

# Applications of Mathematics

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*Applications of Mathematics*, Vol. 36 (1991), No. 4, 277--283

Persistent URL: <http://dml.cz/dmlcz/104466>

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## NON-NEGATIVE LINEAR PROCESSES

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(Received December 21, 1989)

*Summary.* Conditions under which the linear process is non-negative are investigated in the paper. In the definition of the linear process a strict white noise is used. Explicit results are presented also for the models AR(1) and AR(2).

*Keywords:* autoregressive model, linear process, non-negative process, strict white noise.

*AMS subject classification:* 62M10.

## 1. INTRODUCTION

In this paper we assume that  $\{e_t\}$  is a strict white noise with a finite second moment, i.e. a series of i.i.d. random variables such that  $Ee_t^2 < \infty$ . Let  $F$  be the distribution function of  $e_t$ .

The process

$$(1.1) \quad X_t = \sum_{k=0}^{\infty} c_k e_{t-k}$$

such that  $c_0 = 1$  and  $\sum |c_k| < \infty$ , is called linear. The condition  $\sum |c_k| < \infty$  ensures that (1.1) converges in the quadratic mean even if  $Ee_t \neq 0$ . We shall investigate only linear processes with real coefficients  $c_k$ .

The process  $X_t$  is called non-negative, if  $X_t \geq 0$  with probability one for all  $t$ . Such processes occur frequently in practice (e.g. annual discharge of a river, precipitation, air and water pollution etc.).

If  $c_k \geq 0$  and  $e_t \geq 0$  for all  $k$  and  $t$ , then  $X_t \geq 0$  for all  $t$ . We prove that the condition  $c_k \geq 0$  is also necessary to ensure non-negativity of  $X_t$  (when  $e_t \geq 0$ ) while if  $c_k \geq 0$  the condition  $e_t \geq 0$  is not necessary for  $X_t \geq 0$  (see Theorems 3.1, 3.2). Explicit results are presented for the models AR(1) and AR(2).

## 2. PRELIMINARIES

**Lemma 2.1.** *Let  $A_1, A_2, \dots$  and  $B_1, B_2, \dots$  be two sequences of random events. Assume that the following conditions are fulfilled:*

- a) Events  $A_1, A_2, \dots$  are independent.  
 b) Events  $A_n, B_n$  are independent for each  $n$ .  
 c)  $\lim_{i \rightarrow \infty} P(B_i) = 1$ .  
 d)  $\sum_{i=1}^{\infty} P(A_i) = \infty$ .

Then infinitely many events  $A_i \cap B_i$  occur with probability one.

Proof. See [3].

**Lemma 2.2.** Define

$$Z_m = \sum_{k=m}^{\infty} c_k e_{t-k}.$$

Then for arbitrary  $c > 0$  we have

$$P(|Z_m| \geq c) \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Proof. We assume that  $\sum |c_k| < \infty$ . If  $Ee_t = \mu$  and  $\text{var } e_t = \sigma^2$ , then

$$EZ_m = \mu \sum_{k=m}^{\infty} c_k \rightarrow 0, \quad \text{var } Z_m = \sigma^2 \sum_{k=m}^{\infty} c_k^2 \rightarrow 0.$$

Thus

$$EZ_m^2 = \text{var } Z_m + (EZ_m)^2 \rightarrow 0$$

and

$$P(|Z_m| \geq c) = \int_{|Z_m| \geq c} dP \leq c^{-2} \int Z_m^2 dP = c^{-2} EZ_m^2 \rightarrow 0.$$

### 3. NON-NEGATIVE LINEAR PROCESSES

**Theorem 3.1.** Let  $c_k \geq 0$  for all  $k$  and  $\sum c_k < \infty$ . If there exist  $c > 0$  and  $q \in (0, 1]$  such that

$$P(e_t < -c) = q,$$

then with probability 1 there exist infinitely many indices  $t$  such that  $X_t < 0$ .

Proof. Let  $j_m$  be the smallest integer such that  $j_m q^m \geq 1$  ( $m = 1, 2, \dots$ ). Define sets  $S_1, S_2, \dots$  of positive integers as follows. Let

$$S_1 = \{1, \dots, j_1\}.$$

Let  $S_2$  contain elements of  $j_2$  couples

$$(j_1 + 1, j_1 + 2), \dots, (j_1 + 2j_2 - 1, j_1 + 2j_2),$$

let  $S_3$  contain elements of  $j_3$  triples starting with

$$(j_1 + 2j_2 + 1, j_1 + 2j_2 + 2, j_1 + 2j_2 + 3),$$

and so on. We denote by  $n_1, n_2, \dots$  successively the numbers  $1, \dots, j_1$ , then the last members of the couples, triples, etc. If  $n_i \in S_m$ , we put

$$X_{n_i} = U_{n_i} + Z_{n_i}$$

where

$$U_{n_i} = \sum_{k=0}^{m-1} c_k e_{n_i-k}, \quad Z_{n_i} = \sum_{k=m}^{\infty} c_k e_{n_i-k}.$$

Introduce events

$$A_i = \{U_{n_i} < -c\}, \quad B_i = \{Z_{n_i} < c\}, \quad i = 1, 2, \dots.$$

We have chosen indices in such a way that  $A_1, A_2, \dots$  are independent. It is clear that  $A_i$  and  $B_i$  are also independent. For  $n_i \in S_m$  we have

$$P(A_i) = P\left(\sum_{k=0}^{m-1} c_k e_{n_i-k} < -c\right) \geq P(e_{n_i-k} < -c, k = 0, \dots, m-1) = q^m.$$

Thus

$$\sum_{i=1}^{\infty} P(A_i) = \sum_{m=1}^{\infty} j_m q^m = \infty.$$

Lemma 2.2 yields

$$P(B_i) \geq P(|Z_{n_i}| < c) \rightarrow 1.$$

Now, Theorem 3.1 follows from Lemma 2.1.

We have proved that if  $c_k \geq 0$ , then  $e_t \geq 0$  is a necessary condition for  $X_t \geq 0$ . Our proof follows the ideas of the proof of Lemma 10.2 in [2] where the non-negativity of the AR(1) model is considered.

**Theorem 3.2.** *Let  $e_t \geq 0$ ,  $\text{var } e_t > 0$  and  $\sum |c_k| < \infty$ . Assume that  $F(d) - F(c) < 1$  for all  $0 < c < d < \infty$ . If there exists an index  $k_0$  such that  $c_{k_0} < 0$ , then with probability 1 we have  $X_t < 0$  for infinitely many indices  $t$ .*

**Proof.** Denote  $M = \max |c_i|$ ,  $c = |c_{k_0}|$ . Since  $F(d) - F(c) < 1$  for all  $0 < c < d < \infty$ , at least one of the following cases must occur:

- a) The variables  $e_t \geq 0$  can be arbitrarily small, i.e. for every  $\varepsilon > 0$  there exists  $\gamma > 0$  such that  $P(e_t < \varepsilon) > \gamma$ .
- b) The variable  $e_t$  can be arbitrarily large, i.e. for every  $\varepsilon > 0$  there exists  $\gamma > 0$  such that  $P(e_t > \varepsilon) > \gamma$ .

First, consider the case a). Since  $e_t \geq 0$ ,  $\text{var } e_t > 0$ , there exists  $\gamma > 0$  such that  $P(e_t > 2\gamma) = \delta > 0$ . Further,

$$\beta_n = P\left(e_t < \frac{c\gamma}{nM}\right) > 0, \quad n = 1, 2, \dots$$

Thus for  $n > k_0$  we have

$$\begin{aligned} & \mathbb{P}\left(\sum_{k=0}^{n-1} c_k e_{t-k} < -c\gamma\right) \geq \\ & \geq \mathbb{P}\left(e_{t-k_0} > 2\gamma, c_k e_{t-k} < \frac{c\gamma}{n} \text{ for } k = 0, \dots, n-1; k \neq k_0\right) \geq \\ & \geq \mathbb{P}\left(e_{t-k_0} > 2\gamma, e_{t-k} < \frac{c\gamma}{nM} \text{ for } k = 0, \dots, n-1; k \neq k_0\right) = \\ & = \beta_n^{n-1} \delta > 0. \end{aligned}$$

Let  $j_m$  be the smallest integer such that

$$j_m \beta_m^{j_m-1} \delta \geq 1, \quad m > k_0.$$

The remaining part of the proof is now analogous to the proof of Theorem 3.1.

Consider the case b). It is clear that there exists  $\gamma > 0$  such that  $\mathbb{P}(e_t < \gamma/M) = \delta > 0$ . Further,

$$\beta_n = \mathbb{P}(e_t > \gamma n/c) > 0 \quad \text{for } n > k_0.$$

If  $n > k_0$  then

$$\begin{aligned} & \mathbb{P}\left(\sum_{k=0}^{n-1} c_k e_{t-k} < -\gamma\right) \geq \\ & \geq \mathbb{P}\left(e_{t-k_0} > \frac{\gamma n}{c}, c_k e_{t-k} < \gamma \text{ for } k = 0, \dots, n-1; k \neq k_0\right) \geq \\ & \geq \mathbb{P}\left(e_{t-k_0} > \frac{\gamma n}{c}, e_{t-k} < \frac{\gamma}{M} \text{ for } k = 0, \dots, n-1; k \neq k_0\right) = \\ & = \beta_n \delta^{n-1} > 0. \end{aligned}$$

Let  $j_m$  be the smallest integer such that

$$j_m \beta_m \delta^{j_m-1} \geq 1, \quad m > k_0.$$

Again, the proof can be completed in the same way as that of Theorem 3.1.

**Remark 3.3.** The assumption  $F(d) - F(c) < 1$  for all  $0 < c < d < \infty$  in Theorem 3.2 cannot be omitted. Define  $c_0 = 1, c_1 = -1, c_2 = 1, c_k = 0$  for  $k \geq 3$ . Let  $e_t$  have the rectangular distribution on the interval  $(2, 3)$ . Then

$$X_t = e_t - e_{t-1} + e_{t-2} > 1$$

and so  $X_t$  is non-negative, although  $e_t \geq 0$  and  $c_1 < 0$ .

#### 4. AUTOREGRESSIVE MODELS

The AR(1) process  $X_t$  is a linear process satisfying

$$X_t = bX_{t-1} + e_t.$$

This process exists if and only if  $b \in (-1, 1)$ . Since

$$X_t = \sum_{n=0}^{\infty} b^n e_{t-n},$$

the conditions of non-negativity of  $X_t$  follow from theorems introduced in Section 3.

**Theorem 4.1.** *Let  $b \in [0, 1)$ . If there exist  $c > 0$  and  $q \in (0, 1]$  such that  $P(e_t < -c) = q$ , then with probability one  $X_t < 0$  for infinitely many indices  $t$ .*

Proof follows from Theorem 3.1.

**Theorem 4.2.** *Let  $F(d) - F(c) < 1$  for all  $0 < c < d < \infty$ . Let  $e_t \geq 0$ ,  $\text{var } e_t > 0$ . If  $b \in (-1, 0)$ , then with probability one  $X_t < 0$  for infinitely many indices  $t$ .*

Proof follows from Theorem 3.2.

Now, consider an AR(2) process

$$X_t = b_1 X_{t-1} + b_2 X_{t-2} + e_t.$$

Let  $z_1, z_2$  be the roots of  $z^2 - b_1 z - b_2 = 0$ . It is known that  $X_t$  exists if and only if  $|z_1| < 1$ ,  $|z_2| < 1$ . This condition is satisfied if and only if  $(b_1, b_2)$  belong to the triangle  $\Delta$  with vertices  $(-2, -1)$ ,  $(2, -1)$ ,  $(0, 1)$ . See Fig. 1.

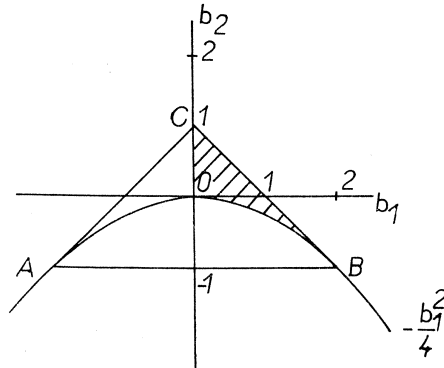


Fig. 1

**Theorem 4.3.** *Let  $F(d) - F(c) < 1$  for all  $0 < c < d < \infty$ . Let  $e_t \geq 0$ . If  $X_t$  is an AR(2) process, then  $X_t \geq 0$  for all  $t$  if and only if  $(b_1, b_2) \in \Delta$ ,  $b_2 \geq -b_1^2/4$ ,  $b_1 \geq 0$ .*

Proof. The process AR(2) is a linear process with coefficients  $c_k$  which coincide with the coefficients in the expansion of the function

$$c(z) = (1 - b_1 z - b_2 z^2)^{-1}$$

in the neighbourhood of zero (see [1]). Let  $\alpha_1, \alpha_2$  be the roots of  $1/c(z)$ . Then

$$1 - b_1 z - b_2 z^2 = -b_2(z - \alpha_1)(z - \alpha_2).$$

Assume that  $\alpha_1 \neq \alpha_2$ . Then

$$c(z) = -b_2^{-1}[A_1(z - \alpha_1)^{-1} + A_2(z - \alpha_2)^{-1}],$$

where

$$A_1 = (\alpha_1 - \alpha_2)^{-1}, \quad A_2 = -(\alpha_1 - \alpha_2)^{-1}.$$

Thus

$$c(z) = b_2^{-1}(\alpha_1 - \alpha_2)^{-1} \sum_{k=0}^{\infty} (\alpha_1^{-k-1} - \alpha_2^{-k-1}) z^k.$$

This implies

$$c_k = b_2^{-1}(\alpha_1 - \alpha_2)^{-1} (\alpha_1^{-k-1} - \alpha_2^{-k-1}).$$

Since

$$\alpha_{1,2} = [-b_1 \pm (b_1^2 + 4b_2)^{1/2}]/(2b_2),$$

we get

$$\alpha_1 \alpha_2 = -1/b_2, \quad \alpha_1 - \alpha_2 = (b_1^2 + 4b_2)^{1/2}/b_2.$$

Then

$$\begin{aligned} c_k &= \left(-\frac{1}{2}\right)^{k+1} \{[-b_1 - (b_1^2 + 4b_2)^{1/2}]^{k+1} - \\ &\quad - [-b_1 + (b_1^2 + 4b_2)^{1/2}]^{k+1}\} (b_1^2 + 4b_2)^{-1/2} = \\ &= 2^{-k} b_1^k \sum_{j=0}^{[k/2]} \binom{k+1}{2j+1} \left(1 + \frac{4b_2}{b_1^2}\right)^j. \end{aligned}$$

If  $b_1 > 0$ ,  $b_2 > -b_1^2/4$ , then  $c_k \geq 0$  for all  $k$ . Since  $c_1 = b_1$ , we have  $c_1 < 0$  if  $b_1 < 0$ . In the case  $b_2 = -b_1^2/4$  we can derive

$$c(z) = \sum_{k=0}^{\infty} 2^{-k}(k+1) b_1^k z^k, \quad c_k = 2^{-k}(k+1) b_1^k.$$

If  $b_2 = -b_1^2/4$  and  $b_1 > 0$ , then  $c_k \geq 0$  for all  $k$ . The case  $b_2 = 0$  is trivial.

If  $b_2 < -b_1^2/4$ , then

$$\alpha_{1,2} = r(\cos \varphi \pm i \sin \varphi),$$

where  $r > 0$ ,  $\varphi \in (0, \pi)$ . After a computation we obtain

$$c_k = r^{-k} \frac{\sin(k+1)\varphi}{\sin \varphi}.$$

Thus  $c_k < 0$  for infinitely many indices  $k$ .

The set of the vectors  $(b_1, b_2)$  which correspond to a non-negative AR(2) process  $X_t$  is depicted in Fig. 1 as OBC.

### References

- [1] *J. Anděl*: Statistical Analysis of Time Series (Czech). SNTL Praha 1976.
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### Souhrn

## NEZÁPORNÉ LINEÁRNÍ PROCESY

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V článku se vyšetřují podmínky, za nichž je lineární proces nezáporný. V definici lineárního procesu se přitom užívá striktní bílý šum. Explicitních výsledků je dosaženo rovněž pro modely AR(1) a AR(2).

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