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AFFINE-INVARIANT MONOTONE ITERATION METHODS WITH APPLICATION TO SYSTEMS OF NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEMS

Rudolf L. Voller

Summary. In this paper we present a new theorem for monotone including iteration methods. The conditions for the operators considered are affine-invariant and no topological properties neither of the linear spaces nor of the operators are used. Furthermore, no inverse-isotony is demanded. As examples we treat some systems of nonlinear ordinary differential equations with two-point boundary conditions.

AMS classification: 65J15, 65L10

1. Introduction

Given the equation

\[ F(u) = 0 \]

where \( F \) is an (in general nonlinear) operator acting from a subset \( D \) of a partially ordered space \( V \) into a real linear space \( W \), we investigate Newton-like iteration schemes of the form

\[
\begin{align*}
S_{1k}(v_{k+1} - v_k) &= -F(v_k) \\
S_{2k}(u_{k+1} - u_k) &= -F(u_k),
\end{align*}
\]

to obtain sequences \( \{u_k\}_{k \in \mathbb{N}_0}, \{v_k\}_{k \in \mathbb{N}_0} \) such that starting with two elements \( u_0 \leq v_0 \) we have for all \( k \in \mathbb{N}_0: u_k \leq u_{k+1} \leq \cdots \leq v_{k+1} \leq v_k \), so that for all \( k \in \mathbb{N} \) and every solution \( u^* \) of \( F(u) = 0 \) from \([u_0, v_0]\): \( u_k \leq u^* \leq v_k \). Here the linearizations \( S_{ik} \), \( i = 1, 2 \) depend on \( u_k \) and/or \( v_k \). In earlier papers it was assumed (comp. [12], [13]) that the \( S_{ik} \) are inverse-isotone or have at least positive left subinverses \( B_{ik} \). These assumptions are weakened in Theorem 2 for the methods from [12] and [13] similar as in [7], [8], so that any monotonicity assumptions are superfluous. Furthermore the new conditions are affine-invariant, so that they are as in [9] fulfilled for \( AF \), iff they
are fulfilled for $F$, where $A$ is a linear, bijective mapping from $W$ into $V$. Beside the results in [7] we can not only drop the inverse-isotony of the $S_{ik}$ or the positivity of their left subinverses, but also the inverse-isotony of the mappings $B_{ik}S_{ik}$. Since the theorem will be proved for a partially ordered space $V$ we also do not use any property depending on the topology neither of $F$ nor the $S_{ik}$ nor of $V$ nor $W$.

In Section 2 we demonstrate how the results can be applied to certain systems of weakly nonlinear two-point boundary value problems. First a system is considered where the discretization by finite differences leads to a nonlinear system of equations whose “natural” linearization is not inverse-isotone, so that the theorems from [1], [10], [11], [12], [13] and [14] are not applicable. Secondly a predator-prey-model is considered.

2. AFFINE-INARIANT MONOTONE ITERATION METHODS
IN PARTIALLY ORDERED LINEAR SPACES

In this section we first present some definitions for operators acting on partially ordered spaces which are used in the following. For a detailed discussion see [4].

Definition 1. Let $V$ be a linear space over $\mathbb{R}$. Then $V$ is called a partially ordered linear space (POS), iff there exists a reflexive, antisymmetric and transitive relation $\leq$ on $V \times V$ with the following properties:

\[ \forall x, y, z \in V : x \leq y \Rightarrow x + z \leq y + z \]
\[ \forall x, y \in V \forall \lambda \in \mathbb{R}_+: x \leq y \Rightarrow \lambda x \leq \lambda y \]

Definition 1. $x \geq y$: $\Leftrightarrow y \leq x$, $x < y$: $\Leftrightarrow x \leq y$ and $x \neq y$ and $x > y$: $\Leftrightarrow y < x$.

Definition 1. Let $V$ be a POS. Then the set $K_V := \{x \in V | 0 \leq x\}$ is called the cone of $V$ with respect to $\leq$.

Definition 1. Let $X$ be a real Banach space. Then $X$ is called a partially ordered Banach space (POB), iff $X$ is a POS and $K_X$ is closed with respect to the norm topology.

Definition 1. Let $V$ be a POS and $X$ be a POB.

(i) Let $x, y \in V$. The subset $\{z \in V | x \leq z \leq y\}$ of $V$ is called an order-interval and is denoted by $[x, y]$.

(ii) $D \subset V$ is called o-bounded, iff there exists two elements $x, y \in V$, such that $\forall z \in D : x \leq z \leq y$ (analogously o-bounded from above or from below).

(iii) $K_V$ is called (strongly) minihedral, iff every pair of elements of $V$ (each subset of $V$, which is o-bounded from above) has a supremum.

(iv) $K_X$ is called normal, iff $\exists C \in \mathbb{R}_+: \forall x, y \in K_X : x \leq y \Rightarrow \|x\| \leq C\|y\|$.

(v) $K_X$ is called regular, iff each o-bounded, monotone sequence in $X$ is convergent.
Remark. Each regular cone is normal.

Definition 1. Let $V, W$ be POS and $A : D \subset V \rightarrow W$ an operator. Then $A$ is called

\[ \text{isotone} : \iff x, y \in D, \ x \leq y \Rightarrow A(x) \leq A(y) \]
\[ \text{inverse - isotone} : \iff x, y \in D, \ A(x) \leq A(y) \Rightarrow x \leq y \]
\[ \text{positive} : \iff x \in D \cap K_V \Rightarrow 0 \leq A(x). \]

Remark. For linear operators isotone and positive are equivalent properties. Each inverse-isotone operator is injective.

Furthermore we need the following definitions:

Definition 1. Let $F$ be an operator from a POS $V$ into a real linear space $W$. Then any operator $F^{-1}_l : W \rightarrow V$ having the property

\[ \forall x \in V : F^{-1}_l F(x) \leq x \]

is called a left subinverse of $F$. Analogously a right subinverse, superinverse etc. are defined.

Definition 1. Let $V, W$ be linear spaces and $F : D \subset V \rightarrow W$ an operator. A mapping $S : D \times D \rightarrow L(V, W)$ is called a divided difference operator (ddo) for $F$, iff

\[ \forall x, y \in D : S(x, y)(x - y) = F(x) - F(y). \]

If $V$ and $W$ are POS and

\[ \forall x, y \in D \text{ with } x \leq y \text{ or } y \leq x : S(x, y)(x - y) \leq F(x) - F(y) \tag{2} \]

holds, then $S$ is called a generalized divided difference operator (gddo). If (2) is only true for $x \leq y$, then $S$ is called a quasi-divided-difference-operator (qddo).

Definition 1. Let $V, W$ be POS, $D$ a convex subset of $V$. Then an operator $F : D \rightarrow W$ is called convex, iff

\[ \forall x, y \in D, \forall t \in (0, 1) : F(tx + (1 - t)y) \leq tF(x) + (1 - t)F(y) \]

and o-convex, iff this inequality holds only in the case $x \leq y$ or $y \leq x$.

If $F$ has a ddo $S$ and

\[ \forall x, y, z \in D \text{ with } x \leq y \leq z : S(x, y) \leq S(x, z) \leq S(y, z), \]

then $F$ is called ddo-convex.
For a further discussion see [12]. In the following it is important to know, that \textit{ddo}-convex operators and operators with an isotope \textit{gdoo} are o-convex (comp. [12], 1.5 Lemma 3), while operators with a \textit{gdoo} are not necessarily o-convex.

Finally we need the following theorem about solvability of the fixed-point-equation and successive approximation for isotope operators

\textbf{Theorem 1} ([4]). Let \( V \) be a POS and \( A : D \subset V \to V \) an operator. Furthermore we may have elements \( x_0, y_0 \in D \) with \( [x_0, y_0] \subset D \) and \( x_0 \leq A(x_0), A(y_0) \leq y_0 \). Let \( A_{[x_0, y_0]} \) be isotope and let one of the following three conditions be true:

(i) The cone of \( V \) is strongly minihedral.

(ii) \( V \) is a POB with a regular cone, and \( A \) is continuous.

(iii) \( V \) is a POB with a normal cone, and \( A \) is completely continuous.

Then \( A \) has at least one fixed point \( x^* \in [x_0, y_0] \) and for the sequences \( \{x_n\}_{n \in \mathbb{N}_0}, \{y_n\}_{n \in \mathbb{N}_0} \) determined by

\[
x_{n+1} = A(x_n), \quad y_{n+1} = A(y_n), \quad n \in \mathbb{N}_0
\]

we have

\[
x_n \leq x_{n+1} \leq \cdots \leq x^* \leq \cdots \leq y_{n+1} \leq y_n.
\]

If (ii) or (iii) holds, both sequences converge against fixed points of \( A \).

Now we are able to prove the main theorem for monotone including iteration-methods, which we obtain by linearizing the operator equations considered, and thus we call them “Newton-like”, which also means that they converge faster than the iteration of Theorem 1.

\textbf{Theorem 2.} Given an equation (1) let \( F \) be a (in general nonlinear) operator acting from a subset \( D \) of a partially ordered space \( V \) into a real linear space \( W \), and let two elements \( u_0 \leq v_0 \) be given with \( [u_0, v_0] \subset D \). The cone of \( V \) is supposed to be strongly minihedral. Furthermore let linearizations \( S_i : [u_0, v_0] \times [u_0, v_0] \to L(V, W) \) be given, such that for all \( u, v \in [u_0, v_0] \) with \( u \leq v \) \( S_i(u, v) \) has a left subinverse \( B_i(u, v) \in L(W, V) \) with the property \( B_i(u, v)w = 0 \Rightarrow w = 0 \), \( i = 1, 2 \).

Let at least one of the following groups of inequalities 1–IV hold:

1. \( B_1(u_0, v_0)F(u_0) \leq 0 \leq B_1(u_0, v_0)F(v_0) \)

   \( B_2(u_0, v_0)F(u_0) \leq 0 \leq B_2(u_0, v_0)F(v_0) \)

   \( \forall u, v \in [u_0, v_0], u \leq v, z \in \{u, v\} : \)

   \( 0 \leq B_1(z, z)[F(u) - F(v) - S_1(v, v)(u - v)] \)

2. \( \forall u, v, w, z \in [u_0, v_0], u \leq w \leq z \leq v : \)

   \( 0 \leq B_1(z, z)[F(u) - F(w) - S_2(u, v)(u - w)] \)

   \( \forall u, v, w \in [u_0, v_0], u \leq w \leq v, z \in \{w, v\} : \)

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\[0 \leq B_2(u, v)[F(u) - F(z) - S_2(u, v)(u - w)]\]

\[\forall u, v, w, z \in [u_0, v_0], \ u \leq w \leq z \leq v:\]

\[0 \leq B_2(w, z)[F(u) - F(w) - S_2(u, v)(u - w)]\]

\[\forall u, v, z \in [u_0, v_0], \ u \leq z \leq v:\]

\[0 \leq B_2(u, z)[F(z) - F(v) - S_1(v, v)(z - v)]\]

II. \[B_1(u_0, v_0)F(u_0) \leq 0 \leq B_1(u_0, v_0)F(v_0)\]

\[B_2(u_0, u_0)F(u_0) \leq 0 \leq B_2(u_0, u_0)F(v_0)\]

\[\forall u, v, z \in [u_0, v_0], \ u \leq z \leq v, \ w \in \{u, z\}:\]

\[0 \leq B_1(u, v)[F(w) - F(v) - S_1(u, v)(z - v)]\]

\[\forall u, v, w, z \in [u_0, v_0], \ u \leq w \leq z \leq v:\]

\[0 \leq B_1(w, z)[F(z) - F(v) - S_1(u, v)(z - v)]\]

\[\forall u, v, z \in [u_0, v_0], \ u \leq z \leq v:\]

\[0 \leq B_1(z, v)[F(u) - F(z) - S_2(u, u)(u - z)]\]

\[\forall u, v \in [u_0, v_0], \ u \leq v, \ z \in \{u, v\}:\]

\[0 \leq B_2(z, z)[F(u) - F(v) - S_2(u, u)(u - v)]\]

\[\forall u, v, w, z \in [u_0, v_0], \ u \leq w \leq z \leq v:\]

\[0 \leq B_2(w, w)[F(z) - F(v) - S_1(u, v)(z - v)]\]

III. \[B_1(u_0, v_0)F(u_0) \leq 0 \leq B_1(u_0, v_0)F(v_0), i = 1, 2\]

\[\forall u, v, w \in [u_0, v_0], \ u \leq w \leq v:\]

\[0 \leq B_1(u, v)[F(w) - F(v) - S_1(u, v)(w - v)]\]

\[\forall u, v, w, z \in [u_0, v_0], \ u \leq w \leq z \leq v:\]

\[0 \leq B_1(w, z)[F(z) - F(v) - S_1(u, v)(z - v)]\]

\[\forall u, v, w, z \in [u_0, v_0], \ u \leq w \leq z \leq v:\]

\[0 \leq B_1(w, z)[F(u) - F(w) - S_2(u, v)(u - w)]\]

\[\forall u, v, w, z \in [u_0, v_0], \ u \leq w \leq v, \ z \in \{w, v\}:\]

\[0 \leq B_2(u, v)[F(u) - F(z) - S_2(u, v)(u - w)]\]

\[\forall u, v, w, z \in [u_0, v_0], \ u \leq w \leq z \leq v:\]

\[0 \leq B_2(w, z)[F(u) - F(w) - S_2(u, v)(u - w)]\]

IV. \[B_1(z_1, z_2)F(u_0) \leq 0 \leq B_1(z_1, z_2)F(v_0), i = 1, 2, \ z_1 = v_0, z_2 = u_0\]

\[\forall u, v \in [u_0, v_0], \ u \leq v, \ z \in \{u, v\}:\]

\[0 \leq B_1(z, v)[F(u) - F(v) - S_1(v, v)(u - v)]\]

\[\forall u, v, z \in [u_0, v_0], \ u \leq z \leq v:\]

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\begin{align*}
0 \leq B_1(v,v)[F(u) - F(z) - S_2(u,u)(u-z)] \\
\forall u, v, w \in [u_0, v_0], \, u \leq w \leq v, \, z \in \{u, v\} : \\
0 \leq B_2(z, z)[F(u) - F(v) - S_2(u,u)(u-w)] \\
\forall u, v, z \in [u_0, v_0], \, u \leq z \leq v : \\
0 \leq B_2(u,u)[F(z) - F(v) - S_1(v,v)(z-v)]
\end{align*}

Then the iteration processes
\begin{align}
S_{1k}(v_{k+1} - v_k) &= -F(v_k) \\
S_{2k}(u_{k+1} - u_k) &= -F(u_k)
\end{align}

where corresponding to our assumptions we have, respectively
\begin{enumerate}
  \item \(S_{1k} := S_1(v_k, v_k), \, S_{2k} := S_2(u_k, v_k)\)
  \item \(S_{1k} := S_1(u_k, v_k), \, S_{2k} := S_2(u_k, u_k)\)
  \item \(S_{1k} := S_1(u_k, v_k), \, S_{2k} := S_2(u_k, v_k)\)
  \item \(S_{1k} := S_1(v_k, v_k), \, S_{2k} := S_2(u_k, u_k)\)
\end{enumerate}

starting with \(u_0\) and \(v_0\) give us monotone including sequences \(\{u_k\}_{k \in \mathbb{N}_0}, \{v_k\}_{k \in \mathbb{N}_0}\), such that for all \(k \in \mathbb{N}: u_k \leq u_{k+1} \leq \cdots \leq v_{k+1} \leq v_k\) and for all \(k \in \mathbb{N}\) and every solution \(u^*\) of \(F(u) = 0\) from \([u_0, v_0]\): \(u_k \leq u^* \leq v_k\).

\textbf{Proof.} We prove our assertions for the case I., all other cases can be shown analogously, where in case II. first equation (4) and then (3) has to be solved.

The proof is done by induction. Since the first step is proved exactly as the step from \(k\) to \(k+1\), we restrict ourselves to the latter one.

First we have to define operators \(T_{ik} : V \to V, \, i = 1, 2\) by
\begin{align*}
T_{1k} h &= h + B_{1k}[F(v_k) - S_{1k} h] \\
T_{2k} h &= h - B_{2k}[F(u_k) + S_{2k} h]
\end{align*}

where \(B_{ik}\) are the left subinverses of the \(S_{ik}, \, i = 1, 2\). Then we show by Theorem 1 that they have at least one fixed-point.

Because of I. we have \(T_{1k} 0 = B_{1k}[F(v_k)] \geq 0\) and \(T_{1k}(v_k - u_k) = v_k - u_k + B_{1k}[F(v_k) - S_{1k}(v_k - u_k)] = v_k - u_k - B_{1k}[F(u_k) - F(v_k) - S_{1k}(u_k - v_k)] + B_{1k} F(u_k) \leq v_k - u_k\) and furthermore \(T_{1k}(v + dv) = T_{1k} v + dv - B_{1k} S_{1k} dv \geq TK_{1k} v\) for \(dv \geq 0\). Thus \(T_{1k}\) fulfills the conditions of Theorem 1 and therefore has at least one fixed-point in \([0, v_k - u_k]\). Since the cone of \(V\) is strongly minihedral, the set \(\{z \in [0, v_k - u_k] \mid T_{1k} z \leq z\}\), which is nonempty, since it contains the fixed-point, has an infimum \(z_{1k} \geq 0\). Because \(T_{1k} z_{1k} \leq T_{1k} z \leq z\) for all \(z\) from this set, \(T_{1k} z_{1k}\) is also a lower bound of this set, which means \(T_{1k} z_{1k} \leq z_{1k}\), so that \(z_{1k}\) lies in this set, but then also
$T_{1k}z_{1k}$, because $T_{1k}(T_{1k}z_{1k}) \leq T_{1k}z_{1k}$ and $T_{1k}$ is isotone. So we have $z_{1k} \leq T_{1k}z_{1k}$, so that $z_{1k}$ is the lowest fixed-point of $T_{1k}$ in $[0, v_k - u_k]$.

By the definition of $T_{1k}$ it follows that $B_{1k}[F(v_k) + S_{1k}z_{1k}] = 0$ and from the assumption about the kernel of $B_{1k}: S_{1k}z_{1k} = -F(v_k)$. Thus we define $v_{k+1} := v_k - z_{1k}$ and obtain by $0 \leq z_{1k} \leq v_k - u_k$:

$$ u_k \leq v_{k+1} \leq v_k. $$

Furthermore, because of I. we have $T_{2k}0 = -B_{2k}[F(u_k)] \geq 0$ and $T_{2k}(v_{k+1} - u_k) = v_{k+1} - u_k - B_{2k}[F(u_k) + S_{2k}(v_{k+1} - u_k)] = v_{k+1} - u_k - B_{2k}[F(u_k) - F(v_k) - S_{2k}(u_k - v_{k+1})] - B_{2k}F(v_k) \leq v_{k+1} - u_k$ as well as $T_{2k}(v + dv) = T_{2k}v + dv - B_{2k}S_{2k}dv \geq T_{2k}v$ for $dv \geq 0$. Then also $T_{2k}$ fulfills the conditions of Theorem 1 and has a fixed-point in $[0, v_{k+1} - u_k]$. Like $T_{1k}$ also $T_{2k}$ has a lowest fixed-point $z_{2k}$ in $[0, v_{k+1} - u_k]$ (which can be proved analogously).

From the definition of $T_{2k}$ it follows that $B_{2k}[F(u_k) - S_{2k}z_{2k}] = 0$ and because of the assumption about the kernel of $B_{2k}: S_{2k}z_{2k} = F(u_k)$. We define $u_{k+1} := u_k + z_{2k}$ and obtain because of $0 \leq z_{2k} \leq v_{k+1} - u_k$:

$$ u_k \leq u_{k+1} \leq v_{k+1} \leq v_k. $$

From

$$ B_{1k+1}F(u_{k+1}) = -B_{1k+1}[F(u_k) - F(u_{k+1}) - S_{2k}(u_k - u_{k+1})] \leq 0 $$

and

$$ B_{1k+1}F(v_{k+1}) = B_{1k+1}[F(v_{k+1}) - F(u_k) - S_{1k}(v_{k+1} - u_k)] \geq 0 $$

as well as

$$ B_{2k+1}F(u_{k+1}) = -B_{2k+1}[F(u_k) - F(u_{k+1}) - S_{2k}(u_k - u_{k+1})] \leq 0 $$

and

$$ B_{2k+1}F(v_{k+1}) = B_{2k+1}[F(v_{k+1}) - F(u_k) - S_{1k}(v_{k+1} - u_k)] \geq 0 $$

it follows finally that

$$ B_{1k+1}F(u_{k+1}) \leq 0 \leq B_{1k+1}F(v_{k+1}) $$

and

$$ B_{2k+1}F(u_{k+1}) \leq 0 \leq B_{2k+1}F(v_{k+1}). $$

If $u^*$ is a solution of $F(u) = 0$ from $[u_k, v_k]$, it yields

$$ T_{1k}(v_k - u^*) = v_k - u^* + B_{1k}[F(v_k) - S_{1k}(v_k - u^*)] $$

$$ \leq v_k - u^* - B_{1k}[F(u^*) - F(v_k) - S_{1k}(u^* - v_k)] $$

$$ \leq v_k - u^*. $$
so that \( v_k - u^* \geq z_{1k} \). But from this yields
\[
u^* \leq v_{k+1}.
\]
Analogously one shows
\[
u_{k+1} \leq u^*,
\]
and all assertions of the theorem are proved.

Remark 1. The condition, that \( K_V \) has to be strongly minihedral, can be
replaced by one of the following assumptions

V is a POB and \( W \) a real Banach space, \( K_V \) is regular. \( F, B_{ik} \) and \( S_{ik} \) are
for \( i = 1, 2 \) and all \( k \in \mathbb{N}_0 \) continuous and the composed mappings \( B_{ik}S_{ik} \) are
inverse-isotone (comp. [8]).

V is a POB and \( W \) a real Banach space, \( K_V \) is normal. \( F \) and \( S_{ik} \) are for
\( i = 1, 2 \) and all \( k \in \mathbb{N}_0 \) continuous and \( B_{ik} \) completely continuous and the composed
mappings \( B_{ik}S_{ik} \) are inverse-isotone (comp. [8]).

Remark 2. From [12] or [13] we have

If \( V \) is a POB with a normal cone, then we have the error estimate
\[
\forall k \in \mathbb{N}_0 : \max\{||u^* - u_k||, ||u^* - v_k||\} \leq C||u_k - v_k||.
\]
If \( K_V \) is regular, then the sequences \( \{u_k\}_{k \in \mathbb{N}_0}, \{v_k\}_{k \in \mathbb{N}_0} \) are convergent.

If \( F \) is continuous and the \( S_{ik} \) are uniformly bounded with respect to \( k \), then the
limits of the sequences are solutions of the equation \( F(u) = 0 \).

Remark 3. The order of convergence of our iteration methods is discussed in
[8] and [12].

The results of Remark 1 were obtained by Potra in [7] for a certain class of iteration
methods including the cases I. for (3), II. for (4) and IV. for (3) and (4). Potra
assumes that \( B_1 = B_2 \) and \( S_1 = S_2 \), so that the same linear operator can be used
to compute both iterated sequences. Then some of the conditions in I.–IV. can be
simplified. This is also possible in our case III., which is not contained in [7] and
which is treated in [8] under stronger conditions than in Theorem 2. By this way
the cpu-time used to solve (3) and (4) is minimized in many cases. On a parallel
computer this is of no importance, moreover the convergence can be improved for
case I. if I. is replaced by
\[ I^*: S_{2k} = S_2(u_k, v_{k+1}) \]
and for II. by
\[ II^*: S_{1k} = S_1(u_{k+1}, v_k). \]
In these cases one first computes \( v_1 \) or respectively, \( u_1 \), and afterwards simultaneously
\( u_k \) and \( v_{k+1} \) or, respectively, \( v_k \) and \( u_{k+1} \). The assumptions I. and II. have to be
modified as follows, where the second condition in I. for \( u_0 \) becomes stronger and for \( v_0 \) is weakened as well as the fourth and the fifth condition (similar for II.).

**Corollary 1.** Let \( F, D, V, W \) as well as \( S_1 \) and \( B_i, i = 1, 2 \) be given as in Theorem 2. Instead of I. and II. we suppose that:

1°. \( B_1(v_0, v_0)F(u_0) \leq 0 \leq B_1(v_0, v_0)F(v_0) \)

\[ \forall z \in [u_0, v_0]: \ B_2(u_0, z)F(u_0) \leq 0 \]

\[ \forall u, v \in [u_0, v_0], u \leq v, z \in \{u, v\} : \]

\[ 0 \leq B_1(z, z)[F(u) - F(v) - S_1(v, v)(u - v)] \]

\[ \forall u, v, w \in [u_0, v_0], u \leq w \leq v : \]

\[ 0 \leq B_1(v, v)[F(u) - F(w) - S_2(u, v)(u - w)] \]

\[ \forall u, v, w \in [u_0, v_0], u \leq w \leq v : \]

\[ 0 \leq B_2(u, v)[F(u) - F(w) - S_2(u, v)(u - w)] \]

\[ \forall u, v, w, z \in [u_0, v_0], u \leq w \leq z \leq v : \]

\[ 0 \leq B_2(w, z)[F(u) - F(w) - S_2(u, v)(u - w)] \]

\[ \forall u, v, z \in [u_0, v_0], u \leq z \leq v : \]

\[ 0 \leq B_2(u, z)[F(z) - F(v) - S_1(v, v)(z - v)] \]

II°. \( \forall z \in [u_0, v_0] : \ 0 \leq B_1(z, v_0)F(v_0) \)

\[ B_2(u_0, u_0)F(u_0) \leq 0 \leq B_2(u_0, u_0)F(v_0) \]

\[ \forall u, v, w \in [u_0, v_0], u \leq w \leq v : \]

\[ 0 \leq B_1(u, v)[F(u) - F(w) - S_1(u, v)(u - w)] \]

\[ \forall u, v, w, z \in [u_0, v_0], u \leq w \leq z \leq v : \]

\[ 0 \leq B_1(w, z)[F(z) - F(v) - S_1(u, v)(z - v)] \]

\[ \forall u, v, z \in [u_0, v_0], u \leq z \leq v : \]

\[ 0 \leq B_1(z, v)[F(u) - F(z) - S_2(u, u)(u - z)] \]

\[ \forall u, v \in [u_0, v_0], u \leq v, z \in \{u, v\} : \]

\[ 0 \leq B_2(z, z)[F(u) - F(v) - S_2(u, u)(u - v)] \]

\[ \forall u, v \in [u_0, v_0], u \leq z \leq v : \]

\[ 0 \leq B_2(u, u)[F(z) - F(v) - S_1(u, v)(z - v)]. \]

Then the assertions of Theorem 2 are valid.

**Proof.** The proof is done again for case I°. The existence of \( v_{k+1} \) follows as in the proof of Theorem 2. From the second condition in I°. it follows that \( B_2(u_0, v_1)F(u_0) \leq 0 \). For \( k > 0 \) this yields that \( B_{2k}F(u_k) = -B_{2k}[F(u_{k-1}) - F(u_k) - S_{2k}(u_{k-1} - u_k)] \leq 0. \)
Now it follows for \( k \geq 0 \) that \( B_{2k} F(v_{k+1}) \geq 0 \) because \( 0 \leq B_{2k}[F(v_{k+1}) - F(v_k) - S_1(v_{k+1} - v_k)] \). By this way we obtain \( T_2(v_{k+1} - u_k) = v_{k+1} - u_k - B_{2k}[F(u_k) - F(v_{k+1}) - S_2(u_k - v_{k+1})] - B_{2k} F(v_{k+1}) \leq v_{k+1} - u_k \), so that the existence of \( u_{k+1} \) with the property (5) can be shown as in the proof of Theorem 2. As in that proof it follows also that \( B_{1k+1} F(u_{k+1}) \leq 0 \leq B_{1k+1} F(v_{k+1}) \). The same inequalities for \( B_{2k+1} \), namely \( B_{2k} F(u_k) \leq 0 \leq B_{2k} F(v_{k+1}) \), has been already shown.

Finally, the fact that every solution of \( F(u) = 0 \) from \([u_0, v_0]\) lies in \([u_k, v_k]\), \( k \in \mathbb{N} \) follows as in the proof of Theorem 2 again. \( \square \)

If the mappings \( S_{ik} \) have additional properties in the sense of Def. 8 or Def. 9, we have one more corollary.

**Corollary 2** ([12], [13]). Let \( F, D, V, u_0 \) and \( v_0 \) be given as in Theorem 2, let \( W \) be a POS. Furthermore, let

\[
F(u_0) \leq 0 \leq F(v_0).
\]

and let the mappings \( S_i: [u_0, v_0] \times [u_0, v_0] \rightarrow L(V, W) \) have the following properties:

\( \forall u, v \in [u_0, v_0] \) with \( u \leq v \):

\[
(6) \quad S_i(u, v)(u - v) \leq F(u) - F(v),
\]

\( S_i(u, v) \) is inverse-isotone and the image of \( V \) under \( S_i(u, v) \) contains that of \([u_0, v_0]\) under \( F \). Finally

\[
\begin{align*}
\text{I'} & & S_1(\cdot, v) & \text{and} & S_2(u, \cdot) & \text{are isotone.} \\
\text{II'} & & -S_1(\cdot, v) & \text{and} & -S_2(u, \cdot) & \text{are isotone.} \\
\text{III'} & & -S_1(\cdot, v) & \text{and} & S_2(u, \cdot) & \text{are isotone.} \\
\text{IV'} & & S_1(\cdot, v) & \text{and} & -S_2(u, \cdot) & \text{are isotone.}
\end{align*}
\]

Then the assertions of Theorem 2 hold. Furthermore we have for all \( k \in \mathbb{N} \):

\( F(u_k) \leq 0 \leq F(v_k) \).

**Proof.** Since the \( S_{ik} \) are inverse-isotone, they are also injective and because \( F([u_0, v_0]) \subset S_{ik} V \), for each \( k \in \mathbb{N} \) there exist positive inverses \( S_{-1}^{ik}: S_{ik} V \cap S_{2k} V \rightarrow V \). Setting \( B_{ik} := S_{ik}^{1} \) it remains to show that \( F(x) - F(y) - S(v, w)(u - z) \leq 0 \) with \( x, y, v, w, u, z \) being chosen corresponding to the inequalities in I.-IV. This can easily be done in case I. except for the fifth condition for \( z = v \). But only a simple modification of the proof of Theorem 2 using the inverse-isotony of \( S_{ik} \) is necessary to show the assertion (comp. Satz 2 in Section 2.1 in [12]). \( \square \)

**Remark 1.** The conditions I'. and II'. are fulfilled if \( F \) is ddo-convex with \( S_1 = S_2 = S \) or \( S_1 = S_2 = -S \). (6) is fulfilled by a ddo, a gddo as well as by a qddo.

**Remark 2.** Rem. 1 and 2 after Theorem 2 remain valid for Corollary 2.

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Most of the theorems on monotone iteration methods are special cases of Theorem 2 and its corollaries (comp. [8], [12]) except some methods in [7], [8], [10], [11], [14], Brown's method [2] and the interval-iteration methods.

Many examples of nonlinear operator equations which can be treated via Theorem 2 and its corollaries can be found in the cited references on monotone including iteration methods. Especially in [9] a simple example is given, where Theorem 2 can be applied, but the \( S_{ik} \) are not inverse-isotone. In [1] and [12] examples are given where \( S_1 \neq S_2 \) and moreover \( F \) is not \( o \)-convex. In the next chapter we apply our theorem to coupled systems of nonlinear two-point boundary value problems.

3. APPLICATION TO NONLINEAR SYSTEMS OF TWO-POINT BOUNDARY VALUE PROBLEMS

In this section problems of the following kind are solved

\[
y'' = f(y, z, z) \\
z'' = g(y, z, z) \\
y(0) = y(b) = z(0) = z(b) = 0,
\]

where \( f, g : [0, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}, \ b \in \mathbb{R}, \ i = 1, 2 \) are continuous functions.

First we consider the following class of nonlinear systems:

\[
y'' = y^2 + y + cz^2 - f \\
z'' = dy^2 + z^2 + 2z - g
\]

with \( b = 1 \), positive real constants \( c, d \) and nonnegative functions \( f, g \). It is easy to see that \( y \equiv z \equiv 0 \) is a subsolution for (7), (8) and \( \bar{y} = c_1 t(1 - t), \bar{z} = c_2 t(1 - t), \ t \in [0, 1] \) are supersolutions, if \( 2c_1 \geq \max_{t \in [0, 1]} f(t) \) and \( 2c_2 \geq \max_{t \in [0, 1]} g(t) \).

It will be shown, that Theorem 2 is applicable to the discrete problem corresponding to (7), (8), while monotonicity results like Corollary 2 are not, if we use the "natural" ddo

\[
S_1[(u, v); (\bar{u}, \bar{v})] := S_2[(u, v); (\bar{u}, \bar{v})] := S[(u, v); (\bar{u}, \bar{v})] :=
\]

\[
\begin{pmatrix}
\frac{1}{h} & 0 & \cdots & c(v_1 + \bar{v}_1) & 0 & \cdots & 0 \\
\frac{1}{h^2} & -\frac{1}{h} & 0 & \cdots & c(v_2 + \bar{v}_2) & 0 & \cdots \\
\frac{1}{h^2} & \frac{1}{h} & -\frac{1}{h} & 0 & \cdots & c(v_3 + \bar{v}_3) & 0 & \cdots \\
\frac{1}{h^2} & \frac{1}{h^2} & \frac{1}{h} & -\frac{1}{h} & \cdots & c(v_4 + \bar{v}_4) & 0 & \cdots \\
\frac{1}{h^2} & \frac{1}{h^2} & \frac{1}{h} & \frac{1}{h} & \cdots & c(v_n + \bar{v}_n) & 0 & \cdots \\
\frac{1}{h^2} & \frac{1}{h^2} & \frac{1}{h} & \frac{1}{h} & \cdots & c(v_{n+1} + \bar{v}_{n+1}) & 0 & \cdots \\
\frac{1}{h^2} & \frac{1}{h^2} & \frac{1}{h} & \frac{1}{h} & \cdots & c(v_{n+2} + \bar{v}_{n+2}) & 0 & \cdots \\
\frac{1}{h^2} & \frac{1}{h^2} & \frac{1}{h} & \frac{1}{h} & \cdots & c(v_{n+3} + \bar{v}_{n+3}) & 0 & \cdots \\
\frac{1}{h^2} & \frac{1}{h^2} & \frac{1}{h} & \frac{1}{h} & \cdots & c(v_{n+4} + \bar{v}_{n+4}) & 0 & \cdots \\
\frac{1}{h^2} & \frac{1}{h^2} & \frac{1}{h} & \frac{1}{h} & \cdots & c(v_{n+5} + \bar{v}_{n+5}) & 0 & \cdots \\
\frac{1}{h^2} & \frac{1}{h^2} & \frac{1}{h} & \frac{1}{h} & \cdots & c(v_{n+6} + \bar{v}_{n+6}) & 0 & \cdots \\
\frac{1}{h^2} & \frac{1}{h^2} & \frac{1}{h} & \frac{1}{h} & \cdots & c(v_{n+7} + \bar{v}_{n+7}) & 0 & \cdots \\
\end{pmatrix}
\]
It is easy to see that $S[(u, v), (\bar{u}, \bar{v})]$ is not an M-matrix, since the $n + 1$-st diagonal consists of positive elements. Moreover, its inverse therefore contains negative elements and thus it cannot be inverse-isotone. Now we verify the assumptions of Theorem 2. We do this for the sake of brevity for the simplest case of discretization by finite differences, i.e. $h = \frac{1}{2}$, but it is true even for each $h = \frac{1}{n}, 1 < n \in \mathbb{N}$. The system to solve is

$$
x_1^2 + 9x_1 + cx_2^2 = f(0.5) \\
dx_1^2 + 10x_2 + x_2^2 = g(0.5)
$$

with

$$
S(x, \bar{x}) = \begin{pmatrix} x_1 + \bar{x}_1 + 9 & c(x_2 + \bar{x}_2) \\
d(x_1 + \bar{x}_1) & x_2 + \bar{x}_2 + 10 \end{pmatrix}
$$

Now a simple calculation shows that if $f, g \in [0, 2]$, then I. of Theorem 2 is fulfilled for $c \in (0, 3.125)$ and $d \in (0, 3.2)$. Similar conditions for $c, d$ ensure the applicability of our theorem if $f, g > 2$ (see the condition for $c_1$ and $c_2$ above) or for smaller $h$. Since the cone obtained by the “natural” componentwise partial order of $\mathbb{R}^n$ is strongly minihedral, the assumptions of Theorem 2 are fulfilled for the discrete problem.

As a numerical example we solve (7), (8) with finite differences, $h = \frac{1}{100}$ for $c = d = 3$ and $f(t) = 2 \cdot \sin \pi t, g(t) = 1 + t$.

Tab. 1 Inclusions for Example (7), (8)

<table>
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<th>$t$</th>
<th>$\bar{x}_1$</th>
<th>$\bar{x}_2$</th>
<th>$y(t)$</th>
<th>$\bar{y}_2$</th>
<th>$\bar{y}_1$</th>
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</thead>
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<tr>
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<td>0.054649</td>
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<table>
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<th>$\bar{x}_2$</th>
<th>$z(t)$</th>
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<td>0.146549</td>
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<tr>
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<tr>
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<td>0.060632</td>
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</table>

In [3] the following predator-prey problem arising from the Volterra-Lotka model is presented

$$
(9) \quad u'' + u(2 - u - v) = 0 \\
(10) \quad v'' + v(3 + u - 6v) = 0
$$

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where \( b := \pi \). The existence of a positive solution is proved by Theorem 4 in [3] by constructing a supersolution and a subsolution, but the particular super- and subsolution given there are wrong. One obtains correctly

\[
\begin{align*}
    u_0(x) &= \frac{4}{7} \sin x, \quad \bar{u}_0 \equiv 2, \\
    v_0(x) &= \frac{3}{7} \sin x, \quad \bar{v}_0 \equiv \frac{5}{6}.
\end{align*}
\]

The supersolution does not satisfy the boundary condition, but nevertheless our method can be applied and using the \textit{qddo} \( S_i \)

\[
S_i[(u, v); (\tilde{u}, \tilde{v})] := \begin{pmatrix}
    -\frac{d^2}{dx^2} - 2 + (u + \tilde{u}) - v & -\tilde{u} \\
    0 & -\frac{d^2}{dx^2} - 3 + 6(v + \tilde{v})
\end{pmatrix}
\]

condition \( \Gamma' \) from Corollary 2 is fulfilled. Moreover, by [5] the \textit{qddo}

\[
\tilde{S}_i[(u, v); (\tilde{u}, \tilde{v})] := \begin{pmatrix}
    -\frac{d^2}{dx^2} + (u + \tilde{u}) - v & -\tilde{u} \\
    0 & -\frac{d^2}{dx^2} + 6(v + \tilde{v})
\end{pmatrix}
\]

is a-priori inverse-isotone and fulfils \( \Gamma' \). Here \( V = C^2[0, \pi] \times C^2[0, \pi] \) with the pointwise order in each component and \( D = \{(u, v) \in V \mid u(0) = v(0) = u(\pi) = v(\pi) = 0\} \). The fixed-point result of Theorem 1 applied to \( T_{ik} \) in the proof of Theorem 2 is obtained by using the equivalent system of integral equations (comp. [4], Section 7.4). We obtain the improved supersolution

\[
\begin{align*}
    \bar{u}_1(x) &= -\frac{154}{85} \cosh \sqrt{\frac{17}{6}} x + \beta \sinh \sqrt{\frac{17}{6}} x + \frac{2}{5} \cosh \sqrt{7} x - \frac{2}{5} \alpha \sinh \sqrt{7} x + \frac{24}{17} \\
    \bar{v}_1(x) &= \frac{5}{6} \left[1 - \cosh \sqrt{7} x + \alpha \sinh \sqrt{7} x \right],
\end{align*}
\]

where

\[
\begin{align*}
    \alpha &= \frac{\cosh \sqrt{7} \pi - 1}{\sinh \sqrt{7} \pi} \\
    \beta &= \frac{1}{\sinh \sqrt{\frac{17}{6}} \pi} \left(\frac{154}{85} \cosh \sqrt{\frac{17}{6}} \pi - \frac{2}{5} \cosh \sqrt{7} \pi + \frac{2}{5} \alpha \sinh \sqrt{7} \pi - \frac{24}{17}\right),
\end{align*}
\]

which fulfills the boundary conditions and reduces the error in the \( \|\cdot\|_{\infty} \)-norm (comp. Rem. 2 after Theorem 2 with \( C = 1 \) for \( C[0, \pi] \)) by more than a half. Further approximations cannot be computed analytically. For \( k > 1 \) numerical methods must be used.

Since the computation of the supersolutions is independent of the subsolutions, we first determine the supersolutions up to five correct digits and use the best computed supersolution instead of \( \bar{u}_k \) and \( \bar{v}_k \) to determine the subsolutions, so we need 24
Table 2: Supersolutions for Example (9), (10)

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\bar{u}_k(0.25\pi)$</th>
<th>$\bar{u}_k(0.5\pi)$</th>
<th>$\bar{\nu}_k(0.25\pi)$</th>
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</table>

Iteration steps, which is less than with the original method. The mesh size is $h = \frac{\pi}{256}$, so that the discretization error is smaller than $10^{-5}$. Again the iteration stops if five digits are stable (after rounding) and the inclusion error becomes

$$||\bar{u}_{27} - \bar{u}_{24}||_{\infty} \leq 10^{-5}, ||\bar{\nu}_{27} - \bar{\nu}_{24}||_{\infty} \leq 10^{-5}. $$

Acknowledgement. This paper contains parts of the author’s Habilitationsschrift.

References

Tab. 3 Subsolutions for Example (9), (10)

<table>
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<th>$u_k(0.5\pi)$</th>
<th>$v_k(0.25\pi)$</th>
<th>$v_k(0.5\pi)$</th>
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Zusammenfassung

AFFIN-INVARIANTE MONOTONE ITERATIONSVERFAHREN MIT
ANWENDUNG AUF SYSTEME NICHTLINEARER
ZWEIPUNKTRANDWERTPROBLEME

RUDOLF L. VOLLMER


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