\xi| = \xi_1 + \xi_2 + \xi_3 + \ldots + \xi_n$, $1 \leq |\xi| \leq k$, such that $\forall \phi \in C^k(\Omega)$,

\begin{equation}
\int_{\Omega} D^\xi \phi u \, dx = (-1)^{|\xi|} \int_{\Omega} \phi \omega_\xi \, dx, \tag{2.54}
\end{equation}

where $D^\xi = \frac{\partial^{|\xi|}}{\partial x_1^{\xi_1} \ldots \partial x_n^{\xi_n}}$.

For $1 \leq p \leq \infty$ we put

\begin{equation}
\|u\|_{W^{k,p}(\Omega)} = \left(\int_{\Omega} \left[ \sum_{1 \leq |\xi| \leq k} |\omega_\xi|^p + |u|^p \right] dx \right)^{1/p} \tag{2.55}
\end{equation}

and for $p = \infty$

\begin{equation}
\|u\|_{k,\infty} = \sup_{x \in \Omega} |u(x)| + \sum_{1 \leq |\xi| \leq k} \sup_{x \in \Omega} |\omega_\xi(x)|. \tag{2.56}
\end{equation}

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Definition. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Let $1 \leq p \leq \infty$. We introduce
\[ W^{k,p}(\Omega) = \overline{C_c^k(\Omega)}^{||.||_{k,p}}. \]

We shall work with
\begin{align*}
(2.56) & \quad w \in L^2(I, W^{2,2}(\Omega, \mathbb{R}^N)), \\
(2.57) & \quad v^0 \in L^2(I, W^{2,2}(\Omega, \mathbb{R}^N)), \\
(2.58) & \quad v^0 \in C^1(\overline{Q}_t), \\
(2.59) & \quad \varrho_0 \in C(\overline{Q}_t), \\
(2.60) & \quad \varrho \in L^\infty(I, L^p(\Omega)), \quad \varrho \geq 0.
\end{align*}

The weak formulation of the equation (2.8), (2.4) will be the following:
\begin{equation}
(2.61) \quad \int_{\Omega_{0_t}} \varrho v_i \frac{\partial z_i}{\partial t} \, dx \, dt - \int_{\Omega_{0_t}} \varrho_0 \delta_i z_i(0) \, dx + \int_0^t \left( (v, z) \right) \, dt - \\
- \int_{\Omega_{0_t}} (\varrho v_i \nu_j + p \delta_{ij}) \frac{\partial z_i}{\partial x_j} \, dx \, dt = 0, \\
- \int_{\Omega_{0_t}} \varrho_0 z_i(0) \, dx - \int_{\Omega_{0_t}} \varrho \frac{\partial z_i}{\partial t} - \int_{\Omega_{0_t}} \varrho v_j \frac{\partial z_i}{\partial x_j} = 0
\end{equation}

for every $z \in C^\infty(\overline{Q}_t, \mathbb{R}^N)$, $z(t) = 0$, $v, z \in W^{2,2}(\Omega, \mathbb{R}^N) \cap W^{1,2}_0(\Omega, \mathbb{R}^N)$, $\delta^0 = v(0)$, $z = 0$ on $\partial \Omega \times (0, t_0)$, $((v, z))$ is defined by (3.1) in the next section.

3. A MODIFIED GALERKIN METHOD

First we construct a sequence of suitable approximations. Let us denote $V = W^{2,2}(\Omega, \mathbb{R}^N) \cap W^{1,2}_0(\Omega, \mathbb{R}^N)$ and for $w, z \in V$ let
\begin{equation}
(3.1) \quad ((w, z)) = \int_{\Omega} \left[ \gamma \, e_{ii}(w) \, e_{kk}(z) + 2\mu \, e_{ij}(w) \, e_{ij}(z) + \\
+ \gamma_1 \frac{\partial}{\partial x_k} e_{ii}(w) \, \frac{\partial}{\partial x_k} e_{pp}(z) + 2\mu_1 \frac{\partial}{\partial x_k} e_{ij}(w) \, \frac{\partial}{\partial x_k} e_{ij}(z) \right] \, dx.
\end{equation}

(3.1) is a scalar product in $V$.

It may be shown that the bilinear form (3.1) is coercive in $V$.

\begin{equation}
(3.2) \quad \int_{\Omega} \left[ \gamma \, e_{ii}^2(w) + 2\mu \, e_{ij}(w) \, e_{ij}(w) \right] \, dx \geq_{\text{we use Korn's ineq., see [4]}} \\
\geq \int_{\Omega} \left[ \gamma \, e_{ii}^2(w) + \mu \left( \frac{\partial w_i}{\partial x_i} \frac{\partial w_i}{\partial x_j} \right) + \frac{1}{2} \left( \frac{\partial w_i}{\partial x_i} \right)^2 \right] \, dx \geq_{\mu > 0, \gamma \geq -2/3\mu}
\end{equation}
\[ \mu \int_{\Omega} \left( |\nabla w|^2 + \frac{1}{2} \left( \frac{\partial w_i}{\partial x_i} \right)^2 - \frac{2}{3} \varepsilon_f^2(w) \right) \, dx \geq \]

\[ \geq \mu \int_{\Omega} (|\nabla w|^2) \, dx \geq k \int_{\Omega} (|\nabla w|^2 + |w|^2) \, dx. \]

(3.3) \[
\int_{\Omega} \left( 2\mu \frac{\partial}{\partial x_k} e_{ij}(w) \frac{\partial}{\partial x_k} e_{ij}(w) + \gamma_1 \frac{\partial}{\partial x_k} e_{il}(w) \frac{\partial}{\partial x_k} e_{ll}(w) \right) \, dx = \\
= \int_{\Omega} \left( 2\mu e_{ij}(\partial w/\partial x_k) e_{ij}(\partial w/\partial x_k) + \gamma_1 e_{il} \left( \frac{\partial w}{\partial x_k} \right) e_{ll} \left( \frac{\partial w}{\partial x_k} \right) \right) \, dx \geq \\
\geq 2\mu \int_{\Omega} \left( e_{ij} \left( \frac{\partial w}{\partial x_k} \right) e_{ij} \left( \frac{\partial w}{\partial x_k} \right) \right) \, dx + \\
+ \frac{\gamma_1}{2\mu} e_{ll} \left( \frac{\partial w}{\partial x_k} \right) e_{ll} \left( \frac{\partial w}{\partial x_k} \right) \, dx \geq (\text{we use } (\gamma_1/2\pi_1) > -1/3) \\
\geq 2\mu \int_{\Omega} \left( e_{ij} \left( \frac{\partial w}{\partial x_k} \right) e_{ij} \left( \frac{\partial w}{\partial x_k} \right) + \left( -\frac{1}{3} + \varepsilon \right) e_{il} \left( \frac{\partial w}{\partial x_k} \right) e_{ll} \left( \frac{\partial w}{\partial x_k} \right) \right) \, dx \geq \\
\geq 2\mu \int_{\Omega} \left( e_{ij} \left( \frac{\partial w}{\partial x_k} \right) e_{ij} \left( \frac{\partial w}{\partial x_k} \right) - e_{ij} \left( \frac{\partial w}{\partial x_k} \right) \left( -3\varepsilon + 1 \right) \right) \, dx \geq \\
\geq 6\mu e \int_{\Omega} \left( e_{ij} \left( \frac{\partial w}{\partial x_k} \right) e_{ij} \left( \frac{\partial w}{\partial x_k} \right) \right) \, dx.
\]

Now

(3.4) \[
6\mu e \int_{\Omega} \left( e_{ij} \left( \frac{\partial w}{\partial x_k} \right) e_{ij} \left( \frac{\partial w}{\partial x_k} \right) \right) \, dx + \frac{\mu_0}{2} \int_{\Omega} |\nabla w|^2 \, dx \geq \\
\geq 6\mu e \int_{\Omega} \left( \sum_{k=1}^{3} e_{ij} \left( \frac{\partial w}{\partial x_k} \right) e_{ij} \left( \frac{\partial w}{\partial x_k} \right) \right) + \frac{\mu_0}{2} \cdot 6 \cdot \varepsilon \cdot \mu_1 \sum_{k=1}^{3} |\nabla w_k|^2 \, dx \geq \\
\geq 6\mu e \sum_{k=1}^{3} \int_{\Omega} \left( e_{ij} \left( \frac{\partial w}{\partial x_k} \right) e_{ij} \left( \frac{\partial w}{\partial x_k} \right) + |\nabla w_k|^2 \right) \, dx \geq c \, 6\mu e \sum_{k=1}^{3} \left\| \frac{\partial w}{\partial x_k} \right\|_{W^{1,2}}.
\]

(we use the coerciveness of deformations, see [4]). (3.2), (3.4) imply that

\[ ((w, w)) \geq c \int_{\Omega} (|w|^2 + |\nabla w|^2 + |\nabla^2 w|^2) \, dx \quad \forall w \in W^{1,2}(\Omega, \mathbb{R}^n) \]

(this implies that the bilinear form is coercive in $V$).

Let $(z_k)_{k=1}^{\infty}$ be a complete orthonormal system of eigenfunctions in $V$ with the scalar product $((\cdot, \cdot))$, we seek the solution to the eigenproblem in $V$

(3.5) \[
((w, z)) = \lambda (w, z) \quad \forall w \in V, \quad \forall z \in V,
\]

where 

\[ (w, z) = \int_{\Omega} w_i z_i \, dx. \]
We have
\[(3.6) \quad ((w, z^k)) = \lambda_k(w, z^k), \quad (z^k, z^l) = \delta_{kl}.
\]
For \(z \in L^2(\Omega, \mathbb{R}^N)\) let us put
\[(3.7) \quad P_m z = \sum_{k=1}^m \lambda_k(z^k, z) z^k.
\]
If
\[
L^2_m = \text{span} \{z^1, \ldots, z^m\} \quad \text{in} \quad L^2,
V_m = \text{span} \{z^1, \ldots, z^m\} \quad \text{in} \quad V,
\]
then \(P_m\) is a projector of \(L^2\) onto \(L^2_m\) and of \(V\) onto \(V_m\). From the regularity of solutions to the linear elliptic problem we get
\[(3.8) \quad z^k \in C^\infty(\bar{\Omega}; \mathbb{R}^N)
\]
from (3.6), see [7].

Remark.
\[
\sqrt{\lambda_m} z^k \quad \text{is an orthonormal system in} \quad L^2.
\]

Remark. For the construction of the base for the Galerkin method we have used the following regularity property of the weak solution to the elliptic problem
\[(3.9) \quad u \in V, \quad f \in L^2(\Omega, \mathbb{R}^N)
\]
\[
((v, u)) = \int_\Omega v f_i \, dx \quad \text{for every} \quad v \in V.
\]

**Theorem 3.1.** Let \(u \in V\) be a solution to (3.9). Then \(u \in W^{4,2}(\Omega, \mathbb{R}^N)\), \(c > 0\) and
\[(3.10) \quad \|u\|_{W^{4,2}(\Omega, \mathbb{R}^N)} \leq c \|f\|_{L^2(\Omega, \mathbb{R}^N)}.
\]

**Proof.** See [7].

By (3.9) one defines the operator \(\mathbf{A}\):
\[(3.11) \quad ((w, z)) = (\mathbf{A}w, z) \quad \text{for every} \quad z \in V.
\]
Its domain of definition is denoted by \(D(\mathbf{A})\), of course \(W^{4,2}_0(\Omega, \mathbb{R}^N) \subset D(\mathbf{A})\). It is a consequence of Theorem 3.1 that
\[(3.11) \quad \|w\|_{W^{4,2}(\Omega, \mathbb{R}^N)} \leq k_1 \|\mathbf{A}w\|_{L^2(\Omega, \mathbb{R}^N)},
\]
\[
k_1 > 0, \quad \forall w \in D(\mathbf{A}) \quad \text{hence}
\]
\[(3.12) \quad \|P_m w\|_{W^{4,2}(\Omega, \mathbb{R}^N)} \leq k_1 \|\mathbf{A}P_m w\|_{L^2(\Omega, \mathbb{R}^N)} \leq \mathbf{A}P_m = P_m \mathbf{A}
\]
\[
\leq k_1 \|P_m \mathbf{A}w\|_{L^2(\Omega, \mathbb{R}^N)} \leq k_2 \|\mathbf{A}w\|_{L^2(\Omega, \mathbb{R}^N)} \leq k_3 \|w\|_{W^{4,2}(\Omega, \mathbb{R}^N)}
\]
for every \(w \in W^{4,2}_0(\Omega, \mathbb{R}^N)\), \(k_1, k_2, k_3 > 0\).

Remark. Proof of the property \(\mathbf{A}P_m = P_m \mathbf{A}\).
Proof.

$$A P_m^w = A \sum_{i=1}^{m} a_i w_i,$$

where

$$a_i = \sqrt{(\lambda_i)(w^i, w)}.$$

This implies

$$A P_m^w = A \sum_{i=1}^{m} a_i w_i = A \sum_{i=1}^{m} \lambda_i(w^i, w) w^i = \sum_{i=1}^{m} \lambda_i(w^i, w) A w^i =$$

$$= \sum_{i=1}^{m} \lambda_i^2(w, w^i) w^i,$$

$$A w = \sum_{i=1}^{\infty} (A w, w^i) \lambda_i w^i,$$

$$P_m A w = \sum_{i=1}^{m} (A w, w^i) \lambda_i w^i = \sum_{i=1}^{m} \lambda_i^2(w, w^i) w^i.$$

Thus

$$P_m A w = A P_m^w. \quad (3.13)$$

Due to the interpolation theorem, see [8], we thus have for every $v \in W_{0}^{3,2}(\Omega)$

$$\|P_m^w\|_{W_{0}^{3,2}(\Omega, R^N)} \leq k_4\|v\|_{W_{0}^{3,2}(\Omega, R^N)}, \quad k_4 > 0. \quad (3.14)$$

Let $c_i \in C^1(I)$, $I = (0, t_0)$ and let us put

$$w^m(t, x) = \sum_{i=1}^{m} c_i(t) z^i(x)$$

and

$$v^m(t, x) = v^0 + w^m(t, x).$$

We suppose that we know the velocity $v^m$ and want to obtain $\varrho_m$. Let us first look for $\varrho_m \in C^1(\bar{Q}_t)$ such that

$$\frac{\partial \varrho_m}{\partial t} + \frac{\partial}{\partial x_i} (\varrho_m v^m_i) = 0. \quad (3.15)$$

We suppose that

$$\varrho_m(0, t) = \varrho_0(x) \in C^1(\Omega), \quad \varrho_0(x) > 0 \text{ in } \bar{\Omega}. \quad (3.16)$$

Let

$$x^m(t) = -v^m(t - \tau) \varrho^m(\tau), \quad \varrho^m(0) = x^0, \quad x \in \Omega. \quad (3.17)$$

$\varrho_m$ may be obtained by integration along characteristics. These characteristics pass through $\bar{Q}_{t_0}$ and start either in $\Omega_0$ or in $\Gamma_{inp}$. Thus it is possible to use the fact that we know $\varrho_m$ on the sets $\Omega_0$, $\Gamma_{inp}$.

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For $\tau \in I_t$ where $I_t \subset I$ and $I_t = (0, t)$, $\tilde{t} > 0$, $x \to x(\tau)$ is a local diffeomorphism of $\tilde{\Omega}$ onto $\Omega$ and for $\sigma_m = \ln q_m$ we have

$$\frac{\partial \sigma_m}{\partial t} + \frac{\partial \sigma_m}{\partial x_i} v^m_i + \sigma_m \frac{\partial v^m_i}{\partial x_i} = 0,$$

$$- \frac{\partial \sigma_m}{\partial t} + \frac{\partial \sigma_m}{\partial x_i} v^m_i = - \frac{\partial \sigma_m}{\partial t} + \frac{\partial \sigma_m}{\partial x_i} \frac{\partial x^m_i}{\partial \tau} = \frac{\partial v^m_i}{\partial x_i} (t - \tau, x^m(\tau)) =$$

$$= \frac{d \sigma_m}{d \tau} (t - \tau, x^m(\tau)).$$

Hence

$$\int_0^t \frac{d}{d \tau} \sigma_m(t - \tau, x^m(\tau)) d \tau = \int_0^t \frac{\partial v^m_i}{\partial x_i} (t - \tau, x^m(\tau)) d \tau.$$

Thus

$$\sigma_m(t, x^m(0)) = \sigma_m(t - \tilde{t}, x^m(\tilde{t})) - \int_0^t \frac{\partial v^m_i}{\partial x_i} (t - \tau, x^m(\tau)) d \tau.$$

There is a unique characteristic passing through the point $[x, t]$. Let us denote by $t$ the time it takes for a particle of liquid to reach the point $[x, t]$ along this characteristic from the initial point of the characteristic (i.e. from $\Omega_0$ or from $\Gamma_{inp}$).

$$\theta_m(t, x) = \theta_0(t - \tilde{t}, x^m(\tilde{t})) \exp \left( - \int_0^t \frac{\partial v^m_i}{\partial x_i} (t - \tau, x^m(\tau)) d \tau \right),$$

where $x = x^m(0)$.

**Theorem 3.2.** $\theta_m \in C(\bar{Q})$, $\theta_m \in W^{1, \infty}(Q_t)$.

**Proof.** For the sake of simplicity we assume that $\Gamma_{inp}$ is an interval and $N = 2$

For the proof that $\theta_m \in C(\bar{Q})$ see [11].

First we prove that $\theta_m \in W^{1, \infty}(Q_1)$ and $\theta_m \in W^{1, \infty}(Q_2)$, where $Q_1 \cup Q_2 \cup S = Q_t$ and $S$ is the surface described by the trajectories of solutions of the equations $x^m = v^m(t, x^m(t))$, $x \in \Gamma_{inp}$ closed.

For $t$ fixed, $t = \tilde{t}$,

$$\frac{\partial \sigma_m}{\partial x_i} (t, x) = \frac{\partial \theta_0}{\partial x_k} \frac{\partial X_k}{\partial x_i} \exp \left( - \int_0^t \frac{\partial v^m_i}{\partial x_i} (t - \tau, x^m(\tau)) d \tau \right) +$$

$$+ \theta_0(0, x^m(t)) \exp \left( - \int_0^t \frac{\partial v^m_i}{\partial x_i} (t - \tau, x^m(\tau)) d \tau \right) \times$$

$$\times \left( - \int_0^t \frac{\partial^2 v^m_i}{\partial x_j \partial x_k} (t - \tau, x^m(\tau)) \frac{\partial x^m_j}{\partial x_k} (t, x) d \tau \right).$$
\[
\begin{align*}
&= \frac{\partial \varrho_0}{\partial X_k} \frac{\partial X_k}{\partial x_i} \exp \left( - \int_0^t \frac{\partial v_0}{\partial x_i} (t - \tau, x^m(\tau)) \, d\tau \right) + \\
&\quad + \varrho_0(0, x^m(t)) \exp \left( - \int_0^t \frac{\partial v_0}{\partial x_i} (t - \tau, x^m(\tau)) \, d\tau \right) \times \\
&\quad \times \left( - \int_0^t \frac{\partial^2 v_0}{\partial x_j \partial x_k} (t - \tau, x^m(\tau)) \frac{\partial x^m}{\partial X_k} \frac{\partial X_i}{\partial x_i} \, d\tau \right).
\end{align*}
\]

We assume that \( \varrho_0 \in C^1(\overline{Q}_1) \), \( \partial X_i/\partial x_i \) is the inverse matrix to \( \partial x_i/\partial X_k = \mathcal{B} \), where \( x^m(\tau) = v(\tau, x^m(\tau)) \).

(3.22) \[ \frac{d}{d\tau} \left( \frac{\partial^m X_i}{\partial X_k} \right) = \frac{\partial v_j}{\partial x_m} \frac{\partial x_m}{\partial X_k}. \]

It implies that \( \partial X_i/\partial x_i \) are bounded and continuous functions depending on parameters, see [11]. Further \( \mathcal{B}^{-1}(\partial X_i/\partial x_m) \) is bounded which follows from conversation of the mass \( \varrho_0(X) \, dX = \varrho(t, x) \, dx \) and \( \varrho(X)/\varrho(t, x) = dx/dX \neq 0 \) because \( \varrho_0 > 0 \) and \( \varrho \neq 0 \) (from (3.20)). Thus \( \varrho_m \) has a continuous first derivative with respect to \( x_i \) and this derivative is bounded on \( Q_1 \) and \( Q_2 \) (analogously for \( \partial \varrho_m/\partial t; \partial \varrho_m/\partial t, \partial \varrho_m/\partial x_i \) on input).

Thus

(3.23) \[ \varrho_m \in W^{1,\infty}(Q_1) \quad \text{and} \quad \varrho_m \in W^{1,\infty}(Q_2). \]

By second, we verify that the surface \( S \) is differentiable. We have to verify, see [13], that \( |D| \neq 0 \) where \( X \) and \( t \) are parameters of the surface and the surface is described by the following equations:

\begin{align*}
x_1 &= x_1(X, t), \\
x_2 &= x_2(X, t), \\
T &= t.
\end{align*}

Then

\[ D = \begin{pmatrix}
\frac{\partial x_1}{\partial X} & \frac{\partial x_2}{\partial X} & \frac{\partial t}{\partial X} \\
\frac{\partial x_1}{\partial t} & \frac{\partial x_2}{\partial t} & \frac{\partial t}{\partial t}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial x_1}{\partial X} & \frac{\partial x_2}{\partial X} & 0 \\
\frac{\partial x_1}{\partial t} & \frac{\partial x_2}{\partial t} & 1
\end{pmatrix}, \]

\[ |D| \neq 0 \quad \text{because} \quad |\mathcal{B}^{-1}| \neq 0. \]

Thirdly, we use the following theorem:

*Let \( \mathcal{S} \in W^{1,\infty}(Q_1), \mathcal{S} \in W^{1,\infty}(Q_2) \), then \( \mathcal{S} \in W^{1,\infty}(Q_t) \).*
Proof. See [12].

Only the outline of the proof is presented. We wish to prove that

\[
\int_{\Omega} \frac{\partial f}{\partial x_1} \phi = - \int_{\Omega} f \frac{\partial \phi}{\partial x_1}, \quad \text{supp } \phi \subset M
\]

and

\[
\int_{M_1} \frac{\partial f}{\partial x_1} \phi = \int_{\Gamma} f \phi \tilde{v}_1 \, dS - \int_{M_1} f \frac{\partial \phi}{\partial x_1},
\]

\[
\int_{M_2} \frac{\partial f}{\partial x_1} \phi = \int_{\Gamma} f \phi v_1 \, dS - \int_{M_2} f \frac{\partial \phi}{\partial x_1},
\]

and \( \tilde{v}_1 = -v_1. \)

Then

\[
\int_{\Omega} \frac{\partial f}{\partial x_1} \phi = \int_{M_1} \frac{\partial f}{\partial x_1} \phi + \int_{M_2} \frac{\partial f}{\partial x_1} = - \int_{M_1} f \frac{\partial \phi}{\partial x_1} - \int_{M_2} f \frac{\partial \phi}{\partial x_1} = \]

\[
= - \int_{\Omega} f \frac{\partial \phi}{\partial x_1}.
\]

Thus \( \varrho_m \in W^{1,\infty}(Q_t). \)

Now, let us look for \( \tilde{v}^m \) such that \( \forall t \in I, \)

\[
(3.24) \quad \int_{\Omega} \left( \varrho_m \frac{\partial \tilde{v}^m}{\partial t} + \varrho_m v_j^m \frac{\partial \tilde{v}^m}{\partial x_j} + \lambda \frac{\partial \varrho_m}{\partial x_i} \right) z_i \, dx = -((\tilde{v}^m, z^k)), \quad k = 1, \ldots, m
\]

(i.e. we suppose that we know \( \varrho_m \) and want to obtain \( \tilde{v}^m \)). This equation is a system of ordinary differential equations where the unknown is \( \tilde{c}^m(t) \). Since \( \varrho_m \in C^{0,1}(Q_t), \) then \( \tilde{c}^m(t) \in C^1(I). \)

The initial conditions are

\[
(3.25) \quad \int_{\Omega} \sum_{j=1}^{m} \tilde{c}_j(0) z_j^i \, dx = \int_{\Omega} v_i(0, x) z_i^i(x) \, dx.
\]
In the sequel we shall assume that
\[(3.26) \quad v(0, x) \in L^2(\Omega, R^N).\]
Because \(\det \int_{\Omega} q_m z_i z_i' \, dx \neq 0\), we can solve (3.24), (3.25) uniquely in \(I\). We have \(c_i \in C^1(\Theta)\). If we start with \(c_i(t)\) in the ball
\[(3.27) \quad \max_{[0, x]} |c_i(t) - c_i(0)| \leq 1, \quad i = 1, 2, \ldots, m,\]
we get
\[(3.28) \quad \max_{[0, x]} |\tilde{c}_i(t) - c_i(0)| \leq 1, \quad i = 1, 2, \ldots, m,\]
\[(3.29) \quad \max_{[0, x]} |\tilde{c}_i'(t)| \leq K(\alpha),\]
provided \(\alpha\) is sufficiently small. Thus applying Schauder's fixed point theorem we obtain \(\tilde{c}_i = c_i\) on \([0, \alpha]\). But for such solutions we obtain
\[(3.30) \quad \int_{\Omega_t} q_m \, dx \leq \int_{\Omega_0} q_0 \, dx + \int_0^t \int_{\Gamma_{inp}} q_0 v_0 v_1 \, dS \, dt + \frac{1}{2} \int_{\Omega_t} q_m |w|^2 \, dx - \frac{1}{2} \int_{\Omega_0} q_0 |w|^2 \, dx + \int_0^t (v^m, w^m) \, dt + \int_{\Omega_t} \left( \frac{\partial v_i^0}{\partial t} q_m w_i^m + q_m (v_j^0 + w_j) w_i^m \frac{\partial v_i^0}{\partial x_j} \right) \, dx \, dt + \lambda \int_{\Omega_t} (q_m \ln q_m - q_m) \, dx - \lambda \int_{\Omega_0} (q_m \ln q_m - q_m) \, dx + \lambda \int_0^t \int_{\Gamma_{inp}} q_m v_i^0 \ln q_m v_i \, dS \, dt + \lambda \int_0^t \int_{\Gamma_{out}} q_m v_i^0 \ln q_m v_i \, dS \, dt + \lambda \int_{\Omega_t} q_m \, dx - \lambda \int_{\Omega_0} q_m \, dx + \lambda \int_0^t \int_{\Omega_t} q_m \frac{\partial v_i^0}{\partial x_i} \, dx \, dt = 0.\]

Now we estimate as in the previous section.

We obtain
\[(3.32) \quad \frac{1}{2} \int_{\Omega_t} q_m |w|^2 \, dx + \int_{\Omega_t} q_m \ln q_m \, dx + c_1 \int_0^t \|w^m\|^2 \, dt \leq c_2 \int_{\Omega_t} q_m |w|^2 \, dx \, dt + c_3 \leq c_4, \quad c_1, c_2, c_3, c_4 > 0.\]

Now we denote by \(M\) the set of the maximal \(\alpha\)'s. \(M\) is closed, which follows from (3.32) and \(M\) is open as follows from the theory of ordinary differential equations, see [11]. This implies that \(\alpha = t_0\).
4. THE LIMIT PASSAGE

Lemma 4.1. Let $B$ be a Banach space, $B_i$ ($i = 0, 1$) reflexive Banach spaces. Let $B_0 \subset \subset B \subset B_1$ (\(\subset \subset\) denotes a compact imbedding), $1 < p_i < \infty$. Let $W = \{v, v \in L^{p_i}(I, B_0), \partial v/\partial t \in L^{p_i}(I, B_1)\}$. Then $W \subset \subset L^{p_0}(I, B)$.

Proof. See [6].

Main theorem. Let (2.2), (2.5), (2.57), (2.59) be satisfied. Then there exists

\begin{align}
(4.1) & \quad q \in L^\infty(I, L_{q}(\Omega)) \quad q \geq 0 \quad \text{a.e. in } Q_t, \\
(4.2) & \quad v \in L^{2}(I, W^{2,2}(\Omega)) \cap W^{1,2}_0(\Omega, R^N)), \\
(4.3) & \quad \frac{\partial q}{\partial t} \in L^{2}(I, W^{-3,2}(\Omega)), \\
(4.4) & \quad \frac{\partial}{\partial t}(qv) \in L^{2}(I, W^{-3,2}(\Omega, R^N))
\end{align}

satisfying (2.3), (2.8) in the sense of distributions and being such that (2.6) holds. Moreover we have

\begin{align}
(4.5) & \quad \|q\|_{L^\infty(I, L_{1}(\Omega))} \leq \int_{\Omega_0} q_0 \, dx, \\
(4.6) & \quad \frac{1}{2} \|q|^2\|_{L^\infty(I, L_{1}(\Omega))} + \|w\|_{L^2(I, W^{2,2}(\Omega, R^N))}^2 + \\
& \quad + \lambda \sup \int_{\Omega} q \ln q \, dx \leq h_1 \quad (h_1 \geq 0).
\end{align}

Proof. Let $0 \leq k \leq 2$ and let $W^{k,2}(\Omega)$, $W^{k,2}_0(\Omega)$ be the usual Sobolev spaces with fractional derivatives, see [2]; let $V^k = V$, where the closure is taken in $W^{k,2}(\Omega, R^N)$; naturally the traces are zero for $k \geq \frac{1}{2}$ only.

Since

\begin{align}
(4.7) & \quad W^{2,2}_0(\Omega) \subset \subset W^{1,2}_0(\Omega) \subset \subset W^{k,2}(\Omega) \subset \subset C(\Omega) \subset C_0(\Omega) \quad \text{for} \\
& \quad N/2 < k_2 < k_1 < 2
\end{align}

we have

\begin{align}
(4.8) & \quad L_{q}(\Omega) \subset \subset W^{-k,2}(\Omega) \subset W^{-k_1,2}(\Omega) \subset W^{-2,2}(\Omega) ;
\end{align}

obviously

\begin{align}
(4.9) & \quad W^{k,2}(\Omega) \subset C_0(\Omega),
\end{align}

hence

\begin{align}
(4.10) & \quad L_{q}(\Omega) \subset (W^{2,2}_0(\Omega))^* \quad N/2 < k.
\end{align}

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It follows from the interpolation theorem (see [8]) that

\begin{equation}
(4.11) \quad \sup_{v \in V^k} \| P_m v \|_{W^{k,2}(\Omega, R^N)} \leq h_2, \quad h_2 > 0 \ (0 \leq k \leq 2),
\end{equation}

where \( \| v \|_{W^{k,2}(\Omega, R^N)} \leq 1 \)

and

\begin{equation}
(4.12) \quad \sup_{v \in (V^k)^*} \| P^*_m v \|_{(V^k)^*} \leq h_3, \quad h_3 > 0 \ (0 \leq k \leq 2)
\end{equation}

\[ \| v \|_{(V^k)^*} \leq 1 \]

\((P^*_m\) is the dual operator to \(P_m\)).

**Lemma 4.2.** Let \(0 \leq k \leq 2\). Then for \(\varepsilon > 0\) there exists \(l_0\) such that for \(l \geq l_0\) we have \(\| u - P_l u \|_{W^{k,2}(\Omega, R^N)} < \varepsilon\) provided \(\| u \|_{W^{0,2}(\Omega, R^N)} \leq 1\).

**Proof.** Let us assume the contrary. Then there exists \(\varepsilon_0 > 0\) and \(P_{l_j}, l_j \to \infty\) and \(\| u_{l_j} \|_{W^{k,2}(\Omega, R^N)} \leq 1\) such that \(\| u_{l_j} - P_{l_j} u_{l_j} \|_{W^{k,2}(\Omega, R^N)} \geq \varepsilon_0\). Since \(W^{0,2}(\Omega, R^N) \subset C \subset W^{k,2}(\Omega, R^N)\), we can assume \(u_{l_j} \to u\) strongly in \(W^{k,2}(\Omega, R^N)\), hence \(P_{l_j} u_{l_j} \to u\) strongly in \(W^{k,2}(\Omega, R^N)\) which is a contradiction.

Let \(q_m, v^m\) be an approximate solution from Section 3. Then there exist subsequences (denoted again by \((q^m_{(m)})_{m=1}^\infty, (v^m_{(m)})_{m=1}^\infty)\) such that

\begin{equation}
(4.13) \quad q_m \to q^* - \text{weakly in} \quad L^2(I, L^q(\Omega)) = (L^2(I, C^0(\Omega)))^*,
\end{equation}

\begin{equation}
(4.14) \quad v^m \to v \quad \text{weakly in} \quad L^2(I, W^{2,2}(\Omega)),
\end{equation}

\begin{equation}
(4.15) \quad v^m \to v, \quad \frac{\partial}{\partial x_i} v^m \to \frac{\partial}{\partial x_i} v, \quad \frac{\partial^2 v^m}{\partial x_i \partial x_j} \to \frac{\partial^2 v}{\partial x_i \partial x_j}, \quad i, j = 1, \ldots, M
\end{equation}

weakly in \(L^2(Q_i)\).

Due to (3.30) and (4.8) we obtain

\begin{equation}
(4.16) \quad \| q_m \|_{L^\infty(I, W^{-k,2}(\Omega))} \leq c_1, \quad c_1 > 0, \quad N/2 < k \leq 2
\end{equation}

\((q_m \in L^\infty(I, L^q(\Omega)) \subset L^\infty(I, W^{-k,2}(\Omega))\).

For \(N = 2, 3\) we have

\begin{equation}
(4.17) \quad \| v^m \|_{C^0(I, R^N)} \leq c_2 \| v^m \|_{W^{2,2}(\Omega, R^N)}, \quad c_2 > 0
\end{equation}

(Sobolev imbedding),

hence

\begin{equation}
(4.18) \quad \| q_m v^m \|_{L^2(I, L^q(\Omega, R^N))} \leq c_3, \quad c_3 > 0
\end{equation}

\(\left( \sup_{\| \Phi \|_{L^q(\Omega, R^N)} \leq 1} \left( \int_\Omega q_m v^m \Phi \right) \leq \| v^m \|_{W^{2,2}(\Omega)} \left( \int_\Omega q_m \Phi \right) \right) \leq \)

\[ \leq c_3 \| v^m \|_{W^{2,2}(\Omega)} \| q_m \| \& \| v^m \|_{W^{2,2}} < \infty \).

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It follows from (3.15) that

\[
\left\| \frac{\partial q_m}{\partial t} \right\|_{L^2(I, W^{-3,2}(\Omega, R^N))} \leq c_4, \quad c_4 > 0
\]

\[
\left( \frac{\partial q}{\partial t}, w \right) = - \int_{\Omega} q_m v_i^m \frac{\partial w_i}{\partial x_i} \, dx,
\]

\[
\sup_n \left\| \frac{\partial q}{\partial t} \right\|_{W^{3,2}} (w) \leq c \max_n \left| q_i \right| \max_n \left| \frac{\partial w_i}{\partial x_j} \right| \leq c \left\| q \right\|_{W^{3,2}} \left\| w \right\|_{W^{3,2}} \leq c \left\| v \right\|_{W^{3,2}}
\]

and this implies \( \partial q_m / \partial t \in L^2(I, W^{-3,2}) \).

Due to Lemma 4.1, \( q_m \to q \) strongly in \( L^2(I, W^{-2,2}(\Omega)) \). By (4.8), (4.18) we get

\[
q_m v^m \to q v \quad \text{in} \quad L^2(I, W^{-2,2}(\Omega)).
\]

Also

\[
\left\| q_m v^m \right\|_{L^2(I, W^{-k,2}(\Omega, R^N))} \leq c_5, \quad c_5 > 0, \quad N/2 < k/2,
\]

hence by (4.12)

\[
\left\| P_m(q_m v^m) \right\|_{L^2(I, W^{-k,2}(\Omega, R^N))} \leq c_6, \quad c_6 > 0.
\]

According to (3.31), \( q_m |v^m|^2 \) is bounded in \( L^\infty(I, L^1(\Omega)) \), therefore

\[
\left\| q_m |v^m|^2 \right\|_{L^2(I, W^{-2,2}(\Omega, R^N))} \leq \left\| q_m |v^m|^2 \right\|_{L^\infty(I, L^1(\Omega))} \leq c_7, \quad c_7 > 0.
\]

By (3.24), (3.14), (3.32), (4.16), (4.22)

\[
\left\| \frac{\partial}{\partial t} (P_m(q_m v^m)) \right\|_{L^2(I, W^{-3,2}(\Omega, R^N))} \leq c_8, \quad c_8 > 0
\]

holds.

Thus, by Lemma 4.1, \( P_m(q_m v^m) \to a \) strongly in \( L^2(I, W^{-2,2}(\Omega, R^N)) \). Let \( z \in L^2(I, W_0^{-2,2}(\Omega, R^N)) \). Because of Lemma 4.2 for \( m \) sufficiently large, \( k < 2 \), we have

\[
\int_0^t \left| q_m \right| \left| v^m \right|^2 \, dt \leq c^2 \int_0^t \left| z \right|_{W^{2,2}(\Omega, R^N)}^2 \, dt,
\]

hence for \( \int_0^t \left| z \right|_{W^{2,2}(\Omega, R^N)}^2 \, dt \leq 1 \), it follows that

\[
\lim_{m \to \infty} \int_0^t \int_{\Omega} \left( P_m \left( q_m v_i^m \right) - q_m v_i^m \right) z_i \, dx \, dt = 0
\]

uniformly with respect to \( z \).

Therefore, \( q_m v^m \) is a Cauchy sequence in \( L^2(I, W^{-2,2}(\Omega, R^N)) \) and \( q_m v^m \to a \) strongly in \( L^2(I, W^{-2,2}(\Omega, R^N)) \). But \( q_m v^m \to q v \) in \( D'(Q) \) in the sense of distributions, hence \( a = q v \). Therefore due to (4.22)

\[
q_m v_i^m z_j \to q v_i z_j \quad \text{weakly in} \quad L^2(I, W^{-2,2}(\Omega, R^N)).
\]
It follows from (3.24) that for every $\phi \in C^\infty(\mathcal{Q}, \mathbb{R}^N)$ satisfying $\phi(t) \in V_m$ for every $t \in [0, T]$ and $\phi(T) = 0$, we have

\[
\int_{\mathcal{Q}_t} q v_i \frac{\partial \phi_i}{\partial t} \, dx \, dt + \int_{\mathcal{Q}_t} q v_i v_j \frac{\partial \phi_i}{\partial x_j} \, dx \, dt + \lambda \int_{\mathcal{Q}_t} q \frac{\partial \phi_i}{\partial x_j} \, dx \, dt = \\
= \int_0^T \left( (\mathbf{v}, \phi) \right) \, dt - \int_{\Omega} q_0 \psi_i \phi_i \, dx
\]

and

\[
- \int_{\Omega_0} q_0 \phi_i(0) \, dx - \int_{\mathcal{Q}_t} q \frac{\partial \phi_i}{\partial t} \, dx \, dt - \int_{\mathcal{Q}_t} q v_i \frac{\partial \phi_i}{\partial x_j} = 0.
\]

Due to the density argument (2.61) holds and (2.8) is satisfied in the sense of distributions. The continuity equation is obviously satisfied in the sense of distributions.

References


Souhrn

GLOBÁLNÍ ŘEŠENÍ IZOTERMICKÉ STLAČITELNÉ BIPOLÁRNÍ TEKUTINY NA KONEČNÉM KANÁLU S NENULOVÝMI VSTUPY A VÝSTUPY

Šárka Matušš-Nečasová

V práci je dokázána globální existence slabého řešení vazké stlačitelné izotermické bipolarní tekutiny smíšené počáteční okrajové úlohy na konečném kanálu.

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