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SUM OF OBSERVABLES IN FUZZY QUANTUM SPACES

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Summary. We introduce the sum of observables in fuzzy quantum spaces which generalize the Kolmogorov probability space using the ideas of fuzzy set theory.

Keywords: Fuzzy quantum space, observable, sum of observables.

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1. Introduction

The main notion of the Kolmogorov classical model of probability theory [5] is a $\sigma$-algebra of subsets of a set. This model has been very useful, however, it does not describe situation in quantum mechanical measurements. There are many axiomatic models of quantum mechanics, and today there is a widespread model of quantum logics, see for example [12]. Two of the most important examples of non-Boolean quantum logic models are the system of all closed subspaces of a Hilbert space [6] and the quantum probability spaces introduced by Suppes [11].

The Kolmogorov probability model may be uniquely represented by a system of characteristic functions of subsets of a set $X$ from the given $\sigma$-algebra $\mathcal{F}$, which have values in the closed interval $[0,1]$. When a quantum mechanical event $a$, say, is described vaguely, then by a fuzzy set $a$, that is a fuzzy event $a$, we shall understand a real-valued function $a: X \rightarrow [0,1]$ which describes the quantum mechanical event $a$: this is a basic idea of Zadeh’s theory [13].
The intersection $\cap$ and the union $\cup$ of fuzzy sets $\{a_i\}$, the complement $\perp$ of a fuzzy set $a$ are defined as

$$\bigcap_i a_i := \inf_i a_i,$$

$$\bigcup_i a_i := \sup_i a_i,$$

$$a\perp := 1 - a.$$

If $f$ and $g$ are two $\mathcal{F}$-measurable functions, then the measurability of the sum $f + g$ may be proved using the following simple relation

$$(1.1) \quad \{x \in X : (f + g)(x) < t\} = \bigcup_{r \in \mathbb{Q}} \{x \in X : f(x) < r\} \cap \{x \in X : g(x) < t - r\},$$

where $\mathbb{Q}$ is the set of all rationals.

Using this fact in the present note, we will define the sum of any pair of $F$-observables of a fuzzy quantum space.

2. Fuzzy quantum spaces

**Definition 2.1.** A fuzzy quantum space is a couple $(X, M)$, where $X$ is a nonempty set and $M \subset [0, 1]^X$ satisfies the following conditions:

(i) if $1(x) = 1$ for any $x \in X$, then $1 \in M$;

(ii) if $a \in M$, then $a\perp := 1 - a \in M$;

(iii) if $\frac{1}{2}(x) = \frac{1}{2}$ for any $x \in X$, then $\frac{1}{2} \notin M$;

(iv) $\bigcup_{n=1}^{\infty} a_n := \sup_n a_n \in M$ for any $\{a_n\}_{n=1}^{\infty} \subset M$.

In the fuzzy sets theory the system $M$ is called a soft $\sigma$-algebra [7].

This structure has been suggested by Riečan [9] as an alternative axiomatic model for quantum mechanics. More general structure assuming that $M$ is closed with respect to the union of any sequence of mutually orthogonal fuzzy sets has been proposed by Pykacz [8] and studied by Dvurečenskij and Chovanec [1]. Some fuzzy sets ideas have been studied also by Guz [4], but his approach is different from ours.

The analogue of a random variable is an $F$-quantum observable: An $f$-observable on a fuzzy quantum space $(X, M)$ is a mapping $x : B(R^1) \rightarrow M$ with the following properties:

(i) $x(E^c) = 1 - x(E)$ for every $E \in B(R^1)$;

(ii) if $\{E_n\}_{n=1}^{\infty} \subset B(R^1)$, then $x(\bigcup_{n=1}^{\infty} E_n) = \bigcup_{n=1}^{\infty} x(E_n)$, where $B(R^1)$ is the Borel $\sigma$-algebra of the real line $R^1$ and $E^c$ denotes the complement of $E$ in $R^1$.

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In particular, for \( a \in M \), the mapping \( x_a : B(R^1) \to M \) defined by

\[
x_a(E) = \begin{cases} 
  a \cap a^\perp & 0, 1 \notin E \\
  a^\perp & 0 \in E, 1 \notin E \\
  a & 0 \notin E, 1 \in E \\
  a \cup a^\perp & 0, 1 \in E 
\end{cases} \quad (E \in B(R^1))
\]

is an \( F \)-observable of \((X, M)\) called the indicator of the fuzzy set \( a \in M \).

If \( f : R^1 \to R^1 \) is a Borel measurable function and \( x \) is an \( F \)-observable, then \( f \circ x : E \to x(f^{-1}(E)) \), \( E \in B(R^1) \), is an \( F \)-observable, too. In particular, if \( \alpha \in R^1 \), then \( \alpha x : E \to x(\{t \in R^1 : \alpha t \in E\}) \) for any \( E \in B(R^1) \).

Let \((X, M)\) be a fuzzy quantum space. The set \( M \) may be regarded as a partially ordered set in which we define \( a \leq b \) iff \( a(x) \leq b(x) \) for any \( x \in X \). Using the complementation \( \perp : a \to a^\perp = 1 - a \) for any fuzzy set \( a \in M \), we see that \( \perp \)

(i) \( (a^\perp)^\perp = a \) for any \( a \in M \);

(ii) if \( a \leq b \), then \( b^\perp \leq a^\perp \). It is evident that \( a \cup a^\perp = 1 \) iff \( a \) is a crisp set. Hence \( M \) is a distributive \( \sigma \)-lattice with the complementation \( \perp \), for which de Morgan laws

\[
(2.1) \quad \left( \bigcup_i a_i \right)^\perp = \bigcap_i a_i^\perp,
\]

\[
(2.2) \quad \left( \bigcap_i a_i \right)^\perp = \bigcup_i a_i^\perp
\]

hold whenever \( \{a_i\} \subset M \).

A nonempty subset \( A \subset M \) is called a Boolean algebra \((\sigma \text{-algebra})\) of a fuzzy quantum space \((X, M)\) if

(i) there are minimal and maximal elements \( 0_A \) and \( 1_A \) from \( A \) such that for any \( a \in A \) \( 0_A \leq a \leq 1_A \) and \( a \cup a^\perp = 1_A \) (we recall that \( 0_A \) and \( 1_A \) are not crisp sets, in general);

(ii) \( A \) is boolean algebra \((\sigma \text{-algebra})\).

It is clear that \( 0_A \neq 1_A \). For example, if \( a \) is a fuzzy set from \( M \), then \( A_a = \{a \cap a^\perp, a^\perp, a, a \cup a^\perp\} \) is a Boolean algebra with the minimal and maximal elements \( 0_{A_a} = a \cap a^\perp \) and \( 1_{A_a} = a \cup a^\perp \), respectively.

In particular, if \( x \) is an \( f \)-observable of \((X, M)\), then the range \( R(x) = \{x(E) : E \in B(R^1)\} \) is a Boolean \( \sigma \)-algebra of \((X, M)\) with the minimal and maximal elements \( 0_{R(x)} = x(\emptyset) \) and \( 1_{R(x)} = x(R^1) \).

In accordance with the theory of quantum logics, we say that two elements \( a, b \in M \) are:
(i) orthogonal if \( a \leq 1 - b \), and we write \( a \perp b \);
(ii) compatible if \( a = a \cap b \cup a \cap b^\perp, b = b \cap a \cup b \cap a^\perp \), and we write \( a \leftrightarrow b \);
(iii) strongly compatible if \( a \leftrightarrow b \leftrightarrow a^\perp \leftrightarrow b^\perp \leftrightarrow a \), and we write \( a \leftrightarrow b \). Two observables \( x \) and \( y \) are compatible if \( x(E) \leftrightarrow y(F) \) for any \( E, F \in B(R^1) \).

The following result has been proved by Dvurečenskij and Riečan [2]:

**Theorem 2.2.** Let \( \{a_t : t \in T\} \) be a system of fuzzy sets from \( M \). The following assertions are equivalent:

(i) \( \{a_t : t \in T\} \) is a system of mutually strongly compatible elements;
(ii) \( a_s \cup a_t^\perp = a_t \cup a_t^\perp \) for any \( s, t \in T \);
(iii) there is a Boolean \( \sigma \)-algebra of \( M \) containing all \( \{a_t : t \in T\} \).

Now we characterize \( F \)-observables of a fuzzy quantum space \( (X, M) \).

**Theorem 2.3.** Let \( x \) be an \( F \)-observable of a fuzzy quantum space \( (X, M) \) and let \( B_x(t) = x((-\infty, t]), t \in R^1 \). Then the system \( \{B_x(t) : t \in R^1\} \) fulfills the following conditions:

(i) \( B_x(s) \leq B_x(t) \) if \( s < t \);
(ii) \( \bigcup_t B_x(t) = a \);
(iii) \( \bigcap_t B_x(t) = a^\perp \);
(iv) \( \bigcup_{t<s} B_x(t) = B_x(s) \);
(v) \( B_x(t) \cup B_x^\perp(t) = a \), where \( a = X(R^1) \) and \( a^\perp = x(\emptyset) \).

Conversely, if a system \( \{B(t) : t \in R^1\} \) of fuzzy sets of a fuzzy quantum space \( (X, M) \) fulfills the conditions (i)-(v) for some \( a \in M \), then there is a unique \( F \)-observable \( x \) such that \( B_x(t) = B(t) \) for any \( t \), and \( x(R^1) = a \).

**Proof.** (i) is trivial.
(ii) Let \( a = x(R^1) \), then \( x((-\infty, t]) \leq a \). For every integer \( n \) we have

\[
x((-\infty, n]) \leq a \quad \text{and} \quad x(R^1) = x\left(\bigcup_{n=1}^{\infty} (-\infty, n]\right) = \bigcup_{n=1}^{\infty} x((-\infty, n]).
\]

Similarly we prove (iii).
(iv) the condition (i) implies \( B_x(t) \leq B_x(s) \) for every \( t \leq s \), so that

\[
B_x(s) = \bigcup_{n=1}^{\infty} B_x\left(s - \frac{1}{n}\right).
\]
(v) It may be proved as follows: $B_x(t) \cup B^+_x(t) = x((-\infty, t) \cup (-\infty, t^c)) = x(R^1)$.

For the fuzzy quantum space $(X, M)$ let now a system $\{B(t) : t \in R^1\}$ satisfying (i)-(v) be given. Due to (v), the system $\{B(t) : T \in R^1\}$ consists of mutually strongly compatible elements of $M$ so that, according to Theorem 2.2, there is a minimal Boolean $\sigma$-algebra $F$ of $M$ containing all $B(t)$'s. By the Loomis-Sikorski theorem [10], there is a measurable space $(\Omega, \mathcal{F})$ and a homomorphism $h$ from $\mathcal{F}$ onto $F$. Let $r_1, r_2, r_3, \ldots$ be any distinct enumeration of the rationals. We claim to construct, by induction, sets $A_1, A_2, \ldots$ from $\mathcal{F}$ such that

(a) $h(A_i) = B(r_i)$;
(b) $A_i \subseteq A_j$ if $r_i < r_j$;
(c) $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

We note that if $A \subseteq B$, $A \in \mathcal{F}$ and if there is a $c \in F$ such that $h(A) \leq c \leq h(B)$, then there is a $C \in \mathcal{F}$ such that $A \subseteq C \subseteq B$, $h(C) = c$. Indeed, since $h$ maps $\mathcal{F}$ onto $F$, there is a $C_1 \in \mathcal{F}$ such that $h(C_1) = c$. If we define $C = (C_1 \cap B) \cup A$ then $C$ has the given property. Let $A_1$ be any set in $\mathcal{F}$ such that $h(A_1) = B(r_1)$. Suppose $A_1, A_2, \ldots, A_n \in \mathcal{F}$ have been constructed so that (a) and (b) hold. We shall construct $A_{n+1}$ as follows. Let $(i_1, \ldots, i_n)$ be the permutation of $(1, \ldots, n)$ such that $r_{i_1} < \ldots < r_{i_n}$. Then only one of the following conditions holds:

(i) $r_{n+1} < r_{i_1}$;

(ii) $r_{n+1} > r_{i_n}$;

(iii) there is a unique $k = 1, \ldots, n - 1$ such that $r_{i_k} < r_{n+1} < r_{i_{k+1}}$.

and by the above observation we can select $A_{n+1}$ such that $h(A_{n+1}) = B(r_{n+1})$ and (i) $A_{n+1} \subseteq A_{i_1}$;
(ii) $A_{n+1} \supseteq A_{i_1}$;
(iii) $A_{i_k} \subseteq A_{n+1} \subseteq A_{i_{k+1}}$,

according to (2.3). Then the system \{\chi, \ldots, A_{n+1}\} fulfills (a) and (b). Thus, by induction, it follows that there is a sequence $\{A_j\}$ of sets in $\mathcal{F}$ with the properties (a) and (b). As

$$h\left(\bigcap_{j=1}^{\infty} A_j\right) = \bigcap_{j=1}^{\infty} h(A_j) = \bigcap_{j=1}^{\infty} B(r_j) = 0_F,$$

we may replacing $A_j$ by $A_j \setminus \bigcap_{i} A_i$ if necessary, assume that $\bigcap_{j} A_j = \emptyset$. We define an $\mathcal{F}$-measurable function $f$ as follows:

$$f(\omega) = \begin{cases} 0 & \text{if } \omega \notin \bigcup_{j=1}^{\infty} A_j \\
\inf \{r_j : \omega \in A_j\} & \text{if } \omega \in \bigcup_{j=1}^{\infty} A_j.\end{cases}$$

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The function $f$ is everywhere well-defined and finite. Moreover,

$$f^{-1}((-(\infty, r_k))) = \begin{cases} \bigcup_{r_j < r_k} A_j & \text{if } r_k < 0 \\ \bigcup_{r_j < r_k} A_j \cup (\Omega - \bigcup_i A_i) & \text{if } r_k > 0, \end{cases}$$

hence $f$ is $\mathcal{F}$-measurable and $h(f^{-1}((-(\infty, r_k)))) = B(r_k)$. If define an observable by $x(E) = h(f^{-1}(E))$, $E \in B(\mathbb{R})$, then $x((-(\infty, t))) = B(t)$ for every $t \in \mathbb{R}$. The equality $x_1((-(\infty, t))) = x_2((-(\infty, t)))$ for every $t \in \mathbb{R}$ implies $x_1 = x_2$, hence, the uniqueness of $x$ is shown and the proof is complete. \hfill \Box

3. Existence of a sum

In accordance with (1.1), we define the sum of two observables as follows.

**Definition 3.1.** Let $x$ and $y$ be two $F$-observables of a fuzzy quantum space $(X, M)$. If the system $\{B_{x+y}(t) : t \in \mathbb{R}\}$

$$B_{x+y}(t) = \bigcup_{r \in \mathbb{Q}} (B_x(r) \cap B_y(t-r)), \quad t \in \mathbb{R}, \quad (3.1)$$

where $\mathbb{Q}$ is the set of all rationals, determines an $F$-observable $z$ of $(X, M)$, then we call it the sum of $x$ and $y$, and we write $z = x + y$.

It is clear that if the sum exists, then it is unique. For the proof of Theorem 3.3 the followings lemma is useful.

**Lemma 3.2.** Let $S$ be a countable set in $\mathbb{R}$. For observables $x$ and $y$ let us denote

$$B_{x+y}^S(t) = \bigcup_{s \in S} (B_x(s) \cap B_y(t-s)), \quad \text{(3.2)}$$

then

$$B_{x+y}^S(t) = B_{x+y}(t) \quad \text{for every } t \in \mathbb{R}. \quad \text{(3.3)}$$

Proof. We can show that if $t_n \uparrow t$, $t_n \in S$, then $B_{x+y}^S(t) = \bigcup_n B_{x+y}^S(t_n)$. Indeed,

$$\bigcup_{n} B_{x+y}^S(t_n) = \bigcup_{n} (B_x(s) \cap B_y(t_n - s)) = \bigcup_{n} (B_x(s) \cap B_y(t_n - s)) = B_{x+y}(t).$$

Let now $n$ be any integer, then for each $s \in S$ there is $r = r(s) \in \mathbb{Q}$ such that we have $s < r < s + \frac{1}{n}$. Therefore, $B_x(s) \cap B_y(t - n^{-1} - s) \leq B_x(r) \cap B_y(t - r)$ and $B_{x+y}(t - n^{-1}) \leq B_{x+y}(t)$, $B_{x+y}^S(t) = \bigcup_n B_{x+y}^S(t - n^{-1}) \leq B_{x+y}(t)$. Similarly we show that $B_{x+y}(t) \leq B_{x+y}^S(t)$.

\hfill \Box
Theorem 3.3. For every two F-observables \( x \) and \( y \) of a fuzzy quantum space \( (X, M) \) their sum exists.

Proof. We show that the system \( \{B_{x+y}(t) : t \in R^1\} \) fulfils the conditions of Theorem 2.3. The proof of (i), (ii) and (iv) is simple, due to the \( \sigma \)-continuity of \( M \), that is, if \( a_1 \leq a_2 \leq \ldots \leq M \), then for any \( b \in M \), \( b \cap \bigcup \{a_i\} = \bigcup \{b \cap a_i\} \).

Calculate:

\[
\begin{align*}
a &= B_{x+y}(t) \cup B_{x+y}^+(t) \\
&= \bigcup_{r \in Q} (B_x(r) \cap B_y(t-r)) \cup \bigcap_{s \in Q} (B_x^+(s) \cup B_y^+(t-s)) \\
&= \bigcap_{r \in Q} (B_x(r) \cap B_y(t-r)) \cup (B_x^+(s) \cup B_y^+(t-s)) \\
&= \bigcap_{r \in Q} \left( (B_x(r) \cap B_y(t-r)) \cup (B_x^+(s) \cup B_y^+(t-s)) \right) \cap (B_y(t-r) \cup B_y^+(t-s)) \\
\end{align*}
\]

Since \( B_x(r) \cup B_x^+(s) = x(R^1) \) and \( B_x(r) \cup B_x^+(s) = x(R^1) \) for \( s \leq r \), and

\[
\begin{align*}
a &= \bigcap_{s \leq r} \left( \bigcup_{r \in Q} (B_x(r) \cup B_x^+(s) \cup B_y^+(t-s)) \cap (B_y(t-r) \cup B_y^+(s) \cup B_y^+(t-s)) \right) \\
&= \bigcap_{s \leq r} \left( (x(R^1) \cup B_y^+(t-s)) \cap (B_y(t-r) \cup B_y^+(s)) \right) \\
&= \bigcap_{s \leq r} \left( (x(R^1) \cup B_y^+(t-s)) \cap (y(R^1) \cup B_x^+(s)) \right)
\end{align*}
\]

we conclude that

\( r \downarrow s \) implies \( t - r \uparrow t - s \) and \( \bigcup_{r \geq s} B_y(t-r) = B_y(t-s) \).

Then

\[
\begin{align*}
a &= \bigcap_{s} \left( (x(R^1) \cup B_y^+(t-s)) \cap \left( \bigcup_{r \geq s} B_y(t-r) \cup B_x^+(s) \cup B_y^+(t-s) \right) \right) \cap \\
&\quad \cap \left( \bigcup_{r < s} (B_x(r) \cup B_x^+(s) \cup B_y^+(t-s)) \cap (y(R^1) \cup B_x^+(s)) \right) \\
&= \bigcap_{s} \left( (x(R^1) \cup B_y^+(t-s)) \cap (y(R^1) \cup B_x^+(s)) \cap (x(R^1) \cup B_y^+(t-s)) \cap (y(R^1) \cup B_x^+(s)) \right) \\
&= \bigcup_{s} \left( (x(R^1) \cup B_y^+(t-s)) \cap (y(R^1) \cup B_x^+(s)) \right) = \\
&= (x(R^1) \cup \bigcap_s B_y(t-s)) \cap (y(R^1) \cup \bigcap_s B_x(s)).
\end{align*}
\]
In other words, we have proved
\[ a = (x(R^1) \cup y(\emptyset)) \cap (y(R^1) \cup x(\emptyset)) = x(R^1) \cap y(R^1), \]
which means the strong compatibility of \( \{B_{x+y}(t) : t \in R^1\} \), too. To prove
\begin{equation}
(3.3) \quad \bigcap_{t \in R^1} B_{x+y}(t) = x(\emptyset) \cup y(\emptyset) = a^\perp,
\end{equation}
we take into account that, by virtue of the property (v) of Theorem 2.3, \( \{B_{x+y}(t) : t \in R^1\} \) is a system of mutually strongly compatible elements of a fuzzy quantum space \((X, M)\). By Lemma 3.2, it suffices to prove (3.3) for \( t \in T \), where \( T \) is a countable dense subset of \( R^1 \). By Theorem 2.2, there exists a Boolean \( \sigma \)-algebra \( A \subset M \) containing all \( B_{x+y}(t) \) for any \( t \in R^1 \). Every Boolean \( \sigma \)-algebra \( A \) of \( M \) is \( \sigma \)-distributive, that is, if \( T \) and \( S \) are countable sets, then
\[ \bigcup_{t \in T} \bigcap_{s \in S} a_{ts} = \bigcap_{s \in S} \bigcup_{t \in T} a_{ts}, \]
for any two indexed sequences \( \{a_{ts} : t \in T, s \in S\} \subset M \).

In particular, by Sikorski [10] a Boolean \( \sigma \)-algebra \( A \) is \( \sigma \)-distributive iff for any \( a \in A, a \neq 0_A \), and any sequence \( \{a_n\} \subset A \) there exists \( \{c(n)\}_{n=1}^{\infty} \in \{0, 1\} \) such that
\[ a \cap \bigcap_{n=1}^{\infty} a_n^{c(n)} \neq 0_A, \]
where \( a_n^0 = a_n^1 \), \( a_n^1 = a_n \), which is easily verifiable in our case. Then
\begin{equation}
(3.4) \quad \bigcap_{t \in T} \bigcup_{r \in Q} (B_x(r) \cap B_y(t-r)) = \bigcap_{q \in Q^T} \bigcup_{t \in T} (B_x(q(t)) \cap B_y(t - q(t))).
\end{equation}
It is clear that
\begin{equation}
(3.5) \quad \bigcap_{t \in T} B_{x+y}(t) \supseteq x(\emptyset) \cup y(\emptyset) = x(R^1) \cap y(\emptyset) \cup x(\emptyset) \cap y(R^1).
\end{equation}
Let \( q \in Q^T \), then
\[ \bigcap_{t \in T} (B_x(q(t)) \cap B_y(t - q(t))) = \bigcap_{t \in T} (B_x(q(t)) \cap \bigcap_{t \in T} B_y(t - q(t))). \]
There are two possible cases:
(a) \[ \inf_{t \in T} q(t) = k > -\infty, \quad \text{then} \quad \bigcap_{t \in T} B_x(q(t)) \cap \bigcap_{t \in T} B_y(t - q(t)) = \bigcap_{t \in T} \left( B_x(q(t)) \cap \bigcap_{t \in T} B_y(t - q(t)) \right) = B_x(k) \cap \bigcap_{t \in T} B_y(t - q(t)) = B_x(k) \cap y(\emptyset) \leq x(R^1) \cap y(\emptyset) \leq x(\emptyset) \cup y(\emptyset). \]
(b) \[ \inf_{t \in T} q(t) = -\infty, \quad \text{then} \quad \bigcap_{t \in T} B_x(q(t)) \cap \bigcap_{t \in T} B_y(t - q(t)) = x(\emptyset) \cap y(R^1) \leq x(\emptyset) \cup y(\emptyset). \]
For every \( \mathcal{C} \in Q^T \), we have
\[
\bigcap_{t \in T} \left( B_x(\mathcal{C}(t)) \cap B_y(t - \mathcal{C}(t)) \right) \subseteq x(\emptyset) \cup y(\emptyset),
\]
and taking into account (3.4) and (3.5), the following inequalities hold:
\[
x(\emptyset) \cup y(\emptyset) \subseteq \bigcap_{t} B_{x+y}(t) \subseteq x(\emptyset) \cup y(\emptyset).
\]

\[ \square \]

4. Properties of the sum

In the present part, we establish some of the basic properties of the sum. We recall that if \( x \leftrightarrow y \), then, according to Dvurečenskij and Riečan [3], there exists an \( F \)-observable \( z \) and two Borel measurable functions \( f \) and \( g \) such that \( x = f \circ z \), \( y = g \circ z \).

**Theorem 4.1.**

(i) \( x + y = y + x \) for any two \( F \)-observables \( x \) and \( y \);

(ii) \( (x + y) + z = x + (y + z) \) for any three \( F \)-observables \( x \), \( y \) and \( z \);

(iii) if \( x \leftrightarrow y \), then \( x + y = (f + g) \circ z \) provided \( x = f \circ z \), \( y = g \circ z \);

(iv) Let \( u \in R^1 \) and put
\[
I_u(E) = \begin{cases} 
1 & u \in E \\
0 & u \notin E,
\end{cases} \quad (E \in B(R^1))
\]
then \( x + I_u = f_u \circ z \), where \( f_u(t) = t + u \);

(v) \( \alpha(x + y) = \alpha x + \alpha y \) for any \( \alpha \in R^1 \) and all \( F \)-observables \( x \) and \( y \).

**Proof.** (i) Let \( t \in R^1 \) and denote \( S_t = \{ t - r : r \in Q \} \). Then \( S_t \) is dense in \( R^1 \) and using Lemma 3.2, we have
\[
B_{x+y}(t) = \bigcup_{r \in Q} \left( B_x(r) \cap B_y(t-r) \right) = \bigcup_{s \in S_t} \left( B_y(s) \cap B_x(t-s) \right) = B_{y+z}(s) = B_{y+z}(t).
\]

(ii) \( B_{(x+y) + z}(t) = \bigcup_{r \in Q} \left( B_{x+y}(r) \cap B_z(t-r) \right) = \)
\[
= \bigcup_{r \in Q} \left( \bigcup_{s \in Q} \left( B_x(s) \cap B_y(r-s) \right) \cap B_z(t-r) \right) = \\
= \bigcup_{s \in Q} B_x(s) \cap \left( \bigcup_{r \in Q} \left( B_y(r-s) \cap B_z(t-s-(r-s)) \right) \right).
\]
We denote $C_r = \{r - s : s \in Q\}$, then $C_r$ is a countable dense set in $R^1$. Hence, by Lemma 3.2, we have

$$
\bigcup_{s \in Q} B_x(s) \cap \left( \bigcup_{r \in Q} B_y(r - s) \cap B_z((t - s) - (r - s)) \right) = \\
= \bigcup_{s \in Q} (B_x(s) \cap B_y+z(r - s)) = B_{x+y+z}(t).
$$

(iii) Calculate:

$$
B_{x+y}(t) = \bigcup_{r \in Q} (z((-\infty, r)) \cap y((-\infty, t - r))) =
$$

$$
= \bigcup_{r \in Q} (z(f^{-1}((-\infty, r))) \cap z(g^{-1}((-\infty, t - r)))) =
$$

$$
= \bigcup_{r \in Q} (z(E) \cap z(F)) = \bigcup_{r \in Q} z(E \cap F) =
$$

$$
= z\left( \bigcup_{r \in Q} (f^{-1}((-\infty, r)) \cap g^{-1}((-\infty, t - r))) \right) =
$$

$$
= z(\{(f + g)^{-1}((-\infty, t))\}) = (f + g) \circ z((-\infty, t)) =
$$

$$
= B_{(f+g)\circ z}(t).
$$

(iv) Since $B_{I_u}(t - r) = 0$ if $t - u < r$ and $B_{I_u}(t - r) = 1$ otherwise, we have

$$
B_{x+I_u}(t) = \bigcup_{r \geq 1-u} (0 \cap B_x(r)) \cup \bigcup_{r \leq 1-u} (1 \cap B_x(r)) =
$$

$$
= \bigcup_{r \leq 1-u} B_x(r) = B_x(t - u) = f_u \circ z((-\infty, t)).
$$

(v) is evident.

\[\square\]

Remark 4.2. If $M$ consist of crisp subsets, that is, $M$ is a $\sigma$-algebra of subsets of $M$ (more precisely, $M$ is a set of all characteristic functions of sets from the given $\sigma$-algebra), then the sum of $F$-observables coincides with the pointwise defined sum. Indeed, in this case for $x$ and $y$ there are unique mappings $u, v : X \rightarrow R^1$ such that $x(E) = u^{-1}(E)$ and $y(F) = v^{-1}(F)$, $E, F \in B(R^1)$, and $(x + y)(E) = (u + v)^{-1}(E)$ for any $E \in B(R^1)$ (see the proof of Theorem 3.3).

Remark 4.3. If $\mathcal{O}(M)$ is the set of all $F$-observables of a fuzzy quantum space $(X, M)$, then $\mathcal{O}(M)$ is a real vector space with respect to the sum.

Remark 4.4. We define the subtraction of $F$-observables $x$ and $y$ as $x - y = x + (-y)$, where $(-y)(E) = y(\{t : -t \in E\})$, $E \in B(R^1)$.
References


Súhrn

SÚČET POZOROVATEĽNÝCH VO FUZZY KVANTOVÝCH PRIESTOROCH

Anatolij Dvurečenskij, Anna Tírpáková

V práci je zavedený súčet pozorovateľných vo fuzzy kvantových priestoroch, ktoré zovšeobecňujú Kolmogorov pravdepodobnostný priestor, použíjúc idey teórie fuzzy množín.

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