Applications of Mathematics

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RECURSIVE ESTIMATES OF QUANTILE BASED ON 0–1 OBSERVATIONS

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Summary. The objective of this paper is to introduce some recursive methods that can be used for estimating an LD-50 value. These methods can be used more generally for the estimation of the γ-quantile of an unknown distribution provided we have 0–1 observations at our disposal. Standard methods based on the Robbins-Monro procedure are introduced together with different approaches of Wu or Mukerjee. Several examples are also mentioned in order to demonstrate the usefulness of the methods presented.

Keywords: LD-50 value, quantile, stochastic approximation, isotonic regression

Problem formulation

The problem is formulated as follows: on the level $x$ ($x \in \mathbb{R}$) a random variable $Y_x$ is observed, distributed according to the 0-1 law with an unknown parameter $F(x)$. Having observations on different levels $x$ we want to estimate the solution of the equation $F(x) = \gamma$, where $\gamma \in (0, 1)$ is a given number. The function $F$ is usually a distribution function; however, the fulfillment of conditions

$$F(x) < \gamma, \quad x < x^*,$$

$$F(x) > \gamma, \quad x > x^*$$

for some real number $x^*$ is sufficient for a majority of algorithms.
EXAMPLES

First we give some practical examples that can be mathematically formulated as the above stated problem.

Example 1. Reliability of material

Let $X$ be a random variable indicating the power under which some material is defected. Performing an experiment on a power level $x$ we get the result $Y_x$ with the following distribution:

$$
Y_x = 0 \quad \text{(no defect) with probability} \quad 1 - F(x),
$$

$$
Y_x = 1 \quad \text{(defect) with probability} \quad F(x),
$$

where $F$ is the distribution function of the variable $X$.

Our task is to determine the maximum power that does not effect the structure of a high percentage of material, i.e. to find the solution of the equation $F(x) = \gamma$, where $\gamma \ll 1$.

Example 2. Explosive testing

The random variable $X$ indicates the pressure under which an explosive will explode. The result on the level $x$ is explosion or no explosion with probability $F(x)$ or $1 - F(x)$, respectively, where $F$ is the distribution function of $X$ as in Example 1. The task is to find the pressure under which 100$\gamma$% of the explosive will explode. In this case $\gamma \gg 0$ is usually taken.

Example 3. Biological experiments

We apply a dose $x$ of a substance to an experimental animal. The result of the experiment is 1 if the animal somehow responds to this dose. Otherwise the result is 0. The response of the individual appears with probability $F(x)$, taking $F$ as the distribution function of the random variable that expresses the amount of the dose necessary for the response. We want to estimate the level of the dose on which 100$\gamma$ % of individuals response. The value for $\gamma$ is usually taken as 0.5 and the corresponding level is then called LD-50.

Example 4. Range of feromon

Let $G(x)$ denote the probability of the event that insects will reach the feromon being $x$ units far from it. The marked insects either reach the investigated feromon or not. We look for the $\gamma$-quantile of the distribution of the random variable $X$ denoting the maximum distance from which the insects reach the feromon. The distribution of this variable is clearly given by the distribution function $F \overset{\text{def}}{=} 1 - G$. Thus we look for the solution of the equation $F(x) = \gamma$. 

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Example 5. Psychological testing

The problem is to construct a sequence of easier and more complicated questions in such a way that the complexity of these questions gradually increases and respects the individual abilities of the person investigated. After some time the individual abilities are judged using the level of complexity of the questions. The following model is used in psychology:

The examiner has a set of questions. The answers can be classified as “good” or “bad”. We denote the answer to the question $i$ by the symbol $u_i$; $u_i = 1$ if the answer is correct, $u_i = 0$ otherwise. The question is characterized by the so called characteristic curve $P_i(\theta) = P[u_i = 1|\theta]$, where $\theta$ is a real parameter denoting the individual abilities. Functions $P(\theta)$ are assumed to be continuous and increasing. The logistic curves are usually considered. The complexity of the question is defined as the “$\gamma$-quantile” of the function $P_i(\theta)$, more precisely as the solution of the equation $P_i(x) = \gamma$, where $\gamma \in (0, 1)$. We put questions to the individual on different levels $i_x$. Taking into account his previous answers we try to find the complexity $x^*$ such that the questions on this complexity level are answered correctly with probability $\gamma$. Psychologists consider this value to be the measure of ability of the individual tested. (See [11] for more details.).

The following two examples have a more mathematical character. Nevertheless, they do not lack their practical sense. Example 6 can be considered as the robust version of the stochastic approximation problem of finding the root of some regression function. Example 7 is a special case of 6 and deals with the problem of finding the $\gamma$-quantile having i.i.d. random variables.

Example 6. Finding a root of the general regression quantile function

Let observations at a point $x \in X \subseteq \mathbb{R}$ have a common distribution function $G_x$.

Let the function $M(x) \equiv G_x^{-1}(\gamma) \equiv \inf\{t; G_x(t) \geq \gamma\}$ satisfy inequalities

$$M(x) < \alpha \quad \text{for} \quad x < x^*,$$

$$M(x) > \alpha \quad \text{for} \quad x > x^*$$

where $x^*$ is a real number. The function $M$ is called the regression quantile function. The problem is to find the point $x^*$ provided the observations $Z_x$, distributed according to $G_x$, $x \in X$, are available.

We transform the variables $Z_x$ to $Y_x \equiv I[Z_x \geq \alpha]$, where $I[A]$ denotes the characteristic function of the set $A$. Thus we have the 0–1 variables on various levels $x$, the probability of success being $F(x) = EI[Z_x \geq \alpha] = 1 - G_x(\alpha)$. For $x < x^*$ we clearly
have \( F(x) < \gamma \), while for \( x > x^* \), \( F(x) \geq \gamma \) (\( F(x) > \gamma \) under some mild assumptions). The problem of finding the solution of the equation \( M(x) = \alpha \) having the observations \( Z_x \) is thus transformed to finding the solution of the equation \( F(x) = \gamma \) using 0–1 observations \( Y_x \).

**Example 7. Estimate of \( \gamma \)-quantile based on a random sample**

Let \( H \) denote a distribution function for which there exists exactly one \( \gamma^* \) such that \( H(\gamma^*) = \gamma \). We desire to estimate recursively the value \( \gamma^* \) getting sequentially the variables \( X_i \) (one per time unit) distributed according to the function \( H \).

If \( X \) is an arbitrary random variable distributed according to \( H \) then \( Z_x \overset{\text{def}}{=} m(x) + X \) is distributed according to \( G_x \overset{\text{def}}{=} H(y - m(x)) \). Any increasing function can be taken in the place of \( m \). Being at time \( i \) we put \( Z_{x_i} = X_i + m(x_i), i = 1, 2, \ldots \).

If \( H \) is strictly monotone then \( G_x^{-1}(\gamma) = m(x) + \gamma^* \) and hence \( G_x^{-1}(\gamma) < 0 \) for \( x < m^{-1}(\gamma^*) \) and \( G_x^{-1}(\gamma) > 0 \) for \( x > m^{-1}(\gamma^*) \). Taking into account that \( m \) is a known function (\( m(x) = x \) is usually considered) we transform the problem of finding the \( \gamma \)-quantile of \( H \) to the problem of finding a root of the regression quantile function. Using the transformation from Example 6 this problem can be further transformed to the problem of finding the solution \( F(x) = \gamma \) having 0–1 observations \( Y_x = I[Z_x \geq 0] \).

Although the examples introduced above have the common mathematical background, the algorithms that are going to be suggested are more suitable in some practical cases and less in other ones. In the sequel we will try to sort the algorithms together according to their mathematical character, giving some notes on their usefulness in specific applications.

**General assumptions**

The common feature of the methods introduced in this paper consists in defining the level (or levels) on which we perform our experiment at the next time unit. For all further developed methods and theorems of convergence we will suppose that our observations at time \( n \) depend on the past only through the levels on which they are performed. We also assume that the observations \( Y_x \) on the levels \( x \) are distributed according to the 0–1 law with a parameter \( 1 - F(x) \), where \( F \) is a real function having values between 0 and 1 (\( 0 \leq F(x) \leq 1 \)).

The majority of the methods cited below preserve their convergence properties also under more general assumptions concerning the distribution of \( Y_x \) and the behavior
of $F$. We do not mention these assumptions here, pointing out that the general results are formulated in the literature cited at the relevant places.

UP AND DOWN METHODS

The so called UP AND DOWN methods are characterized by a discrete set of levels at which the experiments are performed. We shall suppose that these levels form a set

(1) \[ L \equiv \{a + ld; l \text{ integer}\} \]

where $a$, $d$ are given numbers. (Although in practice the set of levels can be of another type it can be transformed to (1) using a one to one transformation.)

Having an observation $Y_{x_n}$ at time $n$ we define the level of experiment at time $n + 1$ as

(2) \[
\begin{align*}
    x_{n+1} &= x_n + d & \text{if} \quad Y_{x_n} = 0, \\
    x_{n+1} &= x_n - d & \text{if} \quad Y_{x_n} = 1.
\end{align*}
\]

This property characterizes the UP AND DOWN methods.

One of the first papers concerning the recursive estimate of the LD-50 variable was that of Dixon and Mood. They combine the UP AND DOWN principle with the maximum likelihood method. Their result consists in the assumption of normality of $F$. Since we introduce only non-parametric methods here we refer to [6] for the full algorithm and its numerical properties. A non-parametric UP AND DOWN method is, for example, that of Derman. Here we propose his algorithm (see [5]) in a little bit more general way.

DERMAN’S APPROACH

Assumptions P. Let there exist $x^* \in \mathbb{R}$ and $\gamma \in (\frac{1}{2}, 1)$ such that

\[
\begin{align*}
    F(x) &< \gamma & \text{pro} \quad x < x^*, \\
    F(x) &> \gamma & \text{pro} \quad x > x^*.
\end{align*}
\]

Let \( \lim \inf_{x \to \pm \infty} |F(x) - \gamma| > 0 \).
Algorithm 1. Perform an observation at time \( n \) on the level \( x_n \), where

\[
\begin{align*}
    x_1 &\in L \quad \text{arbitrary}, \\
    x_{n+1} &= x_n - d \quad \text{with probability } \frac{1}{2\gamma} \quad \text{if } Y_{x_n} = 1, \\
    x_{n+1} &= x_n + d \quad \text{with probability } 1 - \frac{1}{2\gamma} \quad \text{if } Y_{x_n} = 1, \\
    x_{n+1} &= x_n + d \quad \text{with probability } 1 \quad \text{if } Y_{x_n} = 0.
\end{align*}
\]

(For \( \gamma = \frac{1}{2} \) this algorithm coincides with (2).)

We define an estimate of \( x^* \) at time \( n \) as \( \theta_n \) which is the element of

\[
\text{arg}\max_{i\in L} \sum_{k=1}^{\infty} I\{x_k = i\}.
\]

We put \( \theta_n \) to be the arithmetic mean of all the elements of (3) if this set has more than one element.

Assertion 1. Let the assumption \( P \) hold true. If \( x^* \in L \), the relation

\[
\theta_n \in (x^* - d, x^* + d)
\]

holds for all \( n \geq N_0 \), where \( N_0 \) is an a.s. finite random variable. If \( x^* \not\in L \) then \( \theta_N \in ([x^*], [x^* + d]) \), where \([x] \equiv \max\{y \in L; y \leq x\} \).

The proof of the assertion is given in [5] under the assumption that \( F \) is a distribution function. In [3] it was proved that the assertion remains valid also for \( F(x) \xrightarrow{x \to \pm \infty} \gamma \) provided this convergence is sufficiently slow.

The Derman method can be applied also in the case of \( \gamma \in (0, \frac{1}{2}) \). However, in this case the sequence \( \{x_n\}_{n=1}^{\infty} \) is defined in the following manner:

\[
\begin{align*}
    x_{n+1} &= x_n + d \quad \text{with probability } \frac{1}{2(1-\gamma)} \quad \text{if } Y_{x_n} = 0, \\
    x_{n+1} &= x_n - d \quad \text{with probability } 1 - \frac{1}{2(1-\gamma)} \quad \text{if } Y_{x_n} = 0, \\
    x_{n+1} &= x_n - d \quad \text{with probability } 1 \quad \text{if } Y_{x_n} = 1.
\end{align*}
\]

We should notice that the disadvantage of this method consists in the fact that \( \limsup_{n\to\infty} |x_n - x^*| = \infty \). It means that the levels at which we perform observations can be arbitrarily far from the true value \( x^* \). This fact should be taken into account, especially when the price of the experiment increases with the distance of the level from the value \( x^* \).
METHODS OF THE ROBBINS-MONRO TYPE

In this section we suppose that there exist \( x^* \in \mathbb{R}, \gamma \in \mathbb{R} \) such that

\[
\inf_{\varepsilon < |x - x^*| < 1/\varepsilon} (F(x) - \gamma)(x - x^*) > 0,
\]

for all \( \varepsilon > 0 \).

The Robbins-Monro method (RM) seems to be the method most frequently used for finding the root of the regression function. RM is the a recursive scheme, where the level of our experiment at time \( n \) is defined as the estimate of \( x^* \) at this time. In the simplest form the sequence of levels given by RM can be expressed as

\[(5) \quad x_{n+1} = x_n - a_n(Y_{x_n} - \gamma).\]

If \( \sum a_n = \infty \) and \( \sum a_n^2 < \infty \), then \( x_n \to x^* \) almost surely (see [2] for proof).

An interesting insight into this method was given by Robbins, Lai [14]. They pointed out that the level \( x_{n+1} \) can be obtained as the root of the linear regression based on the observations \( (x_i, Y_{x_i})_{i=1}^n \). Now we go briefly through their idea.

Suppose a theoretical model

\[(L) \quad Y_{x_i} = \beta x_i + \alpha + \varepsilon_i,\]

where \( \varepsilon_i \) are independent variables with \( N(0, \sigma^2) \) distribution. If \( \beta \) is supposed to be known then we get the maximum likelihood estimate of the root of the equation \( \beta x + \alpha = 0 \) in the form

\[(6) \quad x_{n+1} = \bar{x}_n - \beta^{-1} \bar{y}_n,\]

where \( \bar{x}_n = \frac{1}{n} \sum x_i, \bar{y}_n = \frac{1}{n} \sum (Y_{x_i} - \gamma) \). Setting \( a_n = \frac{1}{\beta n} \), we get the identity between the sequences defined by the relations (5) and (6). In the paper of Robbins and Lai the parameter \( \beta \) (which is usually unknown in practice) is estimated using the maximum likelihood method again. Thus the recursive scheme suggested in their paper is of the form

\[(7) \quad x_{n+1} = x_n - \frac{Y_{x_n} - \gamma}{nb_n},\]

where \( b_n = b \lor (\hat{\beta}_n^* \land B) \) for \( n \geq n_0, b_n \in (b, B) \) arbitrary for \( n < n_0 \),

\[
0 < b < \beta < B, \quad n_0 = \inf \left\{ k : \sum_{i=1}^k (x_i - \bar{x}_k)^2 > 0 \right\} \quad \text{and} \quad \hat{\beta}_n = \frac{\sum_{i=1}^n Y_{x_i}(x_i - \bar{x}_n)}{\sum_{i=1}^n (x_i - \bar{x}_n)^2}.
\]

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Assertion 2. Let $F'(x^*) > 0$ and let $F$ be continuously differentiable in a neighborhood of $x^*$. Then for the sequence $\{x_n\}_{n=1}^\infty$ defined in (7) the convergencies

$$x_n \to x^* \quad \text{a.s.}$$

$$\sqrt{n}(x_n - x^*) \xrightarrow{D} N(0, \gamma(1 - \gamma)/(F'(x^*))^2)$$

$$\beta_n \to F'(x^*) \quad \text{a.s.}$$

hold true.


Using the results of Sacks [17] the asymptotic normality can be shown also for the sequences $x_n$ defined in (5) with $a_n = \frac{a}{n}$, if $a > \frac{1}{2F'(x^*)}$. In this case the asymptotic variance is equal to $\frac{a^2\gamma(1-\gamma)}{2aF'(x^*)-1}$. This variance is minimized for $a = \left( F'(x^*) \right)^{-1}$. Thus the value $b_n$ from (7) can be interpreted as the estimate of this optimal choice which is usually unknown in practice. The values $b, B$ can be considered as an apriori knowledge concerning the parameter $\beta$.

Another approach to estimating the unknown value $(F'(x^*))^{-1}$ is based on the differences of the function $F$. This method (in literature usually called adaptive) can be used also in the case when the apriori knowledge about $b$ and $B$ is not correct. From the results dealing with adaptive algorithms, that of Fabian [9] is well known:

$$x_{n+1} = x_n - \frac{d_n(Y_n - \gamma)}{n},$$

where $Y_n = (Y_{x_n+c_n} + Y_{x_n-c_n})/2$, $d_n = (\{C_1 \log(n+1)\}^{-1} \lor A_n^{-1}) \land C_2 n^\alpha$,

$$A_n = (n-1)^{-1} \sum_{j=1}^{n-1} \frac{Y_{x_j+c_j} - Y_{x_j-c_j}}{2c_j},$$

$$0 < C_1 < C_2, \quad 0 < \alpha < \frac{1}{2}, \quad c_n = cn^{-\gamma}, \quad \gamma \in (\frac{1}{4}, \frac{1}{2}).$$

In practical realization of (8) we perform two observations at time $n$ on the levels $x_n + c_n, x_n - c_n$.

Assertion 3. Let $F$ have the second bounded derivative in a neighborhood of $x^*$, let $F'(x^*) > 0$. Then we get

$$x_n \to x^* \quad \text{a.s.}$$

$$\sqrt{n}(x_n - x^*) \xrightarrow{D} N(0, \gamma(1 - \gamma)/(F'(x^*))^2).$$

See [9] for proof.
The methods of the RM type belong to the most frequently used recursive statistical approaches. One of their disadvantages in the case of estimating a variable LD-50 may consist in the following fact. The experimental levels at which we can perform our observations usually form a discrete set (e.g. of the type of L). This fact contradicts the demands of the RM type algorithms. The transformation of the RM algorithm to the case of a discrete set of levels of observations can be found in [8]. Another disadvantage of these algorithms can appear when we need to know in advance the levels at which we shall perform our experiments. This situation arises e.g. in a preparation of psychological tests (see Example 5 & [11]). Fixing the number of items in a test (denote this number by k which is usually greater then 25) we need to combine all possibilities of answers to prepare the test on a computer. In the case of RM type methods this represents $2^k$ possibilities. On the other hand, the UP AND DOWN methods need only $\frac{1}{2}k(k+1)$ possibilities.

**Non-parametric methods based on the parametric approach**

The methods of the RM type can be viewed as parameter estimates in the model (L). However, the linear fit is not the best what we can do in the case of 0–1 variables and a regression function which is a distribution one. Therefore other parameterizations were suggested using more suitable models. The convergence results were obtained also in these cases. The general idea of these methods can be expressed as follows: The function $F$ is supposed hypothetically to be of the form $F(x) = H(x|\theta)$. Using observations $(x_i, Y_{x_i})_{i=1}^n$ we evaluate the maximum likelihood estimate $\hat{\theta}_n$ of the parameter $\theta$. For the level of our experiment at time $n + 1$ we take the solution of the equation $\tilde{F}_n(x) = \gamma$, where $\tilde{F}_n(\cdot) = H(\cdot|\hat{\theta}_n)$. This level is denoted by $x_{n+1}$ as in the previous algorithms.

In the case of RM type methods we have $H(x|\theta) = \alpha + \beta x$, $\theta = (\alpha, \beta)$.

Wu in [18] suggested for the LD-50 problem the following logistic parameterization of the function $F : H(x|\theta) = (1 + e^{-\lambda(x-\alpha)})^{-1}$, $\lambda > 0$, $\theta = (\alpha, \lambda)$. In this case the maximum likelihood function is of the form

$$L(\lambda, \alpha|x_i, Y_{x_i}, i = 1, \ldots, n) = \prod_{i=1}^{n} H(x_i|\lambda, \alpha)^{Y_{x_i}} (1 - H(x_i|\lambda, \alpha))^{(1-Y_{x_i})},$$

hence the likelihood equations are

$$\sum_{i=1}^{n} H(x_i|\lambda, \alpha) = \sum_{i=1}^{n} Y_{x_i},$$

$$\sum_{i=1}^{n} x_i H(x_i|\lambda, \alpha) = \sum_{i=1}^{n} x_i Y_{x_i}.$$
Thus the estimate of the $\gamma$-quantile of $F$ is defined as the value $\tilde{L}_\gamma = \tilde{\alpha} - \tilde{\lambda}^{-1} \log(1/\gamma - 1)$, where $\tilde{\alpha}, \tilde{\lambda}$ solve (9). This value defines also the level $x_{n+1}$ on which we perform our experiment at time $n + 1$. For practical purposes we should know under what condition there exists a solution of (9). Due to Silvapulle [19] there exists exactly one solution of (9) if

$$(x_{\min}^+, x_{\max}^+) \cap (x_{\min}^-, x_{\max}^-) \neq \emptyset$$

or

$$x_{\min}^+ < x_{\min}^- = x_{\max}^- < x_{\max}^+$$

or

$$x_{\min}^- < x_{\min}^+ = x_{\max}^+ < x_{\max}^-,$$

where

$$x_{\max}^+ = \max\{x_i; Y_x = 1\}, \quad x_{\min}^+ = \min\{x_i; Y_x = 1\},$$
$$x_{\max}^- = \max\{x_i; Y_x = 0\}, \quad x_{\min}^- = \min\{x_i; Y_x = 0\}.$$  

If this condition is fulfilled for some $n_0$ then it is fulfilled also for all $n \geq n_0$. For $n < n_0$ we should use another method, e.g. of the RM type.

In [18] the convergence of this method is proved for $\lambda$ given. For $\lambda$ obtained as the solution of (9) the asymptotical efficiency of the method is proved provided the a.s. convergence holds true. In his paper Wu performs also some simulation studies according to which the previous algorithm gives better results then the RM method for small and medium $n$. See [18] for some numerical illustrations that exhibit better behavior of this method as compared with RM algorithm for small and medium sample sizes.

**Methods based on a more general parametric model**

Mukerjee in [12] suggested the approximation of the unknown distribution function (in his paper the regression function, generally) by the so called isotonic regression. The set of levels at which we perform our observations is the same as in the case of the UP AND DOWN methods, i.e. the experimental levels are taken from the set $L$ defined in (1).

Let us introduce the following notation for the rest of our paper:
$X_n \ldots$ levels, on which we performed our experiments up to time $n$, 
$n(x) \ldots$ number of observations on the level $x$ made up to time $n$, 
$y_i(x) \ldots$ the value of the $i$-th observation on the level $x$ ($i = 1, \ldots, n(x)$).
Isotonic regression based on the observations \((x_i, Y_i)_{i=1}^n\) is defined as

\[
(10) \quad \arg \min_{t \in RF} \sum_{x \in X_n} \sum_{i=1}^{n(x)} (y_i(x) - t(x))^2,
\]

where \(RF = \{ t : X_n \to R; \; t(x) \leq t(y), \; \text{if} \; x < y \} \). The recursive scheme based on the isotonic regression is described by the following algorithm:

\textbf{Algorithm 2.}

1) Choose \(a < b \in \mathbb{R}\) arbitrarily and set \(X_0 := L \cap (a, b)\), \(n := 0\). At every point of \(X_0\) perform at least one observation.

2) Fit the values obtained up to time \(n\) by isotonic regression, i.e. find the element of \((10)\) and denote it by \(t_n^*\). (Notice that \(t_n^*\) is unique). Define

\[
\begin{align*}
t_n^{\min, \gamma} &\equiv \max\{ \min X_n - d, \max\{ x \in X_n ; t_n^* < \gamma \} \}, \\
t_n^{\max, \gamma} &\equiv \min\{ \max X_n + d, \min\{ x \in X_n ; t_n^* > \gamma \} \},
\end{align*}
\]

where we set \(\max \emptyset = -\infty\), \(\min \emptyset = \infty\).

Choose \(\theta_n \in (t_n^{\min, \gamma}, t_n^{\max, \gamma})\) arbitrarily and perform experiments on the levels \([\theta_n]\), \([\theta_n + d]\), where \([y] = \max\{ x \in L ; x \leq y \}\). Set \(n := n + 1\) and repeat step 2).

The general convergence theorem from the paper of Mukerjee can be rewritten for our purposes in the following way.

\textbf{Assertion 4.} Let there exist \(x_m < x_M \in L\) such that

\[
\begin{align*}
F(x) &< \gamma \quad \text{for} \quad x \leq x_m, \\
F(x) &> \gamma \quad \text{for} \quad x \geq x_M.
\end{align*}
\]

Then for \(\theta_n\) from Algorithm 2 the relation

\[
P[\theta_n \notin (x_m, x_M) \text{ infinitely often}] = 0
\]

is valid.

The proof is given in [12].

For completeness we should mention the algorithm for solving \((10)\) which is, however, given e.g. in [1]. Here we would suggest other algorithms for finding the \(\gamma\)-quantile of the distribution function \(F\) whose character is similar to Mukerjee's algorithm but the problem \((10)\) is simplified.
Instead of looking for the isotonic regression and for the interval \((t_{\min}^{\gamma}, t_{\max}^{\gamma})\) we estimate the \(\gamma\)-quantile directly as any element of

\[
\arg\min_{\theta} \sum_{x \leq \theta} \sum_{i=1}^{n(x)} (1 - \gamma)y_i(x) + \sum_{x > \theta} \sum_{i=1}^{n(x)} \gamma(1 - y_i(x)).
\]

Recalling that \(y_\theta(x)\) is equal to 0 or 1 we get that the set (11) is equal to

\[
\arg\min_{\theta} \sum_{x \leq \theta} \sum_{i=1}^{n(x)} (y_i(x) - \gamma)^+ + \sum_{x > \theta} \sum_{i=1}^{n(x)} (y_i(x) - \gamma)^-,
\]

where \(z^+ = \max(0, z), z^- = \max(0, -z)\). The following algorithm for finding any element of (11), or (12) is analogous to that one of Dupač [7] for finding the root of the quasiisotonic regression.

Algorithm 3. Set \(x_1 := \min X_n\). Let \(r_1 \in L\) be the smallest number such that

\[
\sum_{x \in (x_1, r_1)} \sum_{i=1}^{n(x)} (y_i(x) - \gamma) \leq 0,
\]

\(r_2 > r_1\) the smallest number such that

\[
\sum_{x \in (r_1, r_2]} \sum_{i=1}^{n(x)} (y_i(x) - \gamma) \leq 0,
\]

etc.

We take \(\tilde{\theta} \in (0, x_1)\) arbitrarily if \(r_1\) does not exist. Otherwise we denote by \(r^*\) the last one from the \(r_k\)'s defined and take \(\tilde{\theta} \in (r^*, r^* + d)\) if \(r^* < \max X_n\). If \(r^* = \max X_n\), then select \(\tilde{\theta} \in (r^*, \infty)\). \(\tilde{\theta}\) is an element of (11) or (12).

Let any \(\tilde{\theta} \in (l_1, l_2)\) be an element of (11), where \(l_1, l_2 \in L \cup \{\infty\}\), and let

\[
\sum_{x \in (z, l_1]} \sum_{i=1}^{n(x)} (y_i(x) - \gamma) = 0.
\]

Then any \(\tilde{\theta} \in (z, z + d)\) is also an element of (11).

Denote

\[
\theta_{\min}^{\min, \gamma} = \max\{\min X_n - d; \inf\{\theta; \theta \text{ is an element of (11)}\}\}
\]

and

\[
\theta_{\max}^{\min, \gamma} = \min\{\max X_n + d; \sup\{\theta; \theta \text{ is an element of (11)}\}\}.
\]

The relation between \(\theta_{\min}^{\min, \gamma}\) and \(lt_{\max}^{\min, \gamma}\) is expressed in the following assertion.
Assertion 5. \( \theta_{\min}^{n, \gamma} \geq t_{\min}^{n, \gamma}, \theta_{\max}^{n, \gamma} \leq t_{\max}^{n, \gamma} \).

Proof. We denote \( \theta_{\min} = \theta_{\min}^{n, \gamma} \) and \( t_{\min} = t_{\min}^{n, \gamma} \) for simplicity. We prove the first inequality, the second could be proved similarly. We suppose that \( t_{\min} > \min X_n - d \), otherwise the assertion is clear. Let \( \theta_{\min} < t_{\min} \).

Denote by \( t^* \) the solution of (10). There must exist a natural number \( c \) such that
\[
t^*(\theta_{\min} + d) = t^*(\theta_{\min} + 2d) = \ldots = t^*(\theta_{\min} + cd).
\]
Define
\[
M = \{ \theta_{\min} + d, \theta_{\min} + 2d, \ldots, \theta_{\min} + cd \}
\]
and
\[
\tilde{t}(x) = \min \{ \gamma, t^*(\theta_{\min} + (c + 1)d) \} \quad \text{for} \quad x \in M, \\
\tilde{t}(x) = t^*(x) \quad \text{for} \quad x \in X_n \setminus M.
\]

It is obvious that \( \sum_{x \in M} \sum_{i=1}^{n(x)} (y_i(x) - k)^2 \) is a decreasing function in \( k \) for \( k \leq \frac{n}{n(M)} \sum_{x \in M} \sum_{i=1}^{n(x)} y_i(x) \), where \( n(M) = \sum_{x \in M} 1 \) is the number of observations performed on the levels from \( M \). Further, we get \( \frac{1}{n(M)} \sum_{x \in M} \sum_{i=1}^{n(x)} y_i(x) \geq \gamma \) using the inequality \( \sum_{x \in M} \sum_{i=1}^{n(x)} y_i(x) - \gamma \geq 0 \) which follows from the definition of \( \theta_{\min} \) by virtue of (12). This fact implies that the function \( \sum_{x \in M} \sum_{i=1}^{n(x)} (y_i(x) - k)^2 \) is decreasing for \( k \leq \gamma \).

On the set \( M \) the values \( \tilde{t}, t^* \) are constant, less than \( \gamma \) and \( \tilde{t} > t^* \). This contradicts the fact that \( t^* \) is an element of (10).

The next result is an immediate corollary of the previous assertion and Assertion 4.

Assertion 6. Let the assumptions of Assertion 4 be valid. Replacing \( t_{\min}^{n, \gamma} \) by \( \theta_{\min}^{n, \gamma} \) and \( t_{\max}^{n, \gamma} \) by \( \theta_{\max}^{n, \gamma} \) in Algorithm 2 we get for \( \theta_n \) from this algorithm that
\[
P[\theta_n \notin (x_m, x_M) \text{ for infinite number of } n] = 0.
\]

Algorithm 2 thus can be used taking the estimate of the \( \gamma \)-quantile based on solving (11) or (12). These problems are easier to solve than the isotonic regression problem.
The algorithms studied in the last two sections are particularly useful in such situations when some observations are given in advance on different levels without any order.

At the end we give several numerical results that demonstrate the behavior of the algorithms introduced. Table 1 shows the results when the median of the \(N(0,1)\) distribution was to be found. In Table 2 we consider the problem of finding 70\% quantile of \(\chi^2\) (8) distribution which is equal to 9.524. The methods used were the Derman method using Algorithm 1, Robbins-Monro method (according to (5) with \(a_n = \frac{a}{n}\)), adaptive Robbins-Monro method according to (8). In this method the constants \(d_n\) were replaced by \(d'_n = (A_n^{-1} \vee r_1) \wedge r_2\), where \(r_1, r_2\) are chosen constants and \(A_n\) has the same meaning as in the definition of \(d_n\). Further, the isotonic regression approach using algorithm 2 and Robbins-Lai approach using formula (7) were employed. Each method was applied from time 0 to Stoptime giving one sample trajectory of the algorithm. In order to obtain more knowledge about the asymptotic properties of our procedures we took T such sample paths for each of the procedures. The value of T is given in the first column of the numerical tables. We do not introduce the whole history of the processes but only their values at the times that are given in the second column. In the third column the values of specified procedures are computed at the time moments considered. These values are taken from the last path of our T samples. In the fourth column the averages through the T samples are given. Finally, the sample variances of our procedures at specified time moments are given in the last column. We do not comment the results because they depend very heavily on the parameters of each of our procedures. However, the results probably give some ideas concerning the procedures behavior. For more information the time consumption per one path simulation is given (results were computed on IBM XT compatible without coprocessor). This information can be misleading in the case of the Derman procedure because type of random number generator other than for the other four methods had to be used. We use the Stochastic Approximation program system for our computations (see Charamza [4]). Using this system a lot of simulations under different initial conditions and different parameter values can be easily obtained.
### Table 1

**Derman procedure.**

\(d = 0.1000, \text{StopTime} = 100, \text{Time consumption} = 13.509000s\)

<table>
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<th>variance</th>
</tr>
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**Procedure Robbins-Monroe.**

\(a = -3.0000, \text{StopTime} = 100, \text{Time consumption} = 6.838000s\)

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**Adaptive Robbins-Monro procedure.**

\(c = 1.0000, \gamma = 0.3000, r_1 = 0.1000, r_2 = 5.0000, \text{StopTime} = 100, \text{Time consumption} = 13.534000s\)

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Isotonic regression (see Algorithm 2).

\(d = 0.1000\), \(\text{StopTime} = 100\), Time consumption = 12.883000s

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Procedure of Robbins and Lai.

\(b = 0.1\), \(B = 5.0000\), \(\text{StopTime} = 100\), Time consumption = 8.271500s

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Table 2

Derman procedure.

\[ d = 0.4000, \text{StopTime} = 200, \text{Time consumption} = 161.932000s \]

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Procedure Robbins-Monroe.

\[ a = -5.0000, \text{StopTime} = 200, \text{Time consumption} = 41.689000s \]

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Adaptive Robbins Monro.

\[ c = 1.0000, \gamma = 0.3000, r_1 = 0.1000, r_2 = 5.0000, \text{StopTime} = 200, \text{Time consumption} = 84.525000s \]

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**Isotonic regression.**

\[ d = 0.4000, \text{StopTime} = 200, \text{Time consumption} = 83.092000 \text{s} \]

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**Robbins and Lai procedure.**

\[ b = 0.1, \text{B} = 5.0000, \text{StopTime} = 200, \text{Time consumption} = 43.693000 \text{s} \]

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The algorithms introduced here do not cover all possibilities for solving the problems of LD-50 type. Other algorithms of the RM type were introduced (see [13]) even for the case of dependent observations (see [10]). By their modifications we could gain many other schemes. We should mention at least one reference concerning the non-recursive methods for solving the problem studied above. Several methods of this type were studied intensively in [16].

References

REKURZIVNÍ ODHADY KVANTILŮ ZALOŽENÉ NA 0–1 POZOROVÁNÍCH

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