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ON SOLVING SYSTEMS OF DIFFERENTIAL ALGEBRAIC EQUATIONS

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Summary. In the paper the comparison method is used to prove the convergence of the Picard iterations, the Seidel iterations, as well as some modifications of these methods applied to approximate solution of systems of differential algebraic equations. The both linear and nonlinear comparison equations are employed.

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1. In the paper [2] it was shown that the solution of the system

$$x'(t) = f(t, x(t), y(t)), \qquad x(0) = 0, \tag{1}$$

$$y(t) = g(t, x(t), y(t)) \tag{2}$$

can be found by the Picard iterations

$$x_{k+1}(t) = \int_0^t f(s, x_k(s), y_k(s))ds, \tag{3}$$

$$y_{k+1}(t) = g(t, x_k(t), y_k(t)), \qquad k = 0, 1, \ldots, \tag{4}$$

with x_0 and y_0 arbitrarily chosen, if the functions f and g satisfy the Lipschitz condition with respect to the last two variables and the Lipschitz constant of g with respect to y is less than one.

The aim of the present paper is to present another approach to the problem of solving the system mentioned. We will use the well known comparison method to prove the convergence of the Picard iterations, the Seidel iterations, as well as some modified Picard iterations.

There is a great number of papers devoted to the comparison method and its relation to the Banach contraction principle. The reader interested in the subject can consult the papers [1], [3], [4–8], [10] and the references mentioned there.

2. Assume that B_1 and B_2 are given Banach spaces with norms $\|\cdot\|_1$ and $\|\cdot\|_2$ (further on we will drop the indexes provided it is clear which one is meant). Let $f \in C(J \times B_1 \times B_2, B_1)$ and $g \in C(J \times B_1 \times B_2, B_2)$, $J = [0, T)$ (here $C(X, Y)$ means the space of all continuous functions from the spaces X to Y).

We consider the system

$$x(t) = \int_0^t f(s, x(s), y(s))ds, \tag{5}$$

$$y(t) = g(t, x(t), y(t)), \qquad t \in J, \tag{6}$$

which is equivalent to the system (1)–(2) (observe that a simple change of the variable reduces the general initial condition to that considered here). We will use the following

Assumption H. There are functions $\omega, \Omega \in C(J \times R_+^2, R_+)$, such that:

a) they are nondecreasing with respect to the last two variables,

b) $\omega(t, 0, 0) = 0$, $\Omega(t, 0, 0) = 0$, $t \in J$,

c)

$$\|f(t, x, y) - f(t, \bar{x}, \bar{y})\| \leqslant \Omega(t, \|x - \bar{x}\|, \|y - \bar{y}\|), \tag{7}$$

$$\|g(t, x, y) - g(t, \bar{x}, \bar{y})\| \leqslant \omega(t, \|x - \bar{x}\|, \|y - \bar{y}\|) \tag{8}$$

for any $t \in J$, $x, \bar{x} \in B_1$ and $y, \bar{y} \in B_2$,

d) for every $p, q \in C(J, R_+)$ there exists a maximal solution (u^*, v^*), $u^* \in C(J, R_+)$, $v^* \in C(J, R_+)$, of the comparison system

$$u(t) = \int_0^t \Omega(s, u(s), v(s))ds + p(t), \tag{9}$$

$$v(t) = \omega(t, u(t), v(t)) + q(t), \qquad t \in J, \tag{10}$$

e) $u^*(t) = v^*(t) = 0$ is the only upper semicontinuous solution of the system (9)–(10) when $p(t) = q(t) = 0$ for $t \in J$.

Now, for given $x_0 \in C(J, B_1)$ and $y_0 \in C(J, B_2)$ we take

$$p_0(t) = \left\| \int_0^t f(s, x_0(s), y_0(s))ds - x_0(t) \right\|, \tag{11}$$

$$q_0(t) = \|g(t, x_0(t), y_0(t)) - y_0(t)\| \tag{12}$$

and define the sequences $\{u_k\}, \{v_k\}$:

$$u_{k+1}(t) = \int_0^t \Omega(s, u_k(s), v_k(s))ds, \tag{13}$$

$$v_{k+1}(t) = \omega(t, u_k(t), v_k(t)), \qquad k = 0, 1, \ldots, \tag{14}$$

with (u_0, v_0) being the solution of the system (9)–(10) with p and q replaced by p_0 and q_0. Under Assumption H one can easily show that the sequences $\{u_k\}, \{v_k\}$ are nonincreasing and converge almost uniformly (uniformly on any compact subset of J) to zero. Further, by a standard argument (see [4–7], [10]) one finds the estimates

$$\|x_{k+l}(t) - x_k(t)\| \leqslant u_k(t), \tag{15}$$

$$\|y_{k+l}(t) - y_k(t)\| \leqslant v_k(t), \quad t \in J, \quad k, l = 0, 1, \ldots. \tag{16}$$

Now we can formulate the following

Theorem 1. *Under Assumption H there exists a unique solution (x^*, y^*) of the system* (1)–(2). *The solution can be found by the Picard successive approximation method* (3)–(4). *Moreover, the following error estimates*

$$\|x^*(t) - x_k(t)\| \leqslant u_k(t), \tag{17}$$

$$\|y^*(t) - y_k(t)\| \leqslant v_k(t), \quad t \in J, \quad k = 0, 1, \ldots, \tag{18}$$

hold.

P r o o f. The existence of a solution and the error estimate follow from (15)–(16). The uniquenees of the solution is a consequence of the fact that under Assumption H the only non negative and continuous solution of the system of inequalities

$$u(t) \leqslant \int_0^t \Omega(s, u(s), v(s))ds, \tag{19}$$

$$v(t) \leqslant \omega(t, u(t), v(t)), \qquad t \in J, \tag{20}$$

is $u(t) = 0$ and $v(t) = 0$ for $t \in J$. □

3. Let us now consider a very important special case when the functions Ω and ω are both linear with respect to the last two variables, i.e. the case

$$\Omega(t, u, v) = a(t)u + b(t)v, \qquad \omega(t, u, v) = c(t)u + d(t)v \tag{21}$$

for some $a, b, c, d \in C(J, R_+)$. In this case the system (9)–(10) assumes the form

$$u(t) = \int_0^t [a(s)u(s) + b(s)v(s)]ds + p(t), \tag{22}$$

$$v(t) = c(t)u(t) + d(t)v(t) + q(t). \tag{23}$$

It is easy to see that now Assumption H holds if $d(t) < 1$ for $t \in J$. Indeed, one can solve the equation (23) with respect to v and then v can be eliminated from (22). As a result one gets a linear integral equation which has properties good enough to guarantee that H holds. We can establish the following generalization of the result of the paper [2].

Theorem 2. *If the functions f and g satisfy the Lipschitz conditions with respect to the last two variables with the coefficients a, b, c, d, respectively, and $d(t) < 1$ for $t \in J$, then there is a unique solution of the system* (1)–(2) *and it can be found by the Picard method* (3)–(4).

However, there are many other special cases in which the functions Ω and ω are partially linear, for instance: Ω has the general form and $\omega(t,u,v) = c(t)u + d(t)v$,

$$\Omega(t,u,v) = \Omega_1(t,u) + b(t)v, \qquad \omega(t,u,v) = \omega_1(t,u) + d(t)v,$$

and so on. It is not our intent to formulate further consequences of the main result of Theorem 1. Nonetheless we want to note that the solution (u^*, v^*) appearing in condition d) of Assumption H can be taken as a solution of the system obtained from the system (9)–(10) by replacing the symbols of equality by symbols of inequality "$\geqslant$".

4. Let us now consider the Seidel iterations for finding the solution of the system (1)–(2)

$$\bar{x}_{k+1}(t) = \int_0^t f(s, \bar{x}_k(s), \bar{y}_k(s))ds, \tag{24}$$

$$\bar{y}_{k+1}(t) = g(t, \bar{x}_{k+1}(t), \bar{y}_k(t)), \qquad k = 0, 1, \ldots, \tag{25}$$

with $\bar{x}_0 = x_0$ and $\bar{y}_0 = y_0$.

It is easy to see that these iterations are nothing else than the Picard iterations for the system

$$x(t) = \int_0^t f(s, x(s), y(s))ds, \tag{26}$$

$$y(t) = g\Big(t, \int_0^t f(s, x(s), y(s))ds, y(t)\Big), \tag{27}$$

which is obviously equivalent to the system (1)–(2).

Under Assumption H we now have the comparison system

$$u(t) = \int_0^t \Omega(s, u(s), v(s))ds + p(t), \tag{28}$$

$$v(t) = \omega\Big(t, \int_0^t \Omega(s, u(s), v(s))ds, v(t)\Big) + q(t). \tag{29}$$

Let $(\bar{u}_0, \bar{v}_0)$ be a solution of the system (9)–(10) with p and q replaced by

$$\bar{p}_0(t) = p_0(t), \tag{30}$$

$$\bar{q}_0(t) = \left\| g(t, \int_0^t f(s, x_0(s), y_0(s))ds, y_0(t)) - y_0(t) \right\|, \tag{31}$$

respectively. Define sequences $\{\bar{u}_k\}$ and $\{\bar{v}_k\}$:

$$\bar{u}_{k+1}(t) = \int_0^t \Omega(s, \bar{u}_k(s), \bar{v}_k(s))ds \tag{32}$$

$$\bar{v}_{k+1}(t) = \omega(t, \bar{u}_{k+1}(t), \bar{v}_k(t)), \qquad k = 0, 1, \ldots. \tag{33}$$

It is clear that these sequences converge to zero almost uniformly in J, moreover we have

$$\|x^*(t) - \bar{x}_k(t)\| \leqslant \bar{u}_k(t), \tag{34}$$

$$\|y^*(t) - \bar{y}_k(t)\| \leqslant \bar{v}_k(t), \qquad k = 0, 1, \ldots. \tag{35}$$

Observe that $(\bar{u}_0, \bar{v}_0)$ can be taken as a solution of the system (28)–(29) to get better error estimates. It is worth noticing that starting with initial approximations which give $u_0 = \bar{u}_0$ and $v_0 = \bar{v}_0$ we will get error sequences $\{u_k\}, \{v_k\}, \{\bar{u}_k\}, \{\bar{v}_k\}$ satisfying the relations

$$\bar{u}_k \leqslant u_k, \qquad \bar{v}_k \leqslant v_k, \qquad k = 0, 1, \ldots. \tag{36}$$

To see better the difference in the rate of convergence of these sequences we consider the simple case when both Ω and ω are linear, namely

$$\Omega(t, u, v) = au + bv, \qquad \omega(t, u, v) = cu + dv, \quad d < 1, \tag{37}$$

with a, b, c, d real constants. We also assume that the interval $J = [0, T]$ is compact. Using the weighted norm

$$\|u\|_\lambda = \sup[\|u(t)\| \exp(-\lambda t), \quad t \in J], \quad \lambda > 0, \tag{38}$$

we easily find

$$\|u_{k+1}\|_\lambda \leqslant \lambda^{-1}(a\|u_k\|_\lambda + b\|v_k\|_\lambda), \tag{39}$$

$$\|v_{k+1}\|_\lambda \leqslant c\|u_k\|_\lambda + d\|v_k\|_\lambda, \qquad k = 0, 1, \ldots. \tag{40}$$

The rate of convergence of the sequences $\{\|u_k\|_\lambda\}$, $\{\|v_k\|_\lambda\}$ is determined by the spectral radius ϱ of the matrix (see [9])

$$\begin{pmatrix} a', & b' \\ c, & d \end{pmatrix}, \quad a' = \lambda^{-1}a, \quad b' = \lambda^{-1}b, \tag{41}$$

$$\varrho = 2^{-1}\left(a' + d + \sqrt{(a'+d)^2 - 4(a'd - b'c)}\right). \tag{42}$$

It is clear that ϱ is smaller than one if $d < 1$ and λ is sufficiently large. On the other hand, for the sequences $\{\|\bar{u}_k\|_\lambda\}$, $\{\|\bar{v}_k\|_\lambda\}$ we get a similar relation with a matrix of the form

$$\begin{pmatrix} a', & b' \\ a'c, & b'c + d \end{pmatrix}. \tag{43}$$

It is easy to check that the spectral radius $\bar{\varrho}$ of this matrix is again less than one if $d < 1$ and λ is sufficiently large. Moreover, $\varrho < 1$ implies $\bar{\varrho} < \varrho$ which means that the rate of convergence of the Seidel iterations is higher than that of the Picard iterations. Observe that one can arrive at the same conclusion by the corresponding metrization of the space $C(J, B_1) \times C(J, B_2)$ (see [2], [8]).

In a similar way one can consider another form of the Seidel iterations

$$\widetilde{y}_{k+1}(t) = g(t, \widetilde{x}_k(t), \widetilde{y}_k(t)), \tag{44}$$

$$\widetilde{x}_{k+1}(t) = \int_0^t f(s, \widetilde{x}_k(s), \widetilde{y}_{k+1}(s))ds, \qquad k = 0, 1, \ldots. \tag{45}$$

5. Finally, we consider the iterations

$$\widehat{y}_{k+1}(t) = g(t, \widehat{x}_k(t), \widehat{y}_k(t)), \tag{46}$$

$$\widehat{x}_{k+1}(t) = \int_0^t f(s, \widehat{x}_{k+1}(s), \widehat{y}_{k+1}(s))ds, \qquad k = 0, 1, \ldots, \tag{47}$$

with $\widehat{x}_0 = x_0$ and $\widehat{y}_0 = y_0$. Now we define sequences $\{\widehat{u}_k\}, \{\widehat{v}_k\}$ in the following way:

$$\widehat{v}_{k+1}(t) = \omega(t, \widehat{u}_k(t), \widehat{v}_k(t)), \tag{48}$$

$\widehat{u}_{k+1}$ is the maximal solution of the equation

$$u(t) = \int_0^t \Omega(s, u(s), \omega(s, \widehat{u}_k(s), \widehat{v}_k(s)))ds \tag{49}$$

and $\widehat{u}_0 = u_0, \widehat{v}_0 = v_0$. We assume that p_0 appearing in the definition is strictly positive. Using the results concerning integral inequalities we find that the sequences $\{\widehat{u}_k\}, \{\widehat{v}_k\}$ converge almost uniformly and monotonically to zero.

From Theorem 1 we have

$$\|\widehat{x}_0(t) - x^*(t)\| \leqslant \widehat{u}_0(t), \qquad \|\widehat{y}_0(t) - y^*(t)\| \leqslant \widehat{v}_0(t). \tag{50}$$

On the other hand, from the definition of x_k, y_k, x^*, y^* and from the condition a) of Assumption H we find

$$\|\hat{y}_{k+1}(t) - y^*(t)\| \leqslant \omega(t, \|\hat{x}_k(t) - x^*(t)\|, \|\hat{y}_k(t) - y^*(t)\|),$$
$$\|\hat{x}_{k+1}(t) - x^*(t)\| \leqslant \int_0^t \Omega(s, \|\hat{x}_{k+1}(s) - x^*(s)\|, \|\hat{y}_{k+1}(s) - y^*(s)\|)ds.$$

This together with mathematical induction and results on integral inequalities implies the estimates

$$\|\hat{x}_k(t) - x^*(t)\| \leqslant \hat{u}_k(t), \qquad \|\hat{y}_k(t) - y^*(t)\| \leqslant \hat{v}_k(t) \tag{51}$$

for all $k = 0, 1, \dots$ and $t \in J$. In this way the convergence of the sequence $\{\hat{x}_k, \hat{y}_k\}$ to the solution (x^*, y^*) is established.

Observe that the iterations discussed in this section can be employed for numerical approximate solution of the system (1) -(2). In fact, from (47) it is seen that for given $\hat{x}_k$ and $\hat{y}_k$ the next approximation $\hat{y}_{k+1}$ is found from (46) and then $\hat{x}_{k+1}$ is calculated as the solution of the initial problem

$$x'(t) = f(t, x(t), \hat{y}_{k+1}(t)), \quad x(0) = 0; \tag{52}$$

to this end any numerical method can be employed.

6. We conclude the paper with the following comments: the results mentioned above can be easily extended to the general Volterra systems of the form

$$x'(t) = f(x, y)(t), \qquad x(0) = 0, \tag{53}$$
$$y(t) = g(x, y)(t) \tag{54}$$

with operators $f, g \in C(J \times C(J, B) \times C(J, B), B)$ having the Volterra property and compared with the given functions Ω and ω by the relations

$$\|f(x, y)(t) - f(\bar{x}, \bar{y})(t)\| \leqslant \Omega(t, \|x - \bar{x}\|_t, \|y - \bar{y}\|_t), \tag{55}$$

$$\|g(x, y)(t) - g(\bar{x}, \bar{y})(t)\| \leqslant \omega(t, \|x - \bar{x}\|_t, \|y - \bar{y}\|_t), \tag{56}$$

where $\|x\|_t = \sup[\|x(s)\|, s \in [0, t]]$. Other comparison operators can be considered and the corresponding convergence results can be established.

The case when the operators f and g are not of the Volterra type can be also treated by the comparison method. However, in this case the existence of a solution of the comparison system requires rather strong conditions on the comparison operators. The discussion of this problem is left to another paper.

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