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Bifurcation of heteroclinic orbits for diffeomorphisms


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BIFURCATION OF HETEROCLINIC ORBITS
FOR DIFFEOMORPHISMS

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Summary. The paper deals with the bifurcation phenomena of heteroclinic orbits for diffeomorphisms. The existence of a Melnikov-like function for the two-dimensional case is shown. Simple possibilities of bifurcation of the set of heteroclinic points are described for higher-dimensional cases.

Keywords: bifurcation phenomena, heteroclinic points, discrete dynamical systems.

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1. INTRODUCTION

In this paper we investigate bifurcation of heteroclinic orbits for diffeomorphisms. The results are obtained by the Lyapunov-Schmidt method. This method was used for the study of an analogous problem for ordinary differential equations in [4, 8].

2. TWO-DIMENSIONAL CASE

Let us consider a $C^\infty$-smooth mapping $\Phi: \mathbb{R}^2 \to \mathbb{R}^2$ with the following properties on the set $M = (-1/2, 3/2) \times (-\infty, \infty)$

i) $\Phi$ has the form $\Phi(x, y) = (f(x), g(x, y))$, where $g(x, 0) = 0$ for each $x \in (-1/2, 3/2)$,

ii) the mapping $f: \mathbb{R} \to \mathbb{R}$ has fixed points 0, 1 such that $f'(0) > 1$, $f'(1) < 1$, $f'(\cdot) > 0$ and $g_\nu(\cdot, 0) \neq 0$. Further we assume the existence of a sequence 

\[ \{x_n\}_{n=-\infty}^{+\infty} \subset (0, 1), \quad x_{n+1} = f(x_n), \quad x_n \to 1(0) \text{ as } n \to \infty(-\infty). \]

Thus $\Phi$ has the heteroclinic orbit $\Gamma = \{(x_n, 0)\}_{n=-\infty}^{+\infty}$ from (0, 0) to (1, 0). We note that $\Phi$ also has the family of heteroclinic orbits $\mathcal{M} = \{f^n(x, 0)\}_{n=-\infty}^{+\infty}, \quad x \in (0, 1)$ and this family contains $\Gamma$. We perturb this mapping and try to find heteroclinic orbits near $\Gamma$ for the perturbed mapping.
Let us consider the variational equation of \( \Phi \) around \( F \):
\[
\begin{align*}
    u_{n+1} &= f'(x_n) \cdot u_n + a_n, \\
    v_{n+1} &= g_y(x_n, 0) \cdot v_n + b_n.
\end{align*}
\]

For the mapping \( g \) we have the following four cases:

A. \( |g_y(0, 0)| > 1, \quad |g_y(1, 0)| < 1 \).

**Lemma 2.1.** Let \( X = \left\{ (a_n, b_n) \right\}_{n=0}^{\infty}, a_n, b_n \in \mathbb{R}, \right\| (a_n, b_n) \right\| = \sup \left\{ |a_n|, |b_n| \right\} < \infty \}
and consider the linear operator \( L : X \rightarrow X, \)
\[
L\left( (u_n, v_n) \right) = \left\{ u_{n+1} - f'(x_n) \cdot u_n, \quad v_{n+1} - g_y(x_n, 0) \cdot v_n \right\}.
\]

Then \( \dim \ker L = 2, \quad \text{codim} \text{Im} L = 0 \).

**Proof.** From the equation
\[
\begin{align*}
    u_{n+1} &= f'(x_n) \cdot u_n, \quad v_{n+1} = g_y(x_n, 0) \cdot v_n
\end{align*}
\]
using \( \lim_{n \to \pm \infty} |f'(x_n)| \leq 1 \) and \( \lim_{n \to \pm \infty} |g_y(x_n, 0)| \leq 1 \) we have
\[
\ker L = \mathbb{R}\left( (\Pi_{i=1}^{n} f'(x_i), 0) \right)_{-\infty}^{+\infty} \oplus \mathbb{R}\left( (0, \Pi_{i=0}^{n} g_y(x_i, 0))_{-\infty}^{+\infty} \right),
\]
where
\[
\prod_{1}^{n} a_n = \begin{cases} a_0 \cdots a_{n-1}, & n \geq 1 \\ 1, & n = 0 \\ 1/a_{-1} \cdots 1/a_n, & n < 0. \end{cases}
\]

For \( (a_n, b_n)_{-\infty}^{+\infty} \in X \) we solve the equation
\[
\begin{align*}
    u_{n+1} &= f'(x_n) u_n + a_n, \\
    v_{n+1} &= g_y(x_n, 0) v_n + b_n.
\end{align*}
\]

The first (and similarly the second) equation of (2.2) has the general solution
\[
\begin{align*}
    u_n &= f'(x_{n-1}) \cdots f'(x_0) \left( \Sigma_{i=0}^{n-1} \frac{a_i}{f'(x_i) \cdots f'(x_0)} \right) + K, \quad n \geq 1 \\
    u_0 &= K, \quad u_{-1} = (-a_{-1} + K)f'(x_{-1}), \\
    u_n &= \frac{1}{f'(x_n) \cdots f'(x_{-1})} \left( \Sigma_{i=0}^{n-2} a_i f'(x_{i+1}) \cdots f'(x_{-1}) - a_{-1} + K \right), \quad n \leq -2.
\end{align*}
\]

Since \( \lim_{n \to \infty} |f'(x_n)| < 1 \) we have \( \sup_{n \geq 1} |u_n| < \infty \). The proof of the other cases is similar.

B. \( |g_y(0, 0)| > 1, \quad |g_y(1, 0)| > 1 \).
Lemma 2.2. In this case \( \dim \ker L = 1 \), \( \text{codim } \text{Im } L = 0 \).

Proof. The case \( \dim \ker L = 1 \) is clear. In this case the first equation of (2.2) has a bounded solution for each \( K \). The second has a bounded solution iff the corresponding \( K \) is

\[
K = - \sum_{0}^{+\infty} \frac{b_i}{g_y(x_i, 0) \ldots g_y(x_0, 0)}.
\]

This series is convergent and thus \( \text{codim } \text{Im } L = 0 \).

C. \( |g_y(0, 0)| < 1 \), \( |g_y(1, 0)| > 1 \).

In this case we obtain the same result as in the case B.

D. \( |g_y(0, 0)| < 1 \), \( |g_y(1, 0)| > 1 \).

Lemma 2.3. In this case \( \dim \ker L = 1 \), \( \text{codim } \text{Im } L = 1 \).

Proof. We prove the second part of the lemma. The second equation of (2.2) has a bounded solution for \( n \to \infty \) iff the corresponding \( K \) is

\[
K = - \sum_{-\infty}^{+\infty} \frac{b_i}{g_y(x_i, 0) \ldots g_y(x_0, 0)}.
\]

and for \( n \to -\infty \) iff

\[
K = \sum_{-\infty}^{+\infty} b_i g_y(x_i, 0) \ldots g_y(x_0, 0) + b_{-1}.
\]

Hence

\[
d_{-1} = \sum_{-\infty}^{+\infty} b_i g_y(x_i, 0) \ldots g_y(x_0, 0) + b_{-1} + \sum_{0}^{+\infty} \frac{b_i}{g_y(x_i, 0) \ldots g_y(x_0, 0)} = 0.
\]

We see that (2.2) has a bounded solution if and only if \( d_{-1} = 0 \) and this relation implies \( \text{codim } \text{Im } L = 1 \).

We define the projection \( P: X \to X \), \( P((a_n, b_n)) = (0, d_n) \mapsto -\infty \), where \( d_n = 0 \) for \( n \neq -1 \) and \( d_{-1} \) is defined in the above proof. We see that \( ((a_n, b_n))_{-\infty}^{+\infty} \in \text{Im } L \) if and only if \( P((a_n, b_n)) = 0 \). Thus we define the operator \( K: (I - P) X \to X \),

\[
K((a_n, b_n))_{-\infty}^{+\infty} = ((u_n, v_n))_{-\infty}^{+\infty}, \quad u_0 = 0,
\]

where \( ((u_n, v_n))_{-\infty}^{+\infty} \) is unique bounded solution of (2.2).

The mapping \( \Phi \) has hyperbolic fixed points \( (0, 0) \) and \( (0, 0) \). Hence a perturbed mapping \( \Phi^e: R^2 \to R^2 \) has fixed points \( p_e, q_e \) near them, which are hyperbolic as well. Consider the equation

\[
z_{n+1} = \Phi^e(z_n) = 0
\]

357
on the space $X$. (We assume $\Phi(\cdot) \in C^\infty$.) This equation can be written in the form

\begin{align}
(2.4) \quad u_{n+1} + x_{n+1} &= f(x_n + u_n) + O(e), \\
v_{n+1} &= g(x_n + u_n, v_n) + O(e).
\end{align}

We seek for a bounded solution of (2.4) with $|u_n| + |v_n| + |e| \ll 1$, i.e. we solve the equation (2.4) in $X$ near $0 \in X$ for $e$ small. It is clear that the linearization of (2.4) at $0 \in X$ for $e = 0$ is precisely the operator $L$. According to Lemma 2.3 we have for the case $D$

$$\dim \ker L = 1 \quad \text{and} \quad \text{codim} \im L = 1.$$ 

Hence applying the Lyapunov-Schmidt method [1, 10] we derive a bifurcation equation of the equation (2.4),

\begin{align}
(2.5) \quad Q(c, e) &= 0, \quad Q: U \times U \to R,
\end{align}

where $U$ is a neighbourhood of $0 \in R$. Since for $e = 0$ the equation (2.4) has the solution $u_n = f^n(x) - x_n, v_n = 0$ for each $x \in (0, 1)$, we obtain that $Q(c, 0) = 0$. We note that each small solution of (2.4) yields a heteroclinic orbit of $\Phi$ near $\Gamma$.

**Theorem 2.4.** For the case $D$ we obtain the above bifurcation equation (2.5).

Now we investigate the remaining cases. For these cases we have also the equation (2.4), but according to Lemmas 2.1, 2.2 the linearization of (2.4), which is the operator $L$, satisfies $\text{codim} \im L = 0$, i.e. $L$ is surjective and applying the implicit function theorem we have for $e$ small

**Theorem 2.5.** In the case $A$ there is a three-parametric family of heteroclinic orbits near $\Gamma$, where one parameter is $e$ and the other corresponds to the parameter $x$ from the above mentioned family $\mathcal{M}$ of heteroclinic orbits of $\Phi$.

**Theorem 2.6.** In the cases $B, C$ we have a two-parametric family of heteroclinic orbits near $\Gamma$, where one parameter is $e$ and the other corresponds to the parameter $x$ from the above mentioned family $\mathcal{M}$.

### 3. General Case

**Definition 3.1 (see [6]).** Let $X$ be a Banach space and $\{T_n\}_n \in \mathcal{L}(X)$. We say that $\{T_n\}_n$ has a discrete dichotomy on $I = (Z, Z_+ = N \cup \{0\}, Z_- = -Z_+)$ if there exist positive numbers $M, \theta < 1$ and a sequence of projections $\{P_n\}_n$ such that

i) $T_n P_n = P_{n+1} T_n$,

ii) $T_n / \im P_n$ is an isomorphism from $\im P_n$ into $\im P_{n+1}$.

iii) if $T_{n,m} = T_{n-1} \cdots T_{m+1} T_m$ for $n > m$, $T_{n,n} = \text{Identity}$.
then
\[|T_{n,m}(I - P_m)x| \leq M\theta^{n-m}|x| \quad \text{for} \quad n \geq m,\]
\[|T_{n,m}P_mx| \leq M\theta^{m-n}|x| \quad \text{for} \quad n < m,\]
where \(T_{n,m}P_mx = y\) iff \(P_mx = T_{m,n}y\) for the case \(m > n\).

**Remark 3.2.** If \(T_n\) is a sequence of isomorphisms then the above definition is equivalent to the property that there is a projection \(P \in \mathcal{L}(X)\) such that
\[|T(m)P T^{-1}(s) - M_{m-s}| \leq M\theta^s, \quad m \geq s,\]
\[|T(m)(I - P)T^{-1}(s)| \leq M\theta^{m-s}, \quad s \geq m,\]
where \(T(n) = T_{n-1} \ldots T_0\) for \(n \geq 1\), \(T(n) = T^{-1}_n \ldots T^{-1}_1\) for \(n < 0\), \(T(0) = I\).

**Theorem 3.3.** Let \(\{A_n\}_{n \in \mathbb{Z}}\) be a sequence of invertible matrices \(A_n \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)\) with bounded \(|A_n|, |A_n^{-1}|\) on \(\mathbb{Z}\). We assume that \(\{A_n\}\) has a discrete dichotomy both on \(\mathbb{Z}^+\) and \(\mathbb{Z}^-\). Define the operator
\[L: X \rightarrow X = \{(\{a_n\}_{n=-\infty}^{\infty}, \sup |a_n| < \infty, a_n \in \mathbb{R}^m)\},\]
\[L(\{a_n\}) = a_{n+1} - A_na_n.\]

Then \(L\) is a Fredholm operator and \(\{f_n\} \in \text{Im} L\) iff \(\sum_{k=\infty}^{+\infty} c_k f_n = 0\) for each bounded solution \(\{c_n\}\) of the equation
\[(3.1) \quad c_n = (A_n^*)^{-1} c_{n-1} \quad (* \text{means the transpose}).\]

**Proof.** We consider the equation
\[(3.2) \quad x_{n+1} = A_nx_n.\]

By assumption this equation has a discrete dichotomy on \(\mathbb{Z}^+(-)\) with projection \(P, Q\). (3.2) has the fundamental solution on \(\mathbb{Z}^+\)

\[T(n) = A_{n-1} \ldots A_0, \quad n \geq 1, \quad T(0) = I.\]

The equation (3.1) on the set \(I_1 = \{-1, 0, 1, \ldots\}\) has the fundamental solution

\[S(n) = (A_n^*)^{-1} \ldots I = (T(n+1)^*)^{-1}.\]

We see that (3.1) has a discrete dichotomy on \(I_1\) with the projection \(I - P^*\). Indeed, by Remark 3.2 and using the fact \(|A| = |A^*|\) we have
\[|(T(s + 1)^*)^{-1} P^* T(m + 1)^*| \leq M\theta^{m-s} \quad m \geq s,\]
\[|(T(s + 1)^*)^{-1} (I - P^*) T(m + 1)^*| \leq M\theta^{s-m} \quad s \geq m,\]
i.e.
\[|S(s) P^* S^{-1}(m)| \leq M\theta^{m-s} \quad m \geq s,\]
\[|S(s) (I - P^*) S^{-1}(m)| \leq M\theta^{s-m} \quad s \geq m.\]
Similarly, on the set $I_2 = \{\ldots, -2, -1\}$ the equation (3.1) has a discrete dichotomy with the projection $I - Q^*$. It is clear that $\text{Ker } L \cong V \cap W$, where $V = \text{Im } P$ and $W = \text{Ker } Q$. Hence $\dim \text{Ker } L = \dim V \cap W$. For (3.1) we have $\dim \text{Ker } L^* = \dim V^\perp \cap W^\perp$, where $V^\perp$ is the orthogonal complement of $V$ and $L^*: X \to X$ has the form

$$(L^*\{c_n\}_{-\infty}^{+\infty})_n = c_n - (A_n^*)^{-1}c_{n-1}.$$ 

Using the fact $\dim \text{Ker } L = \dim V^\perp \cap W^\perp$ we see that $\{c_n\}$ is a bounded solution of (3.1) iff $c_0 \in V^\perp \cap W^\perp$, and since $\{A_n^*\}$ has a discrete dichotomy on $I_1$ and $I_2$ we obtain that for each such solution $\{c_n\}$, $c_n$ tends geometrically to zero as $n \to \pm \infty$. Hence $\sum_n f_n$ is convergent for $\{f_n\}_{-\infty}^{+\infty}$ bounded.

For $\{f_n\} \in \text{Im } L$ and a bounded solution $\{c_n\}$ of (3.1) we have

$$a_{n+1} = A_n a_n + f_n.$$ 

Thus

$$\sum_{-\infty}^{+\infty} c_n a_{n+1} = \sum_{-\infty}^{+\infty} (A_n^* a_n + c_n f_n).$$ 

which implies

$$\sum_{-\infty}^{+\infty} a_n^* c_{n-1} = \sum_{-\infty}^{+\infty} a_n^* A_n c_n + \sum_{-\infty}^{+\infty} c_n f_n$$

and

$$0 = \sum_{-\infty}^{+\infty} a_n^* (c_{n-1} - A_n c_n) = \sum_{-\infty}^{+\infty} c_n f_n.$$ 

Conversely, if $\sum_{-\infty}^{+\infty} c_n f_n = 0$ for each bounded solution $\{c_n\}$ of (3.1) then we see that for each $d \in \mathbb{R}^m$ satisfying $d^*(P - (I - Q)) = 0$ and putting $T_j = T(j)$ for $j \geq 0$, $T_j = T(j) = A_j^{-1} \ldots A_{-1}$ for $j < 0$, the sequence

$$(3.3) \quad c_n = (T_{n+1}^k)^{-1} (I - P^*) d, \quad n \geq -1$$

$$c_n = (T_{n+1}^k)^{-1} Q^* d, \quad n \leq -1$$

is the solution of (3.1) and hence

$$d^*(\sum_{-\infty}^{-1} Q(T_{n+1})^{-1} f_n + \sum_{0}^{+\infty} (I - P) (T_{n+1})^{-1} f_n) = 0.$$ 

Thus the following matrix equation has a solution $g$:

$$(P - (I - Q)) g = \sum_{-\infty}^{-1} Q(T_{n+1})^{-1} f_n + \sum_{0}^{+\infty} (I - P) T_{n+1}^{-1} f_n.$$ 

Let us define the sequence $\{x_n\}$ by

$$x_n = T_n P g + \sum_{n-1}^{n} T_n P T_{s+1}^{-1} f_s - \sum_{n}^{+\infty} T_n (I - P) T_{s+1}^{-1} f_s, \quad n \geq 0,$$

$$x_n = T_n (I - Q) g + \sum_{n-1}^{n} T_n Q T_{s+1}^{-1} f_s - \sum_{n}^{-1} T_n (I - Q) T_{s+1}^{-1}, \quad n \leq 0,$$

where we consider $\sum_{p}^{q} \ldots = 0$ for $p > q$. The sequence $\{x_n\}$ is well-defined since $g$ satisfies the above matrix equation. It is not difficult to see that $\{x_n\}$ is a solution of $Lx = f$. Now we proceed in the same way as in [4] and hence we obtain that $\text{codim } \text{Im } L = \dim V^\perp \cap W^\perp$ and index $L = \dim V + \dim W - m$. This completes the proof.
Lemma 3.4. Let \( \{A_n\}_{n \geq 0} \) have a discrete dichotomy on \( \mathbb{Z}_+ \), \( A_n \) being invertible, bounded on \( \mathbb{Z}_+ \), \( A_n \in \mathcal{L}(\mathbb{R}^m) \). Let \( |B_n| \to 0, B_n \in \mathcal{L}(\mathbb{R}^m) \), as \( n \to +\infty \). Further we assume that \( \{A_n + B_n\}_{n \geq 0} \) are invertible. Then \( \{A_n + B_n\}_{n \geq 0} \) has a discrete dichotomy on \( \mathbb{Z}_+ \) and, moreover, if \( P, P' \) are projections of dichotomies for \( \{A_n\}, \{A_n + B_n\} \) (see Remark 3.2), then \( \dim \text{Im } P = \dim \text{Im } P' \).

Proof. For \( e > 0 \) sufficiently small there is \( j \in \mathbb{N} \) such that for each \( n \geq j \) we have \( |B_n| < e \). Hence by [6] the sequence \( \{A_n \times B_n\}_{n \geq j} \) has a discrete dichotomy with projections \( \{P_n\}_{n \geq j} \). If \( \{P_n\}_{n \geq j} \) are projections for \( \{A_n\}_{n \geq 1} \) then by [6] we also have

\[
|P_n - P'_n| < eM_1, \quad n \geq j.
\]

Since \( A_n + B_n \) are invertible we can construct back projections \( P_0', P_1', ..., P_{j-1}' \) such that \( \{A_n + B_n\}_{n \geq 0} \) has a discrete dichotomy on \( \mathbb{Z}_+ \) with the projections \( \{P_n\}_{n \geq 0} \). It is clear that

\[
\dim \text{Im } P_n = \dim \text{Im } P_{n+1} = \dim \text{Im } P, \\
\dim \text{Im } P_n' = \dim \text{Im } P_{n+1}' = \dim \text{Im } P'.
\]

By (3.4) we have

\[
\dim \text{Im } P' = \dim \text{Im } P.
\]

Now we consider a \( C^1 \)-mapping \( G: U \to \mathbb{R}^m \), \( U \) being an open subset of \( \mathbb{R}^m \). We assume that \( G \) has two fixed points \( y_1, y_2 \) which are hyperbolic and there is a subsequence \( \{x_n\}_{n \geq 1} \subset U \) such that

\[
\lim_{n \to -\infty} x_n = y_1, \quad \lim_{n \to +\infty} x_n = y_2, \quad x_{n+1} = G(x_n), \quad \det D G(x_n) \neq 0.
\]

Then we can solve the same problem as in the previous section: we put \( G \) into a smooth family \( G_e: \mathbb{R}^m \to \mathbb{R}^m \) of mappings, \( G_0 = G \). We want to find heteroclinic orbits of \( G_e \) for \( e \) small near \( e = 0 = G \). To this end we consider the equation \( H_e(\cdot) = 0 \), \( H_e: X \to X \),

\[
H_e(\{z_n\}_{n \geq 0}) = z_{n+1} - G_e(z_n).
\]

We see that \( H_0(\Gamma) = 0 \) and \( (D H_0(\Gamma) \{z_n\}_{n \geq 0}) = z_{n+1} - D G(x_n) z_n \), and if we put \( L = D H_0(\Gamma) \), by Theorem 3.3 \( L \) is a Fredholm operator. Since \( x_n \to y_{1(2)} \) as \( n \to +\infty( -\infty) \), applying Lemma 3.4 we have

\[
\text{index } L = m_1 + m_2 - m,
\]

where \( m_{1(2)} \) is the number (counting multiplicities) of the eigenvalues of \( D G(y_{2(1)}) \) with absolute values smaller (greater) than 1. Hence we can reduce the equation \( H_e(z) = 0 \) near \( z = \Gamma \) by using the Lyapunov-Schmidt method to the bifurcation equation

\[
Q(c, e) = 0,
\]

361
where \( Q: U_1 \times U_2 \to \mathbb{R}^{\dim \ker L^*}, \) \( U_1, U_2 \) are open neighbourhoods of \( 0 \in \mathbb{R}^{\dim \ker L^*}, \) \( \mathbb{R} \) respectively, and \( Q(0, 0) = 0. \) Note that \( \dim \ker L^* = \dim \operatorname{codim} \operatorname{Im} L. \) Finally, we can investigate the equation \( Q(c, e) = 0 \) near \( c = 0, \ e = 0 \) by applying the theory of singularities of finite-dimensional mappings \([5, 9]\). We note that each solution of \( H_*(\cdot) = 0 \) near \( \Gamma \) for \( e \) small yields a heteroclinic orbit of \( G_e \) near \( \{x_n\}_{n=0}^{+\infty}. \)

We will follow the above mentioned procedure for special cases of \( G \) in the next section.

4. APPLICATIONS

We generalize the problem from Section 2. Consider a mapping \( f: \mathbb{R} \to \mathbb{R} \) with the same properties as in Section 2. Further, we consider a \( C^3 \)-mapping \( G \)

\[
\begin{align*}
x_1 &= f(x) + o(|y|) \\
y_1 &= A(x) y + o(|y|),
\end{align*}
\]

where \( y \in \mathbb{R}^{m-1}. \) We assume that \( A(\cdot) \in \mathcal{L}(\mathbb{R}^{m-1}), \) \( \det A(\cdot)/\langle 0, 1 \rangle \neq 0 \) and \( A(0), A(1) \) are hyperbolic, i.e. they have no eigenvalues on the unit circle. Then \( G \) has the trajectory \( \Gamma = \{(x, 0)\}_{n=0}^{+\infty} \) and \( (0, 0), (1, 0) \) are hyperbolic fixed points. Consider a perturbed mapping \( G_e: \mathbb{R}^m \to \mathbb{R}^m, \ e \in \mathbb{R}, \ G_0 = G, G(\cdot) \in C^3. \) Now we apply the above mentioned procedure from the end of Section 3, and the relevant operator \( L \) has the index

\[
\text{index } L = 2 \dim \ker L + m_1 + m_2 - m,
\]

where \( \dim \ker L + m_{1(2)} - 1 \) is the number of the eigenvalues of \( A(1, (0)) \) with absolute values smaller (greater) than 1.

We shall investigate two cases:

A. \( \dim \ker L = 1, \ \text{index } L = 0. \)

In this case the bifurcation equation (see the end of Section 3) has the form

\[
Q: U_1 \times U_2 \to \mathbb{R},
\]

where \( U_{1(2)} \) are neighbourhoods of \( 0 \in \mathbb{R} \) and \( Q(c, 0) = 0, \) since \( G_0 = G \) has the family of heteroclinic orbits \( \mathcal{M} = \{(f^n(x), 0)\}_{n=0}^{+\infty}, \) \( x \in (0, 1) \}. \) Hence \( Q(c, e) = e H(c, e) \). Thus a necessary condition for the bifurcation is \( H(0, 0) = 0. \) Moreover, if \( H(0, 0) = 0 \) and \( H_e(0, 0) = 0 \) then by the implicit function theorem we have near \( (0, 0) \)

\[
e \neq 0 \quad \text{and} \quad Q(c, e) = 0 \quad \text{iff} \quad c = c(e), \ c(0) = 0.
\]

Summing up we have proved the following theorem:

**Theorem 4.1.** If \( H(0, 0) = 0 \) and \( H_e(0, 0) = 0 \) then in a neighbourhood of \( \Gamma \) there is a unique trajectory \( \Gamma_e \) of \( G_e \) for \( e \neq 0 \) small. From (4.1) we have \( m \geq 2. \)

B. \( \dim \ker L \geq 2, \ \text{codim } \operatorname{Im} L = 1. \)
From (4.1) we have \( m \geq \dim Ker L + 1 \). In this case the bifurcation equation has the form

\[
Q: U_1 \times U_3 \times U_2 \to R,
\]

where \( U_{1(2)} \) are neighbourhoods of 0 \( \in R \), \( U_3 \) is a neighbourhood of 0 \( \in R^{\dim Ker L-1} \), \( e \in U_2 \). The variable \( c \in U_1 \) corresponds to the family \( \mathcal{M} \). Hence \( Q(c, 0, 0) = 0 \) and since \( Q \) is the bifurcation equation we have \( D_x Q(0, 0, 0) = 0, x \in U_3 \). We assume that \( D_x^2 Q(0, 0, 0) \) is a nondegenerate matrix. Then using the splitting lemma [9] we obtain that \( Q(\cdot, \cdot, \cdot) \) is strongly right equivalent to

\[
Q(c, 0, e) + \langle D_x^2 Q(0, 0, 0) x, x \rangle (1/2),
\]

where \( \langle \cdot, \cdot \rangle \) is the scalar product in \( R^{\dim Ker L-1} \). Since \( Q(c, 0, 0) = 0 \), we obtain

\[
Q(c, 0, e) = e H(c, e).
\]

If we assume that \( H(0, 0) \neq 0 \), then the following theorem holds:

**Theorem 4.2.** Under the above conditions in a neighbourhood of \( \Gamma \) for \( e \) small either there are infinitely many trajectories of \( G_e \) or

i) there is no heteroclinic point near \( (x_0, 0) \in R^m \) for \( e < 0(>0) \),

ii) the set of heteroclinic points of \( G_e \) near \( (x_0, 0) \) lies on \( (0, 1) \times \{0\} \subset R \times R^{m-1} \) and is homeomorphic to \( (0, 1) \) for \( e = 0 \),

iii) the set of heteroclinic points of \( G_e \) near \( (x_0, 0) \) is homeomorphic to \( S^{\dim Ker L-2} \times (0, 1) \) for \( e > 0(<0) \).

(We note that a heteroclinic point is a point which lies on a heteroclinic orbit and \( S^k \) is the k-dimensional sphere.)

**Proof.** Near \((0, 0)\) we must solve in \((c, x)\) the equation

\[
e H(c, e) + \langle D_x^2 Q(0, 0, 0) x, x \rangle (1/2) = 0
\]

for \( e \) small. Since \( H(0, 0) \neq 0 \) and \( D_x^2 Q(0, 0, 0) \) is nondegenerate the structure of solutions near \((0, 0)\) depends mainly on the matrix \( D_x^2 Q(0, 0, 0) \). According as this matrix is indefinite or not we obtain either the first or the second assertion.

**Remark 4.3.** The conditions of regularity from the above theorems 4.1 and 4.2 can be expressed explicitly.

**Remark 4.4.** Using the Morse critical point theory [5] we obtain a precise picture of the set of heteroclinic points of \( G_e \) near \((x_0, 0)\) for \( e \) small in Theorem 4.2. For instance, in the second part of this theorem the sphere, which is homeomorphic to \( S^{\dim Ker L-2} \), in the case iii) shrinks to the point 0 as \( e \to 0 \).

We see that we can use this method for the investigation of local intersections of stable and unstable manifolds. For instance, let \( f: R^m \to R^m \) be a \( C^3 \)-diffeomorphism with hyperbolic fixed points \( y_1, y_2 \) and let us assume that \( m_1 = 1, m_2 = m - 1 \).
The point $y_2$ has a one-dimensional stable manifold $S_0$ and $y_1$ has an $(m - 1)$-dimensional unstable manifold $R_0$. If $R_0 \cap S_0 \ni x_0$ then for the orbit $\{f^n(x_0)\}_{n=0}^{+\infty}$ we have an operator $L$ from Section 3 and index $L = 0$, dim Ker $L \leq 1$. If dim Ker $L = 0$ then $L$ is invertible and for a perturbed smooth mapping $f_\varepsilon : R^m \to R^m$, $R_\varepsilon$ and $S_\varepsilon$ have a transversal intersection near $x_0$ for $\varepsilon$ small, where $R_\varepsilon$, $S_\varepsilon$ are the stable and unstable manifolds of $f_\varepsilon$ near $R_0$, $S_0$, respectively. This follows from the fact that in this case the operator $H_\varepsilon(\cdot)$ (see the end of Section 3) is invertible in $\{f^n(x_0)\}_{n=0}^{+\infty}$. If dim Ker $L = 1$ then for $f_\varepsilon$ we obtain the bifurcation equation $Q(c, \varepsilon) = 0$, $Q : U \times U \to R$, where $U$ is a neighbourhood of $0 \in R$ and $Q(0, 0) = 0$, $Q_\varepsilon(0, 0) = 0$. The generic conditions are $Q_{cc}(0, 0) = \neq 0$ and $Q_\varepsilon(0, 0) = 0$. Under these conditions $R_0$ is tangent to $S_0$ at $x_0$, since $R_\varepsilon$, $S_\varepsilon$ have no intersections near $x_0$ for small $\varepsilon > 0$ ($\varepsilon < 0$), and have precisely a two-point transversal intersection near $x_0$ for small $\varepsilon < 0$ ($\varepsilon > 0$). This last assertion follows from the fact that our assumptions for $Q$ imply that $Q = 0$ is equivalent to $c^2 \neq \varepsilon = 0$.

Now we return to the case D from Section 2. It is a particular case of the case A of this section and we are going to derive the bifurcation equation $Q$ from the end of Section 3. Thus we consider the mapping

\begin{equation}
(4.2) \quad z_{n+1} = f(z_n) + e h(z_n, y_n),
\end{equation}

\begin{equation}
(4.3) \quad y_{n+1} = g(z_n, y_n) + e t(z_n, y_n),
\end{equation}

where $f, g$ have the properties from Section 2, $h, t \in C^3$. We put $v_n = y_n$, $z_n = x_n + ce_n + u_n$, where $\Gamma = \{x_n\}_{n=0}^{+\infty}$, $\{e_n\}_{n=0}^{+\infty} \in$ Ker $L$, $u_0 = 0$. Then

\begin{equation}
(4.4) \quad u_{n+1} = f(x_n + ce_n + u_n) - f(x_n) - ce_{n+1} + eh(\cdot, \cdot)
\end{equation}

\begin{equation}
(4.5) \quad v_{n+1} = g(x_n + ce_n + u_n, v_n) + e t(\cdot, \cdot).
\end{equation}

Using the projection $P$ from Section 2 we have

\begin{equation}
(4.6) \quad u_{n+1} = f(x_n + ce_n + u_n) - f(x_n) - ce_{n+1} + eh(\cdot, \cdot)
\end{equation}

\begin{equation}
(4.7) \quad (I - P) \{v_{n+1} - g(x_n + ce_n + u_n, v_n) - et(\cdot, \cdot)\} = 0
\end{equation}

\begin{equation}
(4.8) \quad P\{v_{n+1} - g(x_n + ce_n + u_n, v_n) - et(\cdot, \cdot)\} = 0,
\end{equation}

where by the implicit function theorem we can solve the first two equations and inserting this solution in the last equation we obtain the bifurcation equation

\begin{equation}
(4.9) \quad Q(c, \varepsilon) = P\{v_{n+1}(c, \varepsilon) - g(x_n + ce_n + u_n(c, \varepsilon), v_n(c, \varepsilon)) - et(\cdot, \cdot)\} = 0.
\end{equation}

As a matter of fact, we have just carried out the Lyapunov-Schmidt procedure for our case.

We see that

\begin{equation}
Q_e(0, 0) = P\{t(x_n, 0)\}.
\end{equation}

Further, using $v_n(c, 0) = 0$, $u_n(c, 0) = (d/dc) u_n(c, 0)|_{c=0} = 0$ we obtain

\begin{equation}
Q_{ce}(0, 0) = P\{-t_x(x_n, 0) e_n - v^e_n(0, 0) g_{yx}(x_n, 0) e_n\}.
\end{equation}
where the sequence \( \{v^e_n(0, 0)\} \) satisfies

\[
(4.3) \quad \{v^e_{n+1}(0, 0) - e_n(0, 0) g_p(x_n, 0)\} = (I - P) \{t(x_n, 0)\}.
\]

Taking the system \( \{x_n(s)\}_{s=-\infty}^{+\infty}, \quad s \in (-\delta, \delta), \quad x_n(s) = f^n(s + x_0) \) we repeat the above procedure and the equation (4.3) assumes the form

\[
\{v^e_{n+1}(s, 0, 0) - e_n(0, 0) g_p(x_n, 0)\} = (I - P(s)) \{t(x_n(0), 0)\},
\]

where \( P(s) \) is the projection from Section 2 corresponding to \( \{x^n(s)\} \). Differentiating the above equation by \( s \) we find

\[
(4.4) \quad \{v^e_{n+1}(0, 0, 0) - v^e_n(0, 0, 0) g_p(x_n, 0) - v^e_{n+1}(0, 0, 0) g_p(x_n, 0) x^e_n(0)\} =
\]

\[
= (I - P(0)) \{t_x(x_n, 0) x^e_n(0)\} - P(0) \{t(x_n, 0)\}.
\]

Note that \( x_n(s) = x_n + se_n + u_n(s, 0) \) for small \( s \), hence

\[
x^e_n(0) = e_n.
\]

Finally, we put

\[
\bar{r}(s) = P(s) \{t(x_n(s), 0)\},
\]

then

\[
\bar{r}(0) = Q_e(0, 0).
\]

From (4.4) we have

\[
Q_{ce}(0, 0) = P\{-t_x(x_n, 0) e_n - v^e_n(0, 0) g_p(x_n, 0) e_n\} =
\]

\[
= P\{-P(0) \{t_x(x_n, 0) e_n\} - P(0) \{t(x_n, 0)\}\} =
\]

\[
= -P(0) \{t_x(x_n, 0) e_n\} - P(0) \{t(x_n, 0)\} = -\bar{r}'(0).
\]

Hence the conditions \( Q_e(0, 0) = 0, \quad Q_{ce}(0, 0) = 0 \) are equivalent to \( r(x_0) = 0, \quad r'(x_0) = 0 \) and \( r \) has the explicit form

\[
(4.5) \quad r(s) = \Sigma_{-\infty}^{+\infty} t(f^i(s), 0) g_p(f^{i+1}(s), 0) \ldots g_p(f^{-1}(s), 0) + t(f^{-1}(s), 0) +
\]

\[
+ \Sigma_0^{+\infty} \frac{t(f^i(s), 0)}{g_p(f^i(s), 0) \ldots g_p(s, 0)}.
\]

Summing up we have proved

**Theorem 4.4.** For the mapping (4.2) the function (4.5) \( r: (0, 1) \to R \) has the following properties: If there is \( s \in (0, 1) \) such that \( r(s) = 0 \) and \( r'(s) = 0 \) then the mapping (4.2) has for \( e \) small an orbit \( \Gamma_e \) near \( \Gamma = \{(f^n(s), 0)\}_{s=-\infty}^{+\infty} \). Moreover, for \( e \neq 0 \), \( \Gamma_e \) is a transversal heteroclinic orbit. Hence the function \( r \) plays the same role as the Melnikov function for ordinary differential equations.

Finally, we consider the quasi-linear mappings

\[
f(x) = \begin{cases} \alpha x, & x \leq 1/2, \quad a > 1, \quad a < 2 \\ (2 - a) x - 1 + a, & x \geq 1/2 \end{cases}
\]

365
\[ g(x, y) = \begin{cases} \eta p, & x \leq 1/2, \ 0 < p < 1 \\ \eta v(x), & 1/2 \leq x = a/2 \\ \eta /d, & x \geq a/2, \ 0 < d < 1, \end{cases} \]

where \( v \in C^3 \) is increasing on \( (1/2, a/2) \) and \( v = p \) for \( x \leq 1/2, v = 1/d \) for \( x \geq a/2, \)

\[
t(x, 0) = \begin{cases} t_1, & x \leq 1/2 \\ w(x), & 1/2 \leq x \leq a/2 \\ t_2, & x \geq a/2, \end{cases} \]

where \( t \in C^3, a, p, d, t_1, t_2 \) are constants. We will apply Theorem 4.4. In this case the sequence \( \{x_n\}_{n=0}^{\infty} \) has the form

\[
x_j = a^j z, \quad j < 0 \\
x_0 = z, \quad z \in (1/2, a/2) \\
x_j = (2 - a)^j (z - 1) + 1, \quad j > 0, \]

and

\[
r(z) = \sum_{-\infty}^{1} p^{[j-1]} t_1 + \frac{w(z)}{v(z)} + \sum_{1}^{\infty} t_2 d^j = t_1 \frac{1}{1 - p} + \left( w(z) + t_2 \frac{d}{1 - d} \right) \frac{1}{v(z)}. \]

Further, if

\[
\begin{align*}
 r(1/2) &= \frac{t_1}{1 - p} + \left( t_1 + t_2 \frac{d}{1 - d} \right) \frac{1}{p} > 0 \quad \text{or} \quad < 0 \\
 r(a/2) &= \frac{t_1}{1 - p} + \left( t_2 + t_2 \frac{d}{1 - d} \right) d < 0 \quad > 0
\end{align*}
\]  

then we obtain the following theorem.

**Theorem 4.5.** If \( f, v, t \) have the above properties, \( h \in C^3(R \times R, R) \), the numbers \( t_1, t_2, p, d \) satisfy the condition (4.6) and \( r'(\cdot) \neq 0 \) on \( (1/2, a/2) \), then the mapping

\[
x_1 = f(x) + e h(x, y), \\
y_1 = y v(x) + e t(x, y)
\]

has at least one transversal heteroclinic orbit for \( e \neq 0 \) small near the set \( (0, 1) \times \times \{0\} \).

Note that for a general \( t \) the function \( r \) has the form

\[
r(z) = \sum_{-\infty}^{1} t(a^j z, 0) \ p^{[j-1]} + \frac{t(z, 0)}{v(z)} + \\
+ \sum_{1}^{\infty} t((2 - a)^j (z - 1) + 1, 0) \frac{d^j}{v(z)}, \quad z \in (1/2, a/2). 
\]
References


Súhrn

BIFURKÁCIA HETEROKLINICKÝCH TRAJEKTÓRIÍ DIFEOMEORFIZMOV

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