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GLOBAL SOLUTION OF VISCOUS COMPRESSIBLE BAROTROPIC MULTIPOLAR GAS IN A FINITE CHANNEL WITH NONZERO INPUT AND OUTPUT

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Summary. The paper contains the proof of global existence of weak solutions of the viscous compressible barotropic gas for the initial-boundary value problem in a finite channel.

Keywords: multipolar gas, compressible fluid, initial boundary-value problem, barotropic gas, global existence of weak solutions

AMS classification: 76N

1. Introduction

This paper is another contribution to the problem of solvability of a viscous compressible liquid closely related to the papers by J. Nečas, A. Novotný, M. Šilhavý [1], J. Nečas, A. Novotný, M. Šilhavý [2], Š. Matušů-Nečasová [3].

The global existence of a strong solution for k-polar liquids with $k \geqslant 3$ is proved here under general initialization data in the time space cylinder $Q_t = I \times \Omega$ (where $I = (0,t)$, $t > 0$ and $\Omega \subset \mathbf{R}^N$ $(N = 2, 3)$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$). Further, we consider nonzero boundary conditions on ϱ and v. Also, a brief comment on the uniqueness of solution is included.

2. Formulation of the problem

For a polytropic gas it is true that $p \in C^1([0,\infty])$ depends only upon ϱ. The function $p(\varrho)$ satisfies the following assumptions:

a) the expression $P(\varrho) = \varrho \int_{\varrho_0}^{\varrho} \frac{p(\sigma)}{\sigma^2}\,\mathrm{d}\sigma$ exists $\forall \varrho > 0$;

b) $\sigma \frac{\mathrm{d}P}{\mathrm{d}\sigma}(\sigma) - P(\sigma) = p(\sigma), \sigma > 0$.

Remark 1.1. The function $p(\varrho) = k\varrho^{\alpha}$, $k > 0$, $\alpha > 0$ satisfies a), b). The isothermic case $p(\varrho) = k\varrho$ is not included.

We consider $V = W^{k,2}(\Omega, \mathbf{R}^N) \cap W_0^{1,2}(\Omega, \mathbf{R}^N)$ and the bounded V-coercitive bilinear form

$$(2.1)\quad ((v,w)) = \int_{\Omega} \sum_{m=1}^{k} A^m_{iji_1i_2\ldots i_mj_1j_2\ldots j_m} \frac{\partial^m v_i}{\partial x_{i_1}\partial x_{i_2}\ldots\partial x_{i_m}} \cdot \frac{\partial^m v_i}{\partial x_{j_1}\partial x_{j_2}\ldots\partial x_{j_m}}\,\mathrm{d}x,$$

where $A^m_{iji_1\ldots i_mj_1\ldots j_m}$, $m = 1, \ldots, k$, i, j, i_l, $j_l = 1, \ldots, N$ are constants; for $m = 1$ we have here merely the combinations $e_{ij}(v)$, $e_{ij}(w)$ $\left(e_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right)$.

Let us assume that $A^m_{iji_1\ldots i_mj_1\ldots j_m}$ are invariant under permutations of subscripts i, j, i_1, $\ldots$, i_m, j_1, $\ldots$, j_m. Evidently, we assume that $\forall v \in V$

$$(2.2)\quad ((v,v)) \geqslant \alpha_1 \|v\|^2_{W^{k,2}(\Omega,\mathbf{R}^N)}, \alpha_1 > 0.$$

This follows, e.g. from the conditions

$$(2.3)\quad A^1_{iji_1j_1} \frac{\partial v_i}{\partial x_{i_1}} \frac{\partial v_j}{\partial x_{j_1}} \geqslant \alpha_2 e_{ii_1}(v) e_{ij_1}(v), \alpha_2 > 0,$$

$$(2.4)\quad \sum_{m=2}^{k} A^m_{iji_1\ldots i_mj_1\ldots j_m} J^i_{i_1\ldots i_m} J^i_{j_1\ldots j_m} \geqslant \alpha_2 \sum_{m=2}^{k} J^i_{i_1\ldots i_m} J^i_{j_1\ldots j_m}$$

for all real vectors $(J^i_{i_1\ldots i_m})^N_{i,i_1,\ldots,i_m} (m = 2, \ldots k)$.

Let us take into account that the problem

$$(2.5)\quad ((v,w)) = \int_{\Omega} f_i w_i\,\mathrm{d}x, \quad f \in L^2(\Omega, \mathbf{R}^N)\ \forall w \in V$$

has a solution belonging to $W^{2k,2}(\Omega, \mathbf{R}^N)$ such that $\|v\|_{W^{2k,2}(\Omega,\mathbf{R}^N)} \leqslant \alpha_3 \|f\|_{L^2(\Omega,\mathbf{R}^N)}$, $\alpha_3 > 0$.

Let us consider the standard symetric stress tensor

$$(2.6)\quad \tau_{ij} = -p\delta_{ij} + \tau^v_{ij}.$$

The continuity equation has the form

$$\frac{\partial \varrho}{\partial t} + \frac{\partial}{\partial x_i}(\varrho v_i) = 0. \tag{2.7}$$

The equation of motion combined with (2.7) yield

$$\frac{\partial}{\partial t}(\varrho v_i) + \frac{\partial}{\partial x_j}(\varrho v_i v_j + p(\varrho)\delta_{ij} - \tau_{ij}^v(v)) = 0. \tag{2.8}$$

External forces will be neglected. In our situation

$$\frac{\partial}{\partial x_j}(\tau_{ij}^v(v)) = \sum_{m=1}^{k}(-1)^{m+1}A^m_{iji_1\dots i_m j_1\dots j_m}\cdot\frac{\partial^{2m} v_i}{\partial x_{i_1}\dots\partial x_{i_m}\partial x_{j_1}\dots\partial x_{j_m}}. \tag{2.9}$$

In addition to the initial conditions

$$\varrho(0) = \varrho_0, \quad v(0) = v_0, \tag{2.10}$$

we consider the conditions on the finite channel as in [3]

$$v = v_0 \quad \text{on } \Gamma_{inp} \cup \Gamma_{out}, \tag{2.11}$$

where

$$v_0\nu < 0 \quad \text{on } \Gamma_{\text{inp}}, \tag{2.11'}$$

$$v_0\nu > 0 \quad \text{on } \Gamma_{\text{out}}, \tag{2.11''}$$

$$v = 0 \quad \text{on } \partial\Omega - (\Gamma_{\text{inp}} \cup \Gamma_{\text{out}}), \tag{2.12}$$

$$\varrho = \varrho_0 \quad \text{on } \Gamma_{\text{inp}}, \tag{2.13}$$

and the unstable boundary conditions

$$\begin{gathered}\sum_{m=1}^{k}\sum_{s=1}^{m-k}(-1)^{s+1}\int_{\partial\Omega} A^m_{iji_1\dots i_m j_1\dots j_m}\frac{\partial^{m+s} v_i}{\partial x_{i_1}\dots\partial x_{i_m}\partial x_{j_1}\dots\partial x_{j_s}} \\ \times\frac{\partial^{m-s-1} z_j}{\partial x_{j_{s+2}}\dots\partial x_{j_m}}\nu_{j_{s+1}}\,\mathrm{d}S = 0 \\ \forall z \in C^\infty(\bar\Omega, \mathbf{R}^N)\cap W_0^{1,2}(\Omega,\mathbf{R}^N).\end{gathered} \tag{2.14}$$

Let us assume that v_0 is a function such that there exists its extension to the entire $Q_t = \Omega \times (0,t)$. Let

$$\frac{\partial v_i}{\partial x_i} \geqslant 0 \tag{2.14'}$$

holds.

First apriori estimates will be treated.

Lemma 2.1. *Let v, v_0 be sufficiently smooth then*

$$\int_{\Omega_t} \varrho \,\mathrm{d}x \leqslant \int_{\Omega_0} \varrho_0 \,\mathrm{d}x - \int_0^t \int_{\Gamma_{\mathrm{inp}}} \varrho_0 v_i \nu_i \,\mathrm{d}S \,\mathrm{d}t \tag{2.15}$$

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_t} \varrho |w|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\Omega_0} \varrho |w|^2 \,\mathrm{d}x \\ &+ \int_{Q_t} \left(\varrho \frac{\partial v_i^0}{\partial t} w_j + \varrho v_j^0 \frac{\partial v_i^0}{\partial x_j} w_i + \frac{\partial v_i^0}{\partial x_j} \varrho w_j w_j \right) \mathrm{d}x \,\mathrm{d}t \\ &+ \int_0^t \int_{\Gamma_{\mathrm{inp}}} v_i^0 \nu_i P(\varrho_0) \,\mathrm{d}t \,\mathrm{d}S + \int_0^t \int_{\Gamma_{\mathrm{out}}} v_i \nu_i P(\varrho) \,\mathrm{d}S \,\mathrm{d}t \\ &- \int_{Q_t} \frac{\partial v_i^0}{\partial x_i} P(\varrho) \,\mathrm{d}x \,\mathrm{d}t + \int_{Q_t} \frac{\partial v_i^0}{\partial x_i} p(\varrho) \,\mathrm{d}x \,\mathrm{d}t \\ &+ \int_{\Omega_t} P(\varrho) \,\mathrm{d}x - \int_{\Omega_0} P(\varrho_0) \,\mathrm{d}x + \int_0^t ((v, w)) \,\mathrm{d}t = 0. \end{aligned} \tag{2.16}$$

Proof. (2.15) see M-N [3]. (2.16) With the aid of [3] we obtain the first five terms; the last term follows from (2.1) and (2.9); to the term $\int_{Q_t} \frac{\partial}{\partial x_j}(p(\varrho)\delta_{ij}) \,\mathrm{d}x \,\mathrm{d}t$ we apply the Green theorem and add the continuity equation (2.7) multiplied by $\frac{\mathrm{d}P}{\mathrm{d}\varrho}$. □

Lemma 2.2. *Let us assume $\varrho_0 > 0$, $\varrho_0 \in C^1(\bar{Q}_t)$, $v_0 \in L^2(I, W^{k,2}(\Omega))$, $v^0 \in C^1(Q_t)$, $\varrho \in L^\infty(I, L^1(\Omega))$, $w \in L^2(I, W^{k,2}(\Omega, \mathbf{R}^N))$ then*

$$\int_{\Omega_t} \frac{1}{2}\varrho |w|^2 + \int_{\Omega_t} P(\varrho) + \int_0^t ((v, w)) \leqslant \int_{\Omega_0} \frac{1}{2}\varrho |w|^2 + \int_{\Omega_0} P(\varrho_0) + C. \tag{2.17}$$

Proof. We start from (2.16). The terms $\int_{Q_t} \varrho \frac{\partial v_i^0}{\partial t} w_j \,\mathrm{d}x \,\mathrm{d}t$, $\int_{Q_t} \varrho w_j \frac{\partial v_i^0}{\partial x_j} v_j^0 \,\mathrm{d}x \,\mathrm{d}t$ are estimated similarly as in [3], i.e. by $c\sqrt{t}\|w\|_{L^2(I, W^{k,2}(\Omega))}$. The term $\int_{Q_t} \varrho w_j \frac{\partial v_i^0}{\partial x_j} w_i$ is estimated with the aid of the term $\int_{\Omega_t} \frac{1}{2}\varrho |w|^2$ with the use of the Gronwall lemma. To the term $\int_0^t ((v, w)) \,\mathrm{d}t$ we apply (2.2). Now $\int_{Q_t} \frac{\partial v_i^0}{\partial t} P(\varrho)$ is estimated again with the aid of the Gronwall lemma (we exploit the term $\int_{\Omega_t} P(\varrho)$). For the term $\int_{Q_t} \frac{\partial v_i^0}{\partial x_i} p(\varrho)$ we make use of the fact that we have $\frac{\partial v_i^0}{\partial x_i} \geqslant 0$ and at the same time $p(\varrho) \geqslant 0$; hence it follows that it is bounded. □

Now we present the weak formulation of the problem (2.9):

$$\int_\Omega \frac{\partial}{\partial t}(\varrho v_i) z_i \,\mathrm{d}x + ((v,z)) = \int_\Omega \Big(\varrho v_i v_j \frac{\partial z_i}{\partial x_j} + p(\varrho)\frac{\partial z_j}{\partial x_j}\Big)\,\mathrm{d}x \tag{2.18}$$

a.e. in I, $\forall z \in V$.

3. The Galerkin method

We select an orthonormal base $\{z^r\}_{r=1}^{+\infty}$ in $L^2(\Omega, \mathbf{R}^N)$ which solves

$$((v, z^r)) = \lambda_r \int_\Omega v_i z_i^r \,\mathrm{d}x \quad \forall z \in V \ (\lambda_1 < \lambda_2 < \lambda_3 < \ldots). \tag{3.1}$$

The regularity of the elliptic problem (3.1) implies that $z^r \in L^\infty(\bar\Omega, \mathbf{R}^N)$ holds. Denote by P_m the orthogonal projector from $L^2(\Omega, \mathbf{R}^N)$ into $L^2_n(\Omega, \mathbf{R}^N) = \operatorname{span}(w^1, w^2, \ldots, w^n)$.

Let $v^n = \sum_{r=1}^n c_r(t) z^r + v^0$. We are looking for the solution (ϱ^n, v^n), $\varrho^n \in C^1(\bar I, C^{k-3}(\bar\Omega))$; $c = (c_1, \ldots, c_n) \in C^1(\bar I, \mathbf{R}^N)$ satisfying

$$\frac{\partial \varrho^n}{\partial t} + \frac{\partial}{\partial x_i}(\varrho^n v_i^n) = 0, \tag{3.2}$$

$$\int_\Omega \frac{\partial}{\partial t}(\varrho^n v_i^n) z_i^r \,\mathrm{d}x + ((v^n, z^r)) = \int_\Omega \Big(\varrho^n v_i^n v_j^n \frac{\partial z_i^r}{\partial x_j} + p(\varrho^n)\frac{\partial z_i^r}{\partial x_j}\Big)\,\mathrm{d}x$$
$$- \int_{\Gamma_{\mathrm{inp}} \cup \Gamma_{\mathrm{out}}} \varrho^n v_i^0 v_j^n \nu_i z_i^r \,\mathrm{d}S - \int_{\partial\Omega} p(\varrho^n) \nu_i z_i^r \,\mathrm{d}S, \tag{3.3}$$

(3.4)
$$\varrho(0) = \varrho_0, \quad \varrho = \varrho_0 \text{ on input}, \quad v = v^0 \text{ on } \Gamma_{\mathrm{inp}} \cup \Gamma_{\mathrm{out}}, \quad c_r(0) = \int_\Omega w^n(0) z_i^r \,\mathrm{d}x.$$

Such solutions exists, see [3].

Now, applying Lemmas 2.1, 2.2 we obtain similarly

$$\frac{1}{2}\int_{\Omega_t} \varrho^n |w^n|^2 \mathrm{d}x + \int_{\Omega_t} P(\varrho^n)\,\mathrm{d}x + \int_0^t ((v^n, w^n))\,\mathrm{d}t$$
$$\leqslant \frac{1}{2}\int_{\Omega_0} \varrho_0 |w^n|^2 \mathrm{d}x + \int_{\Omega_0} P(\varrho_0)\,\mathrm{d}x + \text{const.}, \tag{3.5}$$
$$w^n(0) = \sum_{r=1}^n c_r(0) z^r,$$

where estimates of the following terms are included in the constant:

$$\int_{Q_t} \Big(\varrho^n \frac{\partial v_i^0}{\partial t} w_j^n + \varrho^n v_j^0 \frac{\partial v_i^0}{\partial x_j} w_i^n + \frac{\partial v_i^0}{\partial x_j} \varrho^n w_i^n w_j^n\Big)\, \mathrm{d}x\, \mathrm{d}t,$$

$$\int_0^t \int_{\Gamma_{\mathrm{inp}}} v_i \nu_i P(\varrho_0^n)\, \mathrm{d}S \mathrm{d}t, \quad -\int_{Q_t} 2 \frac{\partial v_i^0}{\partial x_i} P(\varrho^n)\, \mathrm{d}x\, \mathrm{d}t, \quad \int_{Q_t} \frac{\partial v_i^0}{\partial x_i} p(\varrho^n)\, \mathrm{d}x\, \mathrm{d}t,$$

$$\int_0^t \int_{\Gamma_{\mathrm{out}}} v_i \nu_i P(\varrho^n)\, \mathrm{d}S\, \mathrm{d}t.$$

From (3.5) we obtain

$$\|\varrho^n |w^n|^2\|_{L^\infty(I,L^1(\Omega))} \leqslant c_1, \quad c_1 > 0, \tag{3.6}$$

$$\|w^n\|_{L^2(I,W^{k,2}(\Omega,\mathbb{R}^N)} \leqslant c_1, \quad c_1 > 0. \tag{3.7}$$

Evidently, (3.6) and (3.7) yield

$$\|\varrho^n |v^n|^2\|_{L^\infty(I,L^1(\Omega))} \leqslant c_1, \tag{3.6'}$$

$$\|v^n\|_{L^2(I,W^{k,2}(\Omega,\mathbb{R}^N))} \leqslant c_1. \tag{3.7'}$$

Now, similarly as in [3] we compute ϱ^n for the given $v^n = \sum_{r=1}^{n} c_r z_i^r + v_0$. Let

$$\dot{x}(\tau) = -v^n(t-\tau, x^n(\tau)), \quad x^n(0) = x, \quad x \in \Omega. \tag{3.8}$$

ϱ^n can be obtained by integration along the characteristics. These characteristics pass through Q_t and end either in Ω_0 or Γ_{inp}. Here it is possible to exploit the fact that we know ϱ^n on Ω_0, Γ_{inp}. For $\tau \in I_{\tilde{t}}$ where $I_{\tilde{t}} \subset I$ and $I_{\tilde{t}} = (0, \tilde{t}), \tilde{t} > 0, x \to x(\tau)$ is a local difeomorphismus of $\bar{\Omega}$ into $\bar{\Omega}$, and for $\sigma_n = \ln \varrho^n$ we have

$$-\frac{\partial \sigma_n}{\partial t} + \frac{\partial \sigma_n}{\partial x_i} v_i^n = \frac{\partial v_i^n}{\partial x_i}(t-\tau, x^n(\tau)). \tag{3.9}$$

Then we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \sigma_n(t-\tau, x^n(\tau)) = \frac{\partial v_i^n}{\partial x_i}(t-\tau, x^n(\tau)). \tag{3.10}$$

Further, integration of (3.10) yields

$$\varrho^n(t,x) = \varrho_0(t-\tilde{t}, x^n(\tilde{t})) \exp\Big(-\int_0^{\tilde{t}} \frac{\partial}{\partial x_j} v_i^n(\tau, x^n(\tau)) \mathrm{d}\tau\Big), \tag{3.11}$$

where $x = x^n(0)$, $x^n(\tilde{t}) = y$.

Just one characteristic passes through the point $[0,t]$. We denote by $\tilde{t}$ the time where we reach when we go along the characteristic from the point $[0,t]$ to point , which lies in Ω_0 or Γ_{inp}.

The Sobolev imbedding theorem $(W^{k,2}(\Omega,\mathbf{R}^N)) \subset\subset C^1(\bar{\Omega},\mathbf{R}^N))$ for $2k > n$ together with (3.7′) and (3.11) implies

$$\varrho^n \geqslant \varepsilon > 0 \quad \text{almost everywhere in } Q_t, \tag{3.12}$$

$$\|\varrho^n\|_{L^\infty(Q_t)} \leqslant k_2, \quad k_2 > 0. \tag{3.13}$$

Let $\hat{\sigma}^n(t-\tau,y) = \sigma^n(t-\tau,x^n(t,y))$, $\hat{\varrho}^n(t-\tau,y) = \varrho^n(t-\tau,x^n(t,y))$, where $x^n(t,y)$ is a solution of (3.8) such that $x^n(\tilde{t},y) = y$.

Let us put $G = \det \frac{\partial x_r^n}{\partial y}$. From the continuity equation in the Lagrange coordinates where $\varrho(t-\tau,y)G(t-\tau,y) = \varrho(y)$ holds, we have

$$0 < \varepsilon_1 < G(t-\tau,y) < k_3, \qquad k_3 > 0. \tag{3.14}$$

For $k \geqslant 3$ we apply the Gronwall inequality and the imbedding $C^{k-2}(\bar{\Omega}) \subset W^{k,2}(\Omega)$. Hence we obtain, from (3.10),

$$\left\| \frac{\partial^s x_i}{\partial y_1^{s_1} \dots \partial y_N^{s_1}} \right\|_{L^\infty(Q_t)} \leqslant k_4, \tag{3.15}$$
$$s = s_1 + \dots + s_N, \; s \leqslant k-2 \quad (k_4 > 0).$$

By (3.14), for the inverse function $y(t,\cdot)$ we have

$$\left\| \frac{\partial^s y_i}{\partial x_1^{s_1} \dots \partial x_n^{s_N}} \right\|_{L^\infty(Q_t)} \leqslant k_4, \tag{3.16}$$
$$s = s_1 + \dots + s_n, \quad s \leqslant k-2.$$

Now, it is necessary to take into account a certain "nonuniqueness" of ϱ^n in the "corners" of Q_t. If we follow the characteristic from $[x,t]$, we reach either Ω_0 or Γ_{inp}. We use idea of Theorem 3.2 from [3]. A surface S is generated which is described by the trajectories of the equation $\dot{x}^n = v^n(t,x^n(t)), x \in \Gamma_{\text{inp}}$. Here we assume that Γ_{inp} is closed. This surface divides the time-space cylinder into Q_1 and Q_2. Similarly as in [3] it is possible to prove that $\varrho^n \in W^{1,\infty}(Q_1), \varrho^n \in W^{1,\infty}(Q_2)$, and thus $\varrho_n \in W^{1,\infty}(Q_t)$ by [4]. The verification of the fact that S is a surface can again be found in [3]. Consequently, we conclude that $\varrho^n \in W^{1,\infty}(Q_t)$ and $\varrho^n \in C(\bar{Q}_t) \cap C^1(I, C^d(\Omega))$. Now we assume $k = 3$. Then

$$\|\varrho^n\|_{L^\infty(I,W^{k-2,q}(\Omega))} \leqslant k_5, \quad k_5 > 0, \tag{3.17}$$

$$\left\| \frac{\partial \varrho^n}{\partial t} \right\|_{L^2(I,W^{k-3,q}(\Omega))} \leqslant k_5, \quad k_5 > 0, \tag{3.18}$$

where $1 \leqslant q < +\infty$ if $N = 2$, $1 \leqslant q \leqslant 6$ if $N = 3$. For $k > 3$ we have

$$\varrho^n \in W^{1,\infty}(Q_t) \quad \text{and} \quad \varrho^n \in C(\bar{Q}_t) \cap C^1(I, C^d(\Omega)). \tag{3.18'}$$

We multiply (3.3) by c_r, integrate over $(0,t)$ and sum over r ($r = 1, \ldots, n$). Then we obtain (we use test function $\frac{\partial w_i^n}{\partial t}$)

$$\begin{aligned}\int_0^t\int_\Omega \varrho^n \frac{\partial v_i^n}{\partial t}\frac{\partial w_i^n}{\partial t}\,\mathrm{d}x\,\mathrm{d}t &+ ((v^n(t), w^n(t)))\\ &= ((v^n(0), w^n(0))) - \int_0^t\int_\Omega \varrho^n v_j^n \frac{\partial v_i^n}{\partial x_j}\frac{\partial w_i^n}{\partial t}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad - \int_0^t\int_\Omega \frac{\mathrm{d}p}{\mathrm{d}\varrho}(\varrho^n)\frac{\partial \varrho^n}{\partial x_j}\frac{\partial w_i^n}{\partial t}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.19}$$

We modify (3.19) then we obtain

$$\begin{aligned}\int_0^t\int_\Omega \varrho^n \left|\frac{\partial w_i^n}{\partial t}\right|^2 \mathrm{d}x\,\mathrm{d}t &+ ((v^0(t), w^n(t))) + ((w^n(t), w^n(t)))\\ &= ((v^n(0), w^n(0))) - \int_0^t\int_\Omega \Big(\varrho^n \frac{\partial v_i^0}{\partial t}\frac{\partial w_i^n}{\partial t} + \varrho^n v_j^0 \frac{\partial v_i^0}{\partial x_j}\frac{\partial w_i^n}{\partial t}\\ &\quad + \varrho^n w_j^n \frac{\partial v_i^0}{\partial x_j}\frac{\partial w_i^n}{\partial t} + \varrho^n v_j^0 \frac{\partial w_i^n}{\partial x_j}\frac{\partial w_i^n}{\partial t}\\ &\quad + \varrho^n w_j^n \frac{\partial w_i^n}{\partial x_j}\frac{\partial w_i^n}{\partial t} + \frac{dp}{d\varrho}(\varrho^n)\frac{\partial \varrho^n}{\partial x_j}\frac{\partial w_i^n}{\partial t}\Big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The terms on the right-hand side, with the exception of the first, are estimates by $k_6\|\frac{\partial w^n}{\partial t}\|_{L^2(Q_t,\mathbb{R}^N)}$, $k_6 > 0$. Applying the Young inequality, (3.17) and the Cauchy inequality we obtain

$$\left\|\frac{\partial w^n}{\partial t}\right\|_{L^2(Q_t,\mathbb{R}^N)} \leqslant k_7, \quad k_7 > 0, \tag{3.20}$$

$$\|w^n\|_{L^\infty I(W^{k,2}(\Omega,\mathbb{R}^N))} \leqslant k_7. \tag{3.21}$$

Now it is possible to verify that the expression $F_N = P_N(-\frac{\partial}{\partial t}(\varrho^n v^n) + \frac{\partial}{\partial x_j}(\varrho^n v_i^n v_j^n) + p(\varrho^n))$ is bounded in $L^2(Q_t, \mathbb{R}^N)$.

We have $((v^n, w)) = \int_\Omega F_{N_i} w_i\,\mathrm{d}x$; due the regularity of the elliptic system we obtain

$$\|v^n\|_{L^2(I,W^{2k,2}(\Omega,\mathbb{R}^N))} \leqslant k_8, \quad k_8 > 0. \tag{3.22}$$

Now, (3.21), (3.22) and (3.11) yield

$$\left\| \frac{\partial \varrho^n}{\partial t} \right\|_{L^\infty(I, W^{k-3,q}(\Omega))} \leqslant k_9, \quad k_9 > 0 \tag{3.23}$$

and if $\varrho_0 \in C^{2k-3}(\bar{\Omega})$, then also

$$\left\| \frac{\partial \varrho^n}{\partial t} \right\|_{L^2(I, W^{2k-3,q}(\Omega))} \leqslant k_9, \tag{3.24}$$

$$\|\varrho^n\|_{L^\infty(I, W^{2k-2,q}(\Omega))} \leqslant k_9 \quad (\text{for } k = 3), \tag{3.25}$$

where $1 \leqslant q \leqslant 6$ if $N = 3$, $1 \leqslant q < +\infty$ if $N = 2$.

4. Passage to the limit

Lemma 4.1. *Let $\{(\varrho^n, v^n)\}_{n=1}^{+\infty}$ be a sequence of solutions of* (3.2)–(3.4). *Then there exists a subsequence (denoted $\{(\varrho^n, v^n)\}_{n=1}^{+\infty}$ again) such that*

(i) *$\varrho^n \to \varrho$ strongly in $L^4(Q_t)$, $\varrho > \varepsilon > 0$ a.e. in Q_t;*

(ii) $\int_0^t \int_\Omega p(\varrho^n) \frac{\partial \varphi_j}{\partial x_j} \, dx \, dt \to \int_0^t p(\varrho) \frac{\partial \varphi_j}{\partial x_j}$ $\forall \varphi \in L^2(I, V)$;

(iii) *$D^i \varrho^n \to D^i \varrho$ $*$-weakly in $L^\infty(I, L^q))$, $i = 1, \ldots, k-2$, D^i …derivative with respect to the space variables $1 \leqslant q < +\infty$ if $N = 2$, $1 \leqslant q \leqslant 6$ if $N = 3$;*

(iv) *$\frac{\partial \varrho^n}{\partial t} \to \frac{\partial \varrho}{\partial t}$ weakly in $L^2(Q_t)$;*

(v) *$D^i v^n \to D^i v$ weakly in $L^2(Q_t, \mathbf{R}^N)$; $i = 0, \ldots, 2k$;*

(vi) *$v^n \to v$ strongly in $L^2(I, W^{2k-1,2}(\Omega, \mathbf{R}^n)) \cap L^p(I, W^{k-1,2}(\Omega, \mathbf{R}^N))$*

(vii) *$\frac{\partial v^n}{\partial t} \to \frac{\partial v}{\partial t}$ weakly in $L^2(Q_t, \mathbf{R}^N)$;*

(viii) *$\varrho^n v^n \to \varrho v$ strongly in $L^2(Q_t, \mathbf{R}^N)$;*

(ix) $\int_0^t \int_\Omega \varrho^n v_i^n v_j^n \frac{\partial \varphi_i}{\partial x_j} \to \int_0^t \int_\Omega \varrho v_i v_j \frac{\partial \varphi_i}{\partial x_j} \, dx \, dt$ $\forall \varphi \in L^2(I, V)$.

Proof. (i) The first assertion follows from the Lions lemma (see [1]) for $B_0 = W^{1,2}(\Omega)$, $B = L^4(\Omega)$, $B_1 = L^2(\Omega)$, $p_0 = 4$, $p_1 = 2$.

The second assertion is a consenquence of (3.12), see [7]. (ii) $\varrho^n \to \varrho$ a.e. in Q_t follows from (i), $p(\varrho^n)$ is bounded in $L^\infty(Q_t)$ then $p(\varrho^n) \to p(\varrho)$ strongly in $L^p(Q_t)$, $\forall p, 1 < p < +\infty$.

(vi) follows from the Lions lemma with $B_0 = W^{2k,2}$, $B = W^{2k-1,2}$, $p_0 = p_1 = 2$ or $B_0 = W^{k,2}$, $B = W^{k-1,2}$, $B_1 = L^2$, $1 < p_0 < +\infty$, $p_1 = 2$, $p_0 = p$.

(iii), (iv), (vii) are consenquences of (3.17), (3.18), (3.22), (3.20).

(viii)

$$\int_0^t \int_\Omega (\varrho^n v^n - \varrho v) \, dx \, dt \leqslant \|\varrho^n\|_{L^\infty(Q_t)} \|v^n - v\|_{L^2(Q_t, \mathbf{R}^N)} + \|\varrho^n - \varrho\|_{L^4(Q_t)} \|v\|_{L^4(Q_t)}$$

(ix) follows from (vi), (viii).

Now we pass to the limit in (3.2) and (3.3). We verify that (2.7) is satisfied in the sense of distributions also a.e. in Q_t; further, we verify that (2.18) holds and (2.8) is satisfied a.e. in Q_t. □

Theorem 4.1. *Let $k \geqslant 3$, $\varrho_0 \in C^{k-2}(\bar{\Omega})$, $\varrho_0 > \delta > 0$ in $\bar{\Omega}$, $v_0 \in V$, $p \in C^1([0,+\infty])$. Let a), b), (2.1), (2.3), (2.4) hold. Then there exists $\varepsilon > 0$ and a pair (ϱ, v) such that*

$$\varrho \in L^\infty(I, W^{k-2,q}(\Omega)) \cap L^\infty(Q_t), \quad \varrho > \varepsilon \text{ a.e. in } Q_t; \tag{4.1}$$

$$\frac{\partial \varrho}{\partial t} \in L^\infty(I, W^{k-3,q}(\Omega)), \tag{4.2}$$

$1 \leqslant q \leqslant 6$ if $N = 3$, $1 \leqslant q < +\infty$ if $N = 2$ for $k = 3$; for $k > 3$, (3.18′) holds;

$$v \in L^\infty(I, V) \cap L^2(I, W^{2k,2}(\Omega, \mathbf{R}^N)); \tag{4.3}$$

$$\frac{\partial v}{\partial t} \in L^2(Q_t, \mathbf{R}^N) \tag{4.4}$$

so that (2.7) is satisfied a.e. in Q_t, (2.10)–(2.14), (2.18) are satisfied, (2.8) is satisfied a.e. in Q_t.

Proof. It follows directly from (3.24), (3.25) and from Lemma 4.1. □

Theorem 4.2. *Let the assumptions of Theorem 4.1 be satisfied and let $\varrho_0 \in C^{2k-3}(\bar{\Omega})$ $(k = 3)$. Then*

$$\varrho \in L^\infty(I, W^{2k-2,q}(\bar{\Omega})) \cap L^\infty(Q_t); \tag{4.5}$$

$$\frac{\partial \varrho}{\partial t} \in L^\infty(I, W^{2k-3,q}(\Omega)), \tag{4.6}$$

$1 \leqslant q \leqslant 6$ if $N = 3$ and $1 \leqslant q < +\infty$ if $N = 2$.

5. Uniqueness

Theorem 5.1. *Let the assumptions of Theorem 4.1 be satisfied. Then there exists set of solutions satisfying (4.1)–(4.4) and there is at most one solution of the problem (2.7), (2.10)–(2.14), (2.18).*

Proof see [9]. □

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Souhrn

GLOBÁLNÍ ŘEŠENÍ VAZKÉHO STLAČITELNÉHO BAROTROPNÍHO MULTIPOLÁRNIHO PLYNU NA KONEČNÉM KANÁLU S NENULOVÝMI VSTUPY A VÝSTUPY

ŠÁRKA MATUŠŮ-NEČASOVÁ

V práci je dokázána globální existence slabého řešení vazkého barotropního plynu smíšené úlohy na konečném kanálu.

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