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WAVE EQUATION WITH A CONCENTRATED MOVING SOURCE

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Summary. A tempered distribution which is an exact solution of the wave equation with a concentrated moving source on the right-hand side, is obtained in the paper by means of the Cagniard - de Hoop method.

Let $\boldsymbol{x} = (x, y, z)$ be a point in the Euclidean space $\mathbb{R}^3$, $\tau \in \mathbb{R}^1$, let $\mathcal{S}^*(\mathbb{R}^3 \times \mathbb{R}^1)$ be the tempered distributions space, $\delta(\cdot)$ and $H(\cdot)$ the Dirac function and the Heaviside function, respectively.

We also use the notation $R = \sqrt{(x^2 + y^2 + z^2)}$, $r = \sqrt{(x^2 + y^2)}$, and fix the radical branch by the condition $\sqrt{1} = 1$.

The purpose of this paper is to obtain the solution $u(\boldsymbol{x}, \tau) \in \mathcal{S}^*(\mathbb{R}^3 \times \mathbb{R}^1)$ of the following Cauchy problem

$$\left[\Delta - \frac{\partial^2}{\partial \tau^2}\right] u(\boldsymbol{x}, \tau) = -\delta(x)\, \delta(y)\, \delta(\eta\tau - z)\, H(z)\, H(\tau)\,, \tag{1}$$

$$u(\boldsymbol{x}, 0) = u'_\tau(\boldsymbol{x}, 0) = 0\,. \tag{2}$$

Physically the right-hand side of (1) describes the concentrated moving source, which at the moment $\tau = 0$ „switches on" and starts to move with the speed $\eta \neq 1$ from the origin of coordinate to $+\infty$ along the positive z-axis.

The Heaviside functions in (1) distinguish the problem, considered here, from similar problems, described in [1, 2], where concentrated source $\delta(x)\,\delta(y)\,\delta(\eta\tau - z)$ moves along the z-axis from $-\infty$ to $+\infty$ (conditions (2) are of course not satisfied).

Applying the usual integral Laplace transform with respect to τ in (1) and denoting $U(\boldsymbol{x}; p) = L[u(\boldsymbol{x}, \tau)]$, then applying the integral Fourier transforms with respect to x and y with the kernels $e^{ip\alpha x}$, $e^{ip\beta y}$ respectively (according to [3]) and denoting $\tilde{U}(z; \alpha, \beta, p) = F_{xy}[U(\boldsymbol{x}; p)]$, we obtain (taking (2) into account):

$$\frac{d^2\tilde{U}}{dz^2} - p^2\gamma^2\tilde{U} = -\frac{e^{-pz/\eta}\, H(z)}{\eta}\,, \tag{3}$$

where

$$\gamma = \sqrt{(1 + \alpha^2 + \beta^2)} \quad (\operatorname{Re}\gamma \geqq 0)\,.$$

It is easy to find the solution of (3) which is continuous at $z = 0$:

$$(4) \qquad \tilde{U}(z; \alpha, \beta, p) = -\frac{\eta e^{-pz/\eta} H(z)}{p^2(1-\eta^2\gamma^2)} + \frac{e^{-p\gamma|z|}}{2p^2\gamma(1-\eta^2\gamma^2)} + \frac{\eta e^{-p\gamma|z|}(\operatorname{sign} z)}{2p^2(1-\eta^2\gamma^2)}.$$

For brevity we denote the terms in (4) by $\tilde{U}_1, \tilde{U}_2, \tilde{U}_3$, respectively.

The inverse Fourier transform of $\tilde{U}_1$ is applied in the usual way:

$$(5) \qquad U_1(x; p) = -\frac{\eta e^{-p\gamma} H(z)}{4\pi^2\eta^2} \iint_{-\infty}^{\infty} \frac{e^{ip(\alpha x+\beta y)} p^2 \, d\alpha \, d\beta}{1-\eta^2(1+\alpha^2+\beta^2)}.$$

Substituting $x = r\cos\varphi$, $y = r\sin\varphi$, $\alpha = \varrho\cos\theta$, $\beta = \varrho\sin\theta$ in (5), using the well-known representation of the Bessel functions

$$I_m(z) = \frac{(-i)^m}{\pi} \int_0^\pi e^{iz\cos t} \cos mt \, dt \quad (m = 0, 1, 2, \ldots)$$

and the appropriate formulas from [4], we obtain

$$(6) \qquad U_1(x; p) = \begin{cases} \dfrac{H(z)}{2\pi\eta} e^{-pz/\eta} K_0\left(pr\sqrt{\left(1-\dfrac{1}{\eta^2}\right)}\right), & \eta > 1, \\ -\dfrac{H(z)}{4\eta} e^{-pz/\eta} Y_0\left(pr\sqrt{\left(\dfrac{1}{\eta^2}-1\right)}\right), & 0 < \eta < 1, \end{cases}$$

where K_0 and Y_0 are the McDonald and Neumann functions, respectively.

To find the inverse Fourier transform of $\tilde{U}_2$ and $\tilde{U}_3$ we will use the idea of the Cagniard - de Hoop method [3]. We will show this technique only for $\tilde{U}_2$ (the term $\tilde{U}_3$ is treated in the same way). So,

$$(7) \qquad U_2(x; p) = -\frac{1}{8\pi^2} \iint_{-\infty}^{\infty} \frac{e^{-p[(1+\alpha^2+\beta^2)^{1/2}|z| + i(\alpha x+\beta y)]} \, d\alpha \, d\beta}{(1+\alpha^2+\beta^2)^{1/2}\,[1-\eta^2(1+\alpha^2+\beta^2)]}.$$

Substituting $\alpha = (\omega x - qy)/r$, $\beta = (\omega y + qx)/r$, $s = i\omega$ in (7) we obtain

$$(8) \quad U_2(x; p) = -\frac{i}{4\pi^2\eta^2} \int_{-\infty}^{\infty} dq \int_{-i\infty}^{i\infty} \frac{e^{-p[sr+|z|(1+q^2-s^2)^{1/2}]} \, ds}{(1+q^2-s^2)^{1/2}\,[s^2-(1+q^2-1/\eta^2)]}.$$

It is obvious that the interand in (8) (denoted by $\Phi(s)$) on the complex plane S has algebraic branching points

$$s_b^{\pm} = \pm\sqrt{(q^2+1)},$$

and the first-order poles

$$(9) \qquad s_p^{\pm} = \pm\sqrt{(1+q^2-1/\eta^2)} \quad \text{when} \quad \eta > 1,$$

or

$$(10) \qquad S_p^{+} = \begin{cases} \pm i\sqrt{(1/\eta^2-1-q^2)}, & 0 < q < \sqrt{(1/\eta^2-1)}, \quad 0 < \eta < 1, \\ \pm\sqrt{(1+q^2-1/\eta^2)}, & q > \sqrt{(1/\eta^2-1)}, \quad 0 < \eta < 1. \end{cases}$$

Now, to determine the unique radical branch let us make cuts $(s_b^-, -\infty)$, $(s_b^+, +\infty)$ along the real axis, and then, following [3], find the contour Γ in the S-plane, described as follows (Fig. 1):

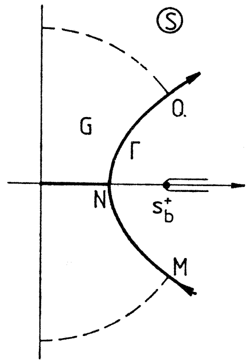

Fig. 1.

$$(11) \qquad S = \begin{cases} \dfrac{\tau r - |z| \sqrt{((1+q^2) R^2 - \tau^2)}}{R^2}, & |z| \sqrt{(1+q^2)} < \tau < R \sqrt{(1+q^2)}, \\ \dfrac{\tau r \pm \mathrm{i} |z| \sqrt{(\tau^2 - (1+q^2) R^2)}}{R^2}, & \tau > R \sqrt{(1+q^2)}, \end{cases}$$

along which the expression

$$(12) \qquad \tau = sr + |z| \sqrt{(1 + q^2 - s^2)}$$

should be real.

The general idea of the Cagniard - de Hoop method is to replace in (8), by means of the well-known Cauchy theorem and the Jordan lemma, the integral along the imaginary axis by an integral along Γ (more precisely, along the curve MNQ) with respect to the variable τ, of the following kind

$$W(q) = \int_{R\sqrt{(q^2+1)}}^{\infty} \frac{A(\tau; q)\, \mathrm{e}^{-p\tau}\, \mathrm{d}\tau}{B(\tau; q)},$$

in order to represent it as a Laplace transform of the function

$$\frac{A(\tau; q)\, H(\tau - R\sqrt{(q^2+1)})}{B(\tau; q)}.$$

The main difference between the "classical" treatment [3] and ours is that in the problem considered both in the supersonic ($\eta > 1$) and subsonic ($0 < \eta < 1$) cases the integrand possesses poles, the positions of which depend on q (i.e., on the variable of the external integration in (8)), so that according to (9)–(11) the integrand $\Phi(s)$ may have two, one or no pole inside the region G (Fig. 1).

Remark that it turns out that these are the poles which give us most of the information about the solution's front.

It is easy to see that

1) if {either $[(\eta > 1)$ and $(R/\eta\,|z| > 1)$ and $(0 < q < \sqrt{((R/\eta z)^2 - 1)})$ or $[(0 < \eta < 1)$ and $(\sqrt{(1/\eta^2 - 1)} < q < \sqrt{((R/\eta z)^2 - 1)})]\}$ then $(0 < s_p^+ < s_r)$, i.e., the integrand's pole $s_p^+ \in G$;
2) if {either $[(\eta > 1)$ and $(R/\eta\,|z| < 1)]$ or $[(\eta > 1)$ and $(R/\eta\,|z| > 1)$ and $(q > \sqrt{((R/\eta z)^2 - 1)})]$ or $[(0 < \eta < 1)$ and $q > \sqrt{((R/\eta z)^2 - 1)}]\}$ then $(s_p^+ > s_r)$, i.e., the integrand's pole $s_p^+ \notin G$.

Besides, from (10) we see that when $0 < \eta < 1$ and $0 < q < \sqrt{(1/\eta^2 - 1)}$ both of the integrand's poles are imaginary, so in this case the internal integral in (8) should be considered in the sense of the principal value.

Taking all this information into account and doing all the necessary procedures accompanying the usage of Cauchy theorem, we transform (8) to one of the following forms:

$$(13)\qquad -\frac{1}{2\pi^2\eta^2}\int_0^\infty W(q)\,\mathrm{d}q\,,\quad \eta > 1\,,\quad 0 < \frac{R}{\eta|z|} < 1\,;$$

$$(14)\qquad -\frac{1}{2\pi^2\eta^2}\int_0^{\sqrt{((R/\eta z)^2 - 1)}}\{\pi \operatorname*{Res}_{s=\sqrt{(q^2+1-1/\eta^2)}}\Phi(s) + W(q)\}\,\mathrm{d}q - $$
$$-\frac{1}{2\pi^2\eta^2}\int_{\sqrt{((R/\eta z)^2-1)}}^\infty W(q)\,\mathrm{d}q\,,\quad \eta > 1\,,\quad \frac{R}{\eta|z|} > 1\,;$$

$$(15)\qquad -\frac{1}{4\pi^2\eta^2}\int_0^{\sqrt{(1/\eta^2-1)}}\{\pi \operatorname*{Res}_{s=-\mathrm{i}\sqrt{(1/\eta^2-1-q^2)}}\Phi(s) + \pi \operatorname*{Res}_{s=\mathrm{i}\sqrt{(1/\eta^2-1-q^2)}}\Phi(s) + $$
$$+ W(q)\}\,\mathrm{d}q - \frac{1}{2\pi^2\eta^2}\int_{\sqrt{((R/\eta z)^2-1)}}^\infty W(q)\,\mathrm{d}q - $$
$$-\frac{1}{2\pi^2\eta^2}\int_{\sqrt{(1/\eta^2-1)}}^{\sqrt{((R/\eta z)^2-1)}}\{\pi \operatorname*{Res}_{s=\sqrt{(q^2+1-1/\eta^2)}}\Phi(s) + W(q)\}\,\mathrm{d}q\,,$$

where

$$W(q) = \int_{R\sqrt{(q^2+1)}}^\infty \frac{A(\tau, q)\,\mathrm{e}^{-p\tau}\,\mathrm{d}\tau}{B(\tau, q)}\,,$$

and

$$(16)\qquad A(\tau, q) = \frac{|z|\,\tau}{R^4}\{\eta^2 r^2[\tau^2 - (q^2 + 1)\,R^2] + \eta^2 z^2 \tau^2 - R^4\}\,,$$

$$B(\tau, q) = \frac{\sqrt{(\tau^2 - (q^2 + 1)\,R^2)}}{R^6}\{(R^2 - \eta|z|\,\tau)^2 + \eta^2 r^2[\tau^2 - (q^2 + 1)\,R^2]\}\,\times$$
$$\times\,\{(R^2 + \eta|z|\,\tau)^2 + \eta^2 r^2[\tau^2 - (q^2 + 1)\,R^2]\}\,.$$

Since

$$\operatorname*{Res}_{s=\sqrt{(q^2+1-1/\eta^2)}}\Phi(s) = \frac{\eta\mathrm{e}^{-p[|z|/\eta + r\sqrt{(1+q^2-1/\eta^2)}]}}{2\sqrt{(1 + q^2 - 1/\eta^2)}}\,,$$

then applying the substitution $\tau = |z|/\eta + r\sqrt{(1 + q^2 - 1/\eta^2)}$ to the corresponding integrals in (14), (15) we obtain for (14):

$$(17)\qquad -\frac{1}{2\pi^2\eta^2}\int_0^{\sqrt{((R/\eta z)^2-1)}} \pi \operatorname*{Res}_{s=\sqrt{(q^2+1-1/\eta^2)}} \Phi(s)\,dq =$$

$$= -\int_0^\infty \frac{H(\tau - |z|/\eta - r\sqrt{(1-1/\eta^2)})\,[1 - H(\tau - R^2/\eta|z|)]\,e^{-p\tau}\,d\tau}{2\pi\sqrt{((\eta\tau - |z|)^2 - r^2(\eta^2 - 1))}}$$

and similarly for (15):

$$(18)\qquad -\frac{1}{2\pi^2\eta^2}\int_{\sqrt{(1/\eta^2-1)}}^{\sqrt{((R/\eta z)^2-1)}} \pi \operatorname*{Res}_{s=\sqrt{(q^2+1-1/\eta^2)}} \Phi(s)\,dq =$$

$$= -\int_0^\infty \frac{[H(\tau - |z|/\eta) - H(\tau - R^2/\eta|z|)]\,e^{-p\tau}\,d\tau}{4\pi\sqrt{((\eta\tau - |z|)^2 + r^2(1 - \eta^2))}}$$

(the Heaviside functions in (17), (18) are used for chaning the original integration bounds to $(0, \infty)$).

Then, since

$$\operatorname*{Res}_{s=-i\sqrt{(1/\eta^2-1-q^2)}} \Phi(s) + \operatorname*{Res}_{s=i\sqrt{(1/\eta^2-1-q^2)}} \Phi(s) =$$

$$= -\frac{\eta\,e^{-p|z|/\eta}\sin(pr\sqrt{(1/\eta^2 - 1 - q^2)})}{\sqrt{(1/\eta^2 - 1 - q^2)}},$$

we obtain in the subsonic case (using [5]):

$$\int_0^{\sqrt{(1/\eta^2-1)}} \frac{\sin(pr\sqrt{(1/\eta^2 - 1 - q^2)})\,dq}{\sqrt{(1/\eta^2 - 1 - q^2)}} = \frac{\pi}{2}\,\mathbf{H}_0(pr\sqrt{(1/\eta^2 - 1)}),$$

where $\mathbf{H}_0$ is the Strouve function.

Before we deal with the iterated integrals in (13)–(15), let us notice the following:

i) to simplify all the subsequent procedures we may consider the Laplace transform parameter p real and positive (cf. [6]);

ii) the similar iterated integrals in (14), (15) cannot be united into one integral with respect to q, since (as is easy to see) in the cases

$$0 < q < \sqrt{((R/\eta z)^2 - 1)} \quad\text{and}\quad \sqrt{((R/\eta z)^2 - 1)} < q < \infty$$

the radicals contained in the integrand have opposite signs;

iii) the possibility of changing the order of integration in (14), (15) is almost obvious;

iv) the integral $\int [A(\tau, q)/B(\tau, q)]\,dq$ with the functions (16) may be found analytically.

So, taking into account i)–iii), calculating the integral in iv), applying [7] to the inverse Laplace transform of (6), (19), and noticing that

$$1 \pm \operatorname{sign} z = 2H(\pm z)$$

we obtain for $u(x, \tau) = u_1(x, \tau) + u_2(x, \tau) + u_3(x, \tau)$ after elementary but tedious calculations:

in the supersonic case $(\eta > 1)$

$$(20) \quad \hat{u}(x,\tau) = \begin{cases} \dfrac{H(z)\, H(\tau - z/\eta - r\sqrt{(1 - 1/\eta^2)})}{2\pi \sqrt{((\eta\tau - z)^2 - r^2(\eta^2 - 1))}} - \dfrac{H(\tau - R)\,[H(z) - H(-z)]}{4\pi \sqrt{((\eta\tau - z)^2 - r^2(\eta^2 - 1))}}, & 0 < R < \eta|z|, \\ \dfrac{H(\tau - R)}{4\pi \sqrt{((\eta\tau - z)^2 - r^2(\eta^2 - 1))}}, \quad R > \eta|z|; \end{cases}$$

2) in the subsonic case $(0 < \eta < 1)$:

$$(21) \qquad \check{u}(x, \tau) = \frac{H(\tau - R)}{4\pi \sqrt{((\eta\tau - z)^2 + r^2(1 - \eta^2))}}.$$

It is obvious that

$$\operatorname{supp} \check{u}(x, \tau) = \{(x, \tau) \in R^3 \times R^1 : 0 < R < \tau\},$$
$$\operatorname{supp} \hat{u}(x, \tau) = \{(x, \tau) \in R^3 \times R^1 : \tau > R > \eta|z|\} \cup$$
$$\cup \{(x, \tau) \in R^3 \times R^1 : z \geqq 0,\ \tau > z/\eta + r\sqrt{(1 - 1/\eta^2)},\ 0 < R < \eta z\} \cup$$
$$\cup \{(x, \tau) \in R^3 \times R^1 : z < 0,\ 0 < R < \eta|z|\}.$$

Remark that the solution (20) is discontinuous at the points $\{(x, \tau) \in R^3 \times R^1 : \tau = R,\ z > 0,\ 0 < R < \eta z\}$.

Using the well-known technique of differentiation of homogeneous distributions and distributions supported by a cone [8] one can check that the functions (20) and (21) satisfy (1) and (2).

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