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A NEW APPROACH TO REPRESENTATION OF OBSERVABLES ON FUZZY QUANTUM POSETS

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Summary. We give a representation of an observable on a fuzzy quantum poset of type II by a pointwise defined real-valued function. This method is inspired by that of Kolesárová [6] and Mesiar [7], and our results extend representations given by the author and Dvurečenskij [4]. Moreover, we show that in this model, the converse representation fails, in general.

Keywords: Fuzzy quantum poset, fuzzy quantum space, q-σ-algebra, observable

AMS classification: 81P15

1. INTRODUCTION

By a fuzzy set we understand a real-valued function a from a given non-void set Ω into the interval $[0,1]$, and we say that

$$\bigcap_i a_i := \inf_i a_i,$$

$$\bigcup_i a_i := \sup_i a_i,$$

$$a^\perp := 1 - a$$

are the fuzzy intersection, the fuzzy union of the fuzzy sets a_i's, and the fuzzy complement of the fuzzy set a, respectively.

Two models of fuzzy quantum posets were considered by A. Dvurečenskij, F. Chovanec, F. Kôpka and L. B. Long in [1, 2, 3, 4, 8], where two fuzzy sets a and b are said to be orthogonal, notation $a \perp b$ iff $a + b \leqslant 1$, and fuzzy orthogonal, notation $a \perp_F b$ iff $a \cap b \leqslant \frac{1}{2}$. (See also Mesiar [9].)

By a model I and model II of a fuzzy quantum poset we understand a couple (Ω, M), where Ω is a non-void set and $M \subset [0,1]^\Omega$ is a system of fuzzy sets such that

(i) If $\mathbf{1}(\omega) = 1$ for any $\omega \in \Omega$, then $\mathbf{1} \in M$;

(ii) if $a \in M$, then $a^\perp := 1 - a \in M$;

(iii) if $\frac{1}{2}(\omega) = \frac{1}{2}$ for any $\omega \in \Omega$, then $\frac{1}{2} \notin M$;

(iv) (model I) if $\{a_n\}_{n=1}^\infty \subseteq M$, $a_n \perp_F a_m$ for any $n \neq m$ then $\bigcup_{n=1}^\infty a_n \in M$;

(model II) if $\{a_n\}_{n=1}^\infty \subseteq M$, $a_n \perp a_m$ for any $n \neq m$ then $\bigcup_{n=1}^\infty a_n \in M$.

If (iv) is replaced by a stronger form (iv)* $\bigcup_{n=1}^\infty a_n \in M$ for any sequence $\{a_n\}_{n=1}^\infty \subseteq M$, then (Σ, M) is said to be a fuzzy quantum space; this model was originally introduced by Riečan [12] as a new axiomatic model of quantum mechanics. Similarly, the model II was suggested by J. Pykacz [11].

It is obvious that a fuzzy quantum space is a fuzzy quantum poset, and a model I of a fuzzy quantum poset is a model II, but the converse is not true, in general, as we can see below.

Example 1. Put $\Omega = [0, 1]$. Consider

$$a(\omega) = \begin{cases} 0.7 \text{ if } & 0 \leqslant \omega < 0.6 \\ 0.3 \text{ if } & 0.6 \leqslant \omega \leqslant 1, \end{cases}$$

$$b(\omega) = \begin{cases} 0.4 \text{ if } & 0 \leqslant \omega < 0.8 \\ 0.6 \text{ if } & 0.8 \leqslant \omega \leqslant 1, \end{cases}$$

$$c = a \cup a^\perp; \quad d = b \cup b^\perp.$$

Put $M = \{0, 1, a, b, c, d, a^\perp, b^\perp, c^\perp, d^\perp\}$, then (Ω, M) is a model II of a fuzzy quantum poset.

On the other hand, we see that $a \perp_F b$ but $a \cup b \notin M$, hence (Ω, M) is not a model I.

2. Representations of observables

We recall that an observable X on (Ω, M) is a function from $B(R)$, the σ-algebra of Borel sets of the real numbers R, into M such that

(i) $X(E^c) = X(E)^\perp$ for any $E \in B(R)$.

(ii) $X(\cup_{i=1}^\infty E_i) = \bigcup_{i=1}^\infty X(E_i)$ for any $E_i \in B(R)$.

A simple example of an observable is a mapping X_a, where a is a fixed fuzzy element of M, defined by

$$X_a(E) = \begin{cases} a \cup a^{\perp} & \text{if } 0,1 \in E \\ a & \text{if } 0 \notin E, 1 \in E \\ a^{\perp} & \text{if } 0 \in E, 1 \notin E \\ a \cap a^{\perp} & \text{if } 0,1 \notin E \end{cases} \tag{1}$$

for any $E \in B(R)$.

X_a plays the role of the indicator of the fuzzy set $a \in M$. If X is an observable on M and $E, F \in B(R)$, $E \cap F = \emptyset$, then $X(E) \perp X(F)$ as well as $X(E) \perp_F X(F)$.

Indeed, we have $X(F^c) = X(E \cup (F^c \cap E^c)) = X(E) \cup X(F^c \cap E^c)$, which entails the orthogonality of $X(E)$ and $X(F)$.

Let us define

$$K(M) := \left\{ A \subseteq \Omega;\ \exists a \in M;\ \{a > \tfrac{1}{2}\} \subseteq A \subseteq \{a \geqslant \tfrac{1}{2}\} \right\},$$

where $\{a > \frac{1}{2}\} := \{\omega \in \Omega;\ a(\omega) > \frac{1}{2}\}$, analogously for $\{a \geqslant \frac{1}{2}\}$.

Let (Ω, M) be a model I or II of fuzzy quantum posets. Let a be a given fuzzy set of M. Put

$$\begin{aligned} M_a &= \{b \in M;\ b \cup b^{\perp} = a \cup a^{\perp}\}, \\ \Omega_a &= \{\omega \in \Omega;\ a(\omega) \neq \frac{1}{2}\}, \\ \Omega_a(b) &= \{\omega \in \Omega_a : b(\omega) = (a \cup a^{\perp})(\omega)\} \\ &= \{\omega \in \Omega_a : b(\omega) > \frac{1}{2}\}, \text{ for any } b \in M_a, \\ Q_a &= \{\Omega_a(b);\ b \in M_a\}. \end{aligned}$$

We see that if $b \cup b^{\perp} = a \cup a^{\perp}$, then $a \perp b$ iff $a \perp_F b$. Therefore, Theorems and Lemmas 2.1, 2.2, 2.3, 2.4, 3.1, 3.2, 3.3 for model I proved in [4] (see also A. Dvurečenskij, F. Chovanec, F. Kôpka [2]) are still valid for model II.

We recall that C is called a q-σ-algebra of subsets of a non-void set Ω if

(i) $\Omega \in$ C;

(ii) if $A \in$ C, then $\Omega - A \in$ C;

(iii) if $\{A_i\}_{i=1}^{\infty} \subseteq$ C, $A_i \cap A_j = \emptyset$ for any $i \neq j$, then $\bigcup_{i=1}^{\infty} A_i \in$ C.

The following theorems for model II of fuzzy quantum posets can be proved by methods analogous to those in [4] and, therefore, their proofs are omitted.

Theorem 2. (i) *Q_a is a q-σ-algebra, for any $a \in M$.*

(ii) *The mapping $\Omega_a(.)\colon M_a \to Q_a$ defined by $b \mapsto \Omega_a(b)$ is a σ-orthoisomorphism, i.e., it is bijective and preserves the maximal elements, complements and joins of any sequences of mutually orthogonal elements.*

Theorem 3. *Let X be an observable on (Ω, M), then there is a unique function $\varphi\colon \Omega_{X(R)} \to R$ such that φ is $Q_{X(R)}$-measurable and*

$$\Omega_{X(R)}(X(E)) = \varphi^{-1}(E), \qquad E \in B(R). \tag{2}$$

Conversely, for any Q_a-measurable mapping $\varphi\colon \Omega_a \to R$, where $a \in M$, there is a unique observable X of (Ω, M) with $X(R) = a \cup a^{\perp}$ such that (2) holds.

Theorem 4. *Let X be an observable of a fuzzy quantum poset (Ω, M), and let Q be the set of all rational numbers. For any $r \in Q$ denote $B_X(r) = X((-\infty, r))$. The system $\{B_X(r);\, r \in Q\}$ fulfills the following conditions:*

(i) *$B_X(s) < B_X(t)$ if $s < t$; $s, t \in Q$;*

(ii) *$\bigcup_{r \in Q} B_X(r) = a$; $\bigcap_{r \in Q} B_X(r) = a^{\perp}$;*

(iii) *$\bigcup_{s<r} B_X(s) = B_X(r)$, $r \in Q$;*

(iv) *$B_X(r) \bigcup B_X(r)^{\perp} = a$, $r \in Q$,*

where $a = X(R)$, $a^{\perp} = X(\emptyset)$.

Conversely, let $\{B(r);\, r \in Q\}$ be a system of fuzzy sets from M fulfilling the conditions (i)–(iv) *for some $a \in M$. Then there is a unique observable X on (Ω, M) such that $B_X(r) = B(r)$ for any $r \in Q$ and $X(R) = a$.*

A representation of fuzzy observables in model I of fuzzy quantum posets was given by the author and A. Dvurečenskij [4], and for fuzzy quantum spaces by A. Dvurečenskij [5]. In both cases the proofs have used an embeddings of M onto orthocomplemented, σ-orthocomplete, orthomodular poset and a Boolean σ-algebra M/J_0 (see [4, 5]), respectively, and the representation of M/J_0 by $K(M)$'s. An interesting direct method of representation of observables via pointwise, $K(M)$-measurable real-valued functions, has been presented by Kolesárová [6] and Kolesárová and Mesiar [7]. Applying this method we will give a representation of fuzzy observables on model II.

Theorem 5. *Let (Ω, M) be a model II of a fuzzy quantum poset, let X be an observable on M. Then there is a $K(M)$-measurable function $f\colon \Omega \to R$ such that*

$$\{X(E) > \tfrac{1}{2}\} \subseteq f^{-1}(E) \subseteq \{X(E) \geqslant \tfrac{1}{2}\} \tag{3}$$

for any Borel set E. Moreover, if g is any $K(M)$-measurable, real-valued function on Ω, then g fulfills (3) *iff*

$$\{\omega \in \Omega;\ f(\omega) \neq g(\omega)\} \subseteq \{X(\emptyset) = \tfrac{1}{2}\}.$$

Proof. According to Kolesárová [6], for any given ω from Ω we consider $X(\omega, .)\colon R \to [0,1]$ defined by

$$T \mapsto X(\omega, t) = X((-\infty, t))(\omega).$$

From the properties of fuzzy observables (Theorem 4) it follows that $X(\omega, .)$ is a non-decreasing function with two values, $X(R)(\omega)$ or $1 - X(R)(\omega)$. Therefore, there exists $a_\omega \in R$ such that

$$X(\omega, t) = \begin{cases} X(R)(\omega) & \text{if } t > a_\omega, \\ 1 - X(R)(\omega) & \text{if } t \leqslant a_\omega, \end{cases}$$

$a_\omega = \sup\{t \in R;\ X(\omega, t) = 1 - X(R)(\omega)\}$ if $X(R)(\omega) \neq \frac{1}{2}$. In the case $X(R)(\omega) = \frac{1}{2}$, a_ω can be chosen arbitrarily.

It is clear that $X((-\infty, a_\omega])(\omega) = X((-\infty, a_\omega))(\omega) \cup X(\{a_\omega\})(\omega)$. Thus

$$X(\{a_\omega\})(\omega) = X(R)(\omega).$$

Further, $X(E)(\omega)$ assumes only two values, $X(R)(\omega)$ or $1 - X(R)(\omega)$. Hence X can be written in the form

$$X(E)(\omega) = \begin{cases} X(R)(\omega) & \text{if } a_\omega \in E \\ 1 - X(R)(\omega) & \text{if } a_\omega \notin E; \end{cases} \tag{4}$$

Now we consider a function $f\colon \Omega \to R$ defined by

$$\omega \mapsto f(\omega) = a_\omega.$$

We claim to prove that f fulfills the conditions of the theorem. To prove that, it suffices to verify that

$$\{X(E) > \tfrac{1}{2}\} \subseteq f^{-1}(E) \subseteq \{X(E) \geqslant \tfrac{1}{2}\}.$$

This is straightforward from the definition of f and (4).

Now, let g be a $K(M)$-measurable function with the condition (3). Then

$$\begin{aligned}\{\omega\in\Omega;\ f(\omega)<g(\omega)\} &= \bigcup_{r\in Q}\{\omega\in\Omega;\ f(\omega)<r<g(\omega)\}\\ &= \bigcup_{r\in Q}\{\omega\in\Omega;\ f(\omega)<r\}\cap\{\omega\in\Omega;\ g(\omega)>r\}\\ &\subseteq \bigcup_{r\in Q}\{\omega\in\Omega;\ f(\omega)<r\}\cap\{\omega\in\Omega;\ g(\omega)\geqslant r\}\\ &\subseteq \bigcup_{r\in Q}\{\omega\in\Omega;\ X((-\infty,r)\cap[r,\infty))(\omega)\geqslant \tfrac{1}{2}\}\\ &= \{\omega\in\Omega;\ X(\emptyset)(\omega)=\tfrac{1}{2}\},\end{aligned}$$

where Q is the set of all rational numbers. Similarly, $\{\omega\in\Omega;\ f(\omega)>g(\omega)\}\subseteq\{\omega\in\Omega;\ X(\emptyset)(\omega)=\frac{1}{2}\}$. Thus $\{\omega\in\Omega;\ f(\omega)\neq g(\omega)\}\subseteq\{\omega\in\Omega;\ X(\emptyset)(\omega)=\frac{1}{2}\}$. Conversely, if g is a $K(M)$-measurable function from Ω into R such that $A:=\{\omega\in\Omega;\ f(\omega)\neq g(\omega)\}\subseteq\{\omega\in\Omega;\ X(\emptyset)(\omega)=\frac{1}{2}\}$ we claim to verify the condition (3). It is clear that if $X(E)(\omega)>\frac{1}{2}$ then $\omega\notin A$. So $\omega\in f^{-1}(E)\cap A^c=g^{-1}(E)\cap A^c$, this means that $\{X(E)>\frac{1}{2}\}\subseteq g^{-1}(E)$. On the other hand, if $\omega\in g^{-1}(E)$, there are two cases:

(i) $\omega\in A$, then $X(\emptyset)(\omega)=\frac{1}{2}=X(E)(\omega)$;

(ii) $\omega\notin A$, Then $\omega\in f^{-1}(E)$, therefore $X(E)(\omega)\geqslant\frac{1}{2}$ which entails $g^{-1}(E)\subseteq\{X(E)\geqslant\frac{1}{2}\}$. □

The converse of Theorem 5 for model I of fuzzy quantum posets was proved in [4], but it is not true for model II, in general, as we show below.

Counterexample 6. Let $\Omega=[0,1]$, a, b, c, d be as in Example 1.

$$\text{Put}\quad e(\omega)=\begin{cases}0.1 & \text{if } 0\leqslant\omega<0.6 \text{ or } 0.8\leqslant\omega\leqslant 1,\\ 0.9 & \text{if } 0.6\leqslant\omega<0.8;\end{cases}$$
$$f(\omega)=0.9 \text{ for } 0\leqslant\omega\leqslant 1;$$

and $M=\{0,1,a,b,c,d,e,f,a^\perp,b^\perp,c^\perp,d^\perp,e^\perp,f^\perp\}$. Then (Ω,M) is a type II of fuzzy quantum posets.

We see that $K(M)=\{\emptyset,\Omega,A,B,C,A^c,B^c,C^c\}$, where $A=[0,0.6)$; $B=[0.8,1]$; $C=[0.6,0.8)$. Hence $K(M)$ is a σ-algebra. Therefore, the function $h=I_{A^c}+2I_{B^c}$, where I_{A^c}, I_{B^c} are indicators of the sets A^c and B^c, respectively, is $K(M)$-measurable and such that

$$h^{-1}(\{1\})=A^c-C=B,\ h^{-1}(\{2\})=B^c-C=A,\ h^{-1}(\{3\})=C.$$

On the other hand, there exist unique a, b and e such that

$$\{a > \tfrac{1}{2}\} \subseteq A \subseteq \{a \geqslant \tfrac{1}{2}\},$$
$$\{b > \tfrac{1}{2}\} \subseteq B \subseteq \{b \geqslant \tfrac{1}{2}\},$$
$$\{e > \tfrac{1}{2}\} \subseteq C \subseteq \{e \geqslant \tfrac{1}{2}\},$$

but $a \cup a^{\perp} \neq b \cup b^{\perp} \neq e \cup e^{\perp} \neq a \cup a^{\perp}$. Therefore, there exists no observable X on (Ω, M) such that (3) holds. □

For any sequence $\{a_n\}_{n=1}^{\infty}$ of fuzzy sets of a model II fuzzy quantum poset (Ω, M) there exist $1_K = \bigcap_{n=1}^{\infty} (a_n \cup a_n^{\perp}) \in M$ and $0_K = 1_K^{\perp} \in M$. However, in general $a_n \cap 1_K \cup 0_K$ does not belong to M if (Ω, M) is not a type I, which entails that the converse of Theorem 5 fails for type II, in general. Nevertheless, we have a converse of Theorem 5 in the following case.

Theorem 7. *Let (Ω, M) be a model II fuzzy quantum poset such that*

$$(5) \qquad a_n \cap \left(\bigcup_{m=1}^{\infty} (a_m \cup a_m^{\perp}) \right) \in M$$

for any $n \geqslant 1$ and any sequence $\{a_n\}_{n=1}^{\infty}$ of M. If $f\colon \Omega \to R$ is any $K(M)$-measurable function, then there exists an observable X of (Ω, M) with (3). If Y is any observable of (Ω, M) with (3), then $X(E) \perp_F Y(E^C)$ for any $E \in B(R)$.

The theorem can be proved by a method analogous to the proof of Theorem 5.2 in [4]. Therefore, the proof is omitted.

It should be noted that a model II of a fuzzy quantum poset with the condition (5) need not be a model I, see the following example.

Example 8. Let $\Omega, a, b, c, d,$ be as in Example 1.

$$\text{Put} \quad g(\omega) = \begin{cases} 0.6 & \text{if } 0 \leqslant \omega < 0.6, \\ 0.4 & \text{if } 0.6 \leqslant \omega \leqslant 1, \end{cases}$$

put $M = \{0, 1, a, b, c, d, g, a^{\perp}, b^{\perp}, c^{\perp}, d^{\perp}, g^{\perp}, g \cup b, g^{\perp} \cap b^{\perp}\}$. Then (Ω, M) is a model II of a fuzzy quantum poset with (5) but it is not a model I.

Theorem 9. *Let X be an observable of a model II of fuzzy quantum poset (Ω, M) and let $\varphi\colon \Omega_{X(R)} \to R$ be a unique $Q_{X(R)}$-measurable function on Ω. Then f fulfills* *condition* (3) *of Theorem* 5 *iff*

$$f(\omega) = \begin{cases} \varphi(\omega) & \text{if } \omega \in \Omega_{X(R)}, \\ \varphi_0(\omega) & \text{if } \omega \in \Omega - \Omega_{X(R)}, \end{cases}$$

where φ_0 *is any mapping from* $\Omega - \Omega_{X(R)}$ *into* R.

Proof. Theorem is proved by a method similar to the proof of Theorem 5.4 [4]. □

Following the ideas from [7], we arrive at the following result.

Theorem 10. *Let* X *be an observable on a model II of a fuzzy quantum poset* (Ω, M). *Then there is a* $K(M)$*-measurable, real-valued function* f *and a fuzzy set* $c \in W_1(M)$ *such that*

$$X(E)(\omega) = \begin{cases} c(\omega) & \text{if } \omega \in f^{-1}(E) \\ 1 - c(\omega) & \text{if } \omega \notin f^{-1}(E) \end{cases} \tag{6}$$

for any $E \in B(R)$.

Conversely, if f *is a* $K(M)$*-measurable real-valued function and* $c \in W_1(M)$ *is such that for any* $E \in C$ *the right-hand side of* (6) *determines fuzzy sets* $X(E)$ *from* M, *where* C *is a* σ*-countable generator of* $B(R)$ *which is closed with respect to finite intersection, for example* $C = \{(-\infty, r)\,;\, r \in Q\}$, *then* (6) *defines a unique observable* X *of* (Ω, M).

Proof. If X is an observable, then from Theorem 5 we have an $f\colon \Omega \to R$ such that (3) holds. If we put $c = X(R)$, then (6) is true.

Conversely, let B be the set of all $E \in B(R)$ such that (6) defines $X(E) \in M$. Then

(i) $\emptyset$, $R \in B$, $C \subseteq B$;

(ii) if $E \in B$, then $E^c \in B$ and $X(E^c) = X(E)^{\perp}$;

(iii) if $E, F \in C$ then $E \cap F \in C \subseteq B$;

(iv) if $E, F \in B$, $E \cap F = \emptyset$ then $X(E) \perp X(F)$;

(v) if $\{E_i\}_{i=1}^{\infty} B$, $E_i \cap E_j = \emptyset$, $i \neq j$, then

$$X\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \bigcup_{i=1}^{\infty} X(E_i) \in M.$$

Hence,

$$\bigcup_{i=1}^{\infty} E_i \in B.$$

Due to Proposition 4.13 and Theorem 4.20 by Neubrunn and Riečan [10], we see that $B = B(R)$. Therefore, X is an observable on (Ω, M). □

Remark. If (Ω, M) is a type I of fuzzy quantum posets, then for any $K(M)$-measurable function $f: \Omega \to R$ there exists $c \in W_1(M)$ such that (6) always defines an observable X of (Ω, M).

3. Summability of observables

Let X, Y be two observables on a model I, II of fuzzy a quantum poset (Ω, M). Let Q be the set of all rational numbers.

$$\text{Put } B_z(t) = \bigcup_{r \in Q} B_x(r) \cap B_y(t - r), \qquad t \in Q.$$

We see that if there exists $B_z(t) \in M$ for any $t \in Q$, then $\{B_z(t); t \in Q\}$ fulfills the conditions of Theorem 3. Therefore, there exists a unique observable called the sum of X and Y and we write $Z = X + Y$; X and Y are said to be summable.

Let X be an observable on M. We write $X \sim f$ and $X \approx \varphi_X$ if f is defined by Theorem 4 and φ_X by Theorem 3.

Proposition 11. *Let X, Y be two observables on a model I, II of a fuzzy quantum poset (Ω, M) and let $X \sim f$, $Y \sim g$, $X \approx \varphi_X$, $Y \approx \varphi_Y$. If X and Y are summable then $f+g$ is also $K(M)$-measurable and $X+Y \sim f+g$, $(X+Y)(R) = X(R) \bigcap Y(R)$. Therefore $\varphi_X + \varphi_Y$ is $Q_{(X+Y)(R)}$-measurable and $X + Y \approx \varphi_X + \varphi_Y$.*

Proof. Let $X + Y \sim h$. We see that:

$$\{B_{X+Y}(t) > \tfrac{1}{2}\} \subseteq \{h < t\} \subseteq \{B_{X+Y}(t) \geqslant \tfrac{1}{2}\}.$$

On the other hand, for any $t, r \in Q$.

$$\{B_X(r) \cap B_Y(t - r) > \tfrac{1}{2}\} \subseteq \{f < t\} \cap \{g < t - r\} \subseteq \{B_X(r) \cap B_Y(t - r) \geqslant \tfrac{1}{2}\}.$$

Hence

$$\Big\{\bigcup_{r \in Q} B_X(r) \cap B_Y(t - r) > \tfrac{1}{2}\Big\} \subseteq \bigcup_{r \in Q} \{f < t\} \cap \{g < t - r\}$$
$$\subseteq \Big\{\bigcup_{r \in Q} B_X(r) \cap B_Y(t - r) \geqslant \tfrac{1}{2}\Big\},$$

i.e. $\{B_{X+Y}(t) > \frac{1}{2}\} \subseteq \{f+g<t\} \subseteq \{B_{X+Y}(t) \geqslant \frac{1}{2}\}$. Therefore

$$\begin{aligned}\{f+g<h\} &\subseteq \bigcup_{r\in Q}\{f+g<r\}\cap\{h>r\}\\ &\subseteq \bigcup_{r\in Q}\{B_{X+Y}(t)\geqslant \tfrac{1}{2}\}\cap\{B_{X+Y}(t)\geqslant \tfrac{1}{2}\}^c\\ &\subseteq \bigcup_{r\in Q}\{B_{X+Y}(t)=\tfrac{1}{2}\}=\{(X+Y)(R)=\tfrac{1}{2}\}\end{aligned}$$

In the same way, we see also that $\{f+g>h\} \subseteq \{(X+Y)(R) = \frac{1}{2}\}$. Hence $A := \{h = f+g\} \subseteq \{(X+Y)(R) = \frac{1}{2}\}$. Thus $A^c \supseteq \{(X+Y)(E) > \frac{1}{2}\}$ for any Borel set E. We see that $(f+g)^{-1}(E) \cap A \in K(M)$ for any Borel set E. So $(f+g)^{-1}(E) \cap A^c = h^{-1}(E) \cap A^c$

$$\{(X+Y)(E) > \tfrac{1}{2}\} \subseteq h^{-1}(E)\cap A^c \subseteq \{(X+Y)(E) \geqslant \tfrac{1}{2}\}.$$

On the other hand,

$$(f+g)^{-1}(E) = ((f+g)^{-1}(E)\cap A)\cup((f+g)^{-1}(E)\cap A^c).$$

Hence $\{(X+Y)(E) > \frac{1}{2}\} \subseteq (f+g)^{-1}(E) \subseteq \{(X+Y)(E) \geqslant \frac{1}{2}\}$ for any Borel set E, which entails $K(M)$-measurability of $f+g$ and $X+Y \sim f+g$.

It can be proved that if X, Y are summable then $(X+Y)(R) = X(R)\cap Y(R)$. Therefore, $\varphi_X + \varphi_Y$ is $Q_{(X+Y)(R)}$-measurable and $X+Y \approx \varphi_X + \varphi_Y$, by Theorem 9. □

As we can see from the following, the sum of two observables on a fuzzy quantum poset need not always exist.

Example 12. Let $\Omega = [0,1]$. Put $M = \{0, 1, a_0, b_0, c, d, a_0^\perp, b_0^\perp, c^\perp, d^\perp\}$, where

$$a_0(\omega) = \begin{cases} 0.7 & \text{if } 0 \leqslant \omega < 0.6,\\ 0.3 & \text{if } 0.6 \leqslant \omega \leqslant 1,\end{cases}$$

$$b_0(\omega) = \begin{cases} 0.4 & \text{if } 0 \leqslant \omega < 0.8,\\ 0.6 & \text{if } 0.8 \leqslant \omega \leqslant 1,\end{cases}$$

$$c(\omega) = 0.7 \quad 0 \leqslant \omega \leqslant 1,$$
$$d(\omega) = 0.6 \quad 0 \leqslant \omega \leqslant 1.$$

Then (Ω, M) is a model II of a fuzzy quantum poset, and

$$K(M) = (\emptyset, \Omega, A, B, A^c, B^c),$$

where $A = [0, 0.6)$; $B = [0.8, 1]$. $K(M)$ consists only from two proper sub σ-algebras $\{\emptyset, \Omega, A, A^c\}$, $\{\emptyset, \Omega, B, B^c\}$. Therefore $f\colon \Omega \to R$ is $K(M)$-measurable iff $f = \alpha I_A + \beta I_{A^c}$ or $f = \gamma I_B + \delta I_{B^c}$, where $\alpha, \beta, \gamma, \delta \in R$.

Let X_{a_0}, X_{b_0} be two observables on (Ω, M) defined via (1). Then it is clear that $X_{a_0} \sim I_A$, $X_{b_0} \sim I_B$. But $I_A + I_B$ is not $K(M)$-measurable. Therefore, it follows from Proposition 10 that X_{a_0} and X_{b_0} are not summable on (Ω, M). □

Definition 12. Let (Ω, M) be a model I or II of a fuzzy quantum poset. Let $X_i \sim f_i$ for $i = 1, 2, \ldots, N$, where N can be either integer or ∞, and let $F\colon R^N \to R$ be a Borel measurable function. We define $F(X_1, \ldots, X_N)$ as any observable X of (Ω, M) such that

(i) $F(f_1, \ldots, f_N)$ is $K(M)$-measurable;

(ii) $X \sim F(f_1, \ldots, f_N)$;

(iii) $X(R) = \bigcap_{i=1}^{N} X_i(R)$.

It is easy to verify that such an X is unique. We recall that if $N = \infty$, then by F we mean some limit expressions, or convergence, respectively.

This definition enables us to define a calculus of observables. For example, let $F(u, v) = u + v$; $u, v \in R$. If X and Y are summable and $X \sim f$, $Y \sim g$, then $X + Y \sim F(f, g)$, i.e. $X + Y = F(X, Y)$. If we consider $G(u, v) = u.v$; $u, v \in R$ and there exists $X.Y$ then $X.Y \sim G(f, g)$, $X.Y = G(X, Y)$.

If (Ω, M) is a quantum space, the conditions of Definition 12 are fulfilled for any $F\colon R^N \to R$. Consequently, in this case we can always define $F(X_1, \ldots, X_N)$. In a model I, II of fuzzy quantum posets, this is not true, in general. (It suffices take into account the summation.)

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