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ON AN INEQUALITY AND THE RELATED CHARACTERIZATION OF THE GAMMA DISTRIBUTION

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Summary. In this paper we derive conditions upon the nonnegative random variable ξ under which the inequality $\mathrm{D}g(\xi) \leqslant c\mathrm{E}[g'(\xi)]^2\xi$ holds for a fixed nonnegative constant c and for any absolutely continuous function g. Taking into account the characterization of a Gamma distribution we consider the functional $U_\xi = \sup\limits_g \frac{\mathrm{D}g(\xi)}{\mathrm{E}[g'(\xi)]^2\xi}$ and establishing some of its properties we show that $U_\xi \geqslant 1$ and that $U_\xi = 1$ iff the random variable ξ has a Gamma distribution.

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1. INTRODUCTION

If ξ is $N(0,1)$ and g is an absolutely continuous realvalued function with finite variance, then Chernoff in [1] proved

$$\mathrm{D}g(\xi) \leqslant \mathrm{E}[g'(\xi)]^2. \tag{1.1}$$

Considering this inequality, Borovkov and Utev in [2] characterized the normal distribution by the functional

$$\sup_g \frac{\mathrm{D}g(\xi)}{\mathrm{D}\xi \cdot \mathrm{E}[g'(\xi)]^2} = 1. \tag{1.2}$$

Using the multivariate analogue of (1.1) derived by Chen in [5], Prakasa Rao and Sreehari in [6] characterized the multivariate normality. Cacoullos in [3] gave lower and upper bound inequalities for the variance of $g(\xi)$, even for the discrete random variable ξ. The upper bound inequalities in [3] and those obtained by Papathanasiou and Cacoullos in [4] were used by Prakasa Rao and Sreehari in [6, 7] to characterize the Poisson and multivariate normal distributions. T. Cacoullos and V. Papathanasiou in [8] used the functional

$$\inf_g \frac{\mathrm{D}g(\xi)}{\mathrm{D}\xi \cdot \mathrm{E}^2[w(\xi)g'(\xi)]} = 1 \tag{1.3}$$

to characterize the distribution of ξ through the function $w(x)$ with $\mathrm{E}[w(\xi)] = 1$. The special case of $w(x) = 1$ gives the normal and Poisson distributions, as well as the characterizations in [2, 7]. In [8], it is also noted that both the upper bound inequalities with the sup functional as in (1.2) and the lower bound inequalities with the inf functional as in (1.3) lead to the same characterizing results.

In this paper, we obtain that in a certain sense the equality $w(x) = x$ characterizes the Gamma distribution. We also show conditions under which, for a nonnegative random variable ξ, the inequality $\mathrm{D}g(\xi) \leqslant c\mathrm{E}[g'(\xi)]^2\xi$ holds, where c is a fixed nonnegative constant and g is any absolutely continuous function.

2. Main results

Let H_1 be the set of all absolutely continuous functions on $[0, t]$ for each $t > 0$.

Theorem 1. *Let ξ be a nonnegative random variable with distribution function $F(x)$ and density function $f(x)$. Suppose that for some fixed nonnegative constants x_0 and c the following inequalities hold:*

$$\begin{aligned} \int_x^\infty (y - x_0)\,\mathrm{d}F(y) &\leqslant cxf(x) \quad \text{for } x \geqslant x_0, \\ -\int_0^x (y - x_0)\,\mathrm{d}F(y) &\leqslant cxf(x) \quad \text{for } x < x_0. \end{aligned} \tag{2.1}$$

Then for any $g \in H_1$

$$Dg(\xi) \leqslant cE[g'(\xi)]^2\xi. \tag{2.2}$$

Proof. From the Cauchy-Schwartz inequality we have

$$\begin{aligned} \mathrm{D}g(\xi) \leqslant \mathrm{E}[g(\xi) - g(x_0)]^2 &= \int_0^\infty \Big(\int_{x_0}^x g'(t)\,\mathrm{d}t\Big)^2 \mathrm{d}F(x) \\ &\leqslant \int_{x_0}^\infty \int_{x_0}^x [g'(t)]^2\,\mathrm{d}t(x - x_0)\,\mathrm{d}F(x) + \int_0^{x_0} \int_x^{x_0} [g'(t)]^2\,\mathrm{d}t(x_0 - x)\,\mathrm{d}F(x). \end{aligned}$$

Changing the order of integration we get

$$\int_{x_0}^{\infty}\int_{x_0}^{x}[g'(t)]^2\,\mathrm{d}t(x-x_0)\,\mathrm{d}F(x)=\int_{x_0}^{\infty}[g'(t)]^2\int_t^{\infty}(x-x_0)\,\mathrm{d}F(x)\,\mathrm{d}t$$
$$\leqslant c\int_{x_0}^{\infty}t[g'(t)]^2f(t)\,\mathrm{d}t.$$

In the same way,

$$\int_0^{x_0}\int_x^{x_0}[g'(t)]^2\,\mathrm{d}t(x_0-x)\,\mathrm{d}F(x)\leqslant c\int_0^{x_0}t[g'(t)]^2f(t)\,\mathrm{d}t.$$

Consequently, $\mathrm{D}g(\xi)\leqslant c\int_0^{\infty}t[g'(t)]^2f(t)\,\mathrm{d}t=c\mathrm{E}[g'(\xi)]^2\xi$. □

Remark. If $\mathrm{E}\xi=x_0$, then $\int_0^{\infty}(y-x_0)\,\mathrm{d}F(y)=0$ and condition (2.1) can be represented as an inequality

$$\int_x^{\infty}(y-x_0)\,\mathrm{d}F(y)\leqslant cxf(x),\quad 0\leqslant x<\infty. \tag{2.3}$$

In the following, we assert that the random variable ξ has a Gamma distribution with parameters (α,β) (and will be denoted by $\xi\sim\Gamma(\alpha,\beta)$) if $P\{\xi<x\}=I(x>0)\cdot[\Gamma(\alpha)]^{-1}\int_0^x\beta^{\alpha}y^{\alpha-1}\mathrm{e}^{-\beta y}\,\mathrm{d}y$.

Corollary 1. *If ξ is a Gamma distributed random variable with parameters $(x_0/c,1/c)$ where x_0 and c are fixed positive constants, then the assertion of Theorem 1 holds, i.e.*

$$Dg(\xi)\leqslant c\mathrm{E}[g'(\xi)]^2\xi.$$

This inequality cannot be improved as for linear functions, it becomes the equality.

Proof. In our case the density function is

$$f(x)=\frac{\mathrm{e}^{-\frac{x}{c}}x^{\frac{x_0}{c}-1}}{c^{\frac{x_0}{c}}\Gamma(\frac{x_0}{c})},\quad x\geqslant 0.$$

Easy calculations yield that $\mathrm{E}\xi=x_0$ and if (2.3) holds then applying Theorem 1 we obtain $\mathrm{D}g(\xi)\leqslant c\mathrm{E}[g'(\xi)]^2\xi$. Consequently, the corollary will be proved if we show that if $\xi\sim\Gamma(\frac{x_0}{c},\frac{1}{c})$ then(2.1) is fulfilled. Indeed, the following equality can be obtained:

$$\int_x^{\infty}(y-x_0)\,\mathrm{d}F(y)=cxf(x)\quad 0\leqslant x<\infty. \tag{2.4}$$

To verify this it is sufficient to differentiate (2.4) and to compare the values of both sides in (2.4) for some x (we may choose for example $x = 0$).

Let us return to the inequality (2.2) and consider the following classes of real-valued functions:

$$L_2(\xi) = \{g\colon \mathrm{E}g^2(\xi) < \infty\},$$
$$Q(\xi) = \{g\colon \mathrm{D}g(\xi) > 0\},$$
$$H_1(\xi) = \{H_1 \cap L_2(\xi) \cap Q(\xi)\}.$$

For a nondegenerate random variable ξ, let us define the functional

$$R_\xi = R(F) = \sup_{g \in H_1(\xi)} \frac{\mathrm{D}g(\xi)}{\mathrm{E}[g'(\xi)]^2 \xi}. \tag{2.5}$$

Now the assertion of Theorem 1 and of Corollary 1 can be stated in terms of the functional R_ξ as follows: $R_\xi = R(F) \leqslant c$ and $R[\Gamma(x_0/c, 1/c)] = c$.

In (2.5) we use the class of functions $H_1(\xi)$ to avoid indefiniteness of the types $0/0$ and ∞/∞. □

Theorem 2. *Let ξ be a nonnegative, nondegenerate random variable. The functional R_ξ has the following properties:*

(i) *If $g \in H_1$, $R_\xi < \infty$ and $E[g'(\xi)]^2\xi < \infty$ then $g \in L_2(\xi)$ and $Dg(\xi) \leqslant R_\xi E[g'(\xi)]^2\xi$.*

(ii) *The equality $R_{a\xi} = aR_\xi$ holds for any constant $a \neq 0$.*

(iii) *If $E\xi < \infty$ then $D\xi \leqslant R_\xi E\xi$.*

(iv) *If $R\xi < \infty$ then*

$$E\xi^n \leqslant E\xi(E\xi + R_\xi)\Big(E\xi + \Big(\frac{3}{2}\Big)^2 R_\xi\Big) \ldots \Big(E\xi + \Big(\frac{n}{2}\Big)^2 R_\xi\Big), \quad n = 1, 2, \ldots$$

The proof of (i) is based on an auxiliary assertion which we state in the form of a lemma.

Let us denote $T(g, c) = \mathrm{D}g(\xi) - c\mathrm{E}[g'(\xi)]^2\xi$.

Lemma. *Assume that for $g \in H_1$, $g_n \in H_1 \cap L_2(\xi)$, $n = 1, 2, \ldots$ the following conditions are fulfilled:*

(i) $\lim_{n\to\infty} g_n(x) = g(x)$ *for any* x,

(ii) $T(g_n, c) = Dg_n(\xi) - cE[g_n'(\xi)]^2\xi \leqslant 0, \quad n = 1, 2, \ldots,$

(iii) $\limsup E[g_n'(\xi)]^2\xi \leqslant E[g'(\xi)]^2\xi < \infty$.

Then $g \in L_2(\xi)$ and $T(g, c) \leqslant 0$.

Proof. From the conditions of the lemma it can be easily obtained that

$$\limsup \mathrm{D}g_n(\xi) \leqslant c \limsup \mathrm{E}[g_n'(\xi)]^2\xi \leqslant c\mathrm{E}[g'(\xi)]^2\xi < \infty. \tag{2.6}$$

Now choosing a sequence $\{n'\}$ such that $\lim_{n'\to\infty} \mathrm{E}g_{n'}(\xi) = a$ we obtain from (2.6) according to the Fatou theorem

$$\begin{aligned}\mathrm{E}[g(\xi) - a]^2 = \mathrm{E} \lim_{n'\to\infty} [g_{n'}(\xi) - \mathrm{E}g_{n'}(\xi)]^2 &\leqslant \liminf_{n'} \mathrm{D}g_{n'}(\xi) \\ &\leqslant \limsup \mathrm{D}g_n(\xi) \leqslant c\mathrm{E}[g'(\xi)]^2\xi < \infty.\end{aligned}$$

Consequently $|a| < \infty$ and $g \in L_2(\xi)$. Moreover, we know that $\mathrm{D}g(\xi) \leqslant \mathrm{E}[g(\xi) - a]^2 \leqslant c\mathrm{E}[g'(\xi)]^2\xi < \infty$ for any a. In our terms, this means that $T(g, c) \leqslant 0$.

The lemma is proved. □

Proof of Theorem 2.

(i) We define a sequence of functions

$$g_n(x) = g(0) + \int_0^x \Psi_n(y)g'(y)\,\mathrm{d}y, \quad n = 1, 2, \ldots$$

where $\{\Psi_n\}_{n=1}^{\infty}$ are infinitely differentiable functions satisfying $0 \leqslant \Psi_n \leqslant 1$, $\Psi_n = 1$ for $|x| \leqslant n$, $\Psi_n(x) = 0$ for $|x| \geqslant n + 1$ and $\sup_n \sup_x |\Psi_n'(x)| = c < \infty$.

Now we shall show that the conditions of the lemma are fulfilled for $\{g_n(x)\}_{n=1}^{\infty}$. Since clearly

$$\sup_{x\in[0,\infty)} |g_n(x)| \leqslant |g(0)| + \int_0^{n+1} |g'(y)|\,\mathrm{d}y < \infty$$

then $g_n \in H_1 \cap L_2(\xi)$. As $g_n'(x) = \Psi_n(x)g'(x)$, it is clear that $|g_n'(x)| \leqslant |g'(x)|$ for all $x \geqslant 0$, too. From the definition, it is obvious that $g_n(x) = g(x)$ if $|x| \leqslant n$ and $T(g_n, R_\xi) \leqslant 0$. This means that the conditions (i), (ii) and (iii) of the lemma are fulfilled. We conclude that $g \in L_2(\xi)$ and $T(g, R_\xi) \leqslant 0$.

(ii) If we denote $\Psi(x) = g(ax)$ where $g \in H_1(\xi)$, we easily get

$$\begin{aligned}\mathrm{D}g(a\xi) = \mathrm{D}\Psi(\xi) &\leqslant R_\xi \mathrm{E}[\Psi'(\xi)]^2\xi \\ &= a^2 R_\xi \mathrm{E}[g'(a\xi)]^2\xi = aR_\xi \mathrm{E}[g'(a\xi)]^2 a\xi,\end{aligned}$$

that is $R_{a\xi} \leqslant aR_\xi$. However, since $\xi = \frac{1}{a} \cdot a\xi$ we have $R_\xi \leqslant \frac{1}{a} \cdot R_{a\xi}$. Consequently, $aR_\xi = R_{a\xi}$.

(iii) As for $R_\xi = \infty$ the inequality is obvious, let $R_\xi < \infty$. Now setting $g(x) = x$ we derive the equality $\mathrm{E}[g'(\xi)]^2\xi = \mathrm{E}\xi$ and applying the property (i) we get

$$\mathrm{D}\xi = \mathrm{D}g(\xi) \leqslant R_\xi \cdot \mathrm{E}[g'(\xi)]^2\xi = R_\xi \cdot \mathrm{E}\xi.$$

(iv) The case of $E\xi = \infty$ is trivial. Assume that $E\xi < \infty$.

To prove this property we use induction and Liapunov's inequality. By virtue of the property $R_{a\xi} = aR_\xi$, we may assume that $E\xi = 1$.

Now let $g_n(x) = x^{n/2}$, $n = 1, 2, \dots$

For $n = 2$ we have $g_2(x) = x \in H_1$, $R_\xi < \infty$ and since $E[g_2'(\xi)]^2\xi = E\xi < \infty$ the property (i) implies $g_2 \in L_2(\xi)$, $Dg_2(\xi) \leqslant R_\xi E[g_2'(\xi)]^2\xi$.

Hence $E\xi^2 < \infty$, $E\xi^2 \leqslant (E\xi)^2 + R_\xi \cdot E\xi = E\xi(E\xi + R_\xi) < \infty$.

Assuming that the inequality

$$E\xi^k \leqslant E\xi(E\xi + R_\xi)\Big(E\xi + \Big(\frac{3}{2}\Big)^2 R_\xi\Big)\dots\Big(E\xi + \Big(\frac{k}{2}\Big)^2 R_\xi\Big) < \infty$$

holds for $k = 2, 3, \dots, n$ we will prove now that it is also true for $k = n+1$, i.e.

$$E\xi^{n+1} \leqslant E\xi(E\xi + R_\xi)\Big(E\xi + \Big(\frac{3}{2}\Big)^2 R_\xi\Big)\dots\Big(E\xi + \Big(\frac{n+1}{2}\Big)^2 R_\xi\Big) < \infty.$$

Let us consider $g_{n+1}(x) = x^{(n+1)/2} \in H_1$. Then, as $R_\xi < \infty$ and $E[g_{n+1}'(\xi)]^2\xi = \left(\frac{n+1}{2}\right)^2 E\xi^n < \infty$, using again the property (i) we obtain that $g_{n+1} \in L_2(\xi)$, $Dg_{n+1}(\xi) \leqslant R_\xi E[g_{n+1}'(\xi)]^2\xi$.

After some computations we derive that $E\xi^{n+1} < \infty$ and

$$E\xi^{n+1} \leqslant [E\xi^{(n+1)/2}]^2 + \Big(\frac{n+1}{2}\Big)^2 R_\xi \cdot E\xi^n. \tag{2.7}$$

Applying Liapunov's inequality $(E|\eta|^{k_2})^{k_3-k_1} \leqslant (E|\eta|^{k_1})^{k_3-k_2}(E|\eta|^{k_3})^{k_2-k_1}$, $0 \leqslant k_1 \leqslant k_2 \leqslant k_3$ with $|\eta| = \xi^{1/2}$, $k_1 = 2$, $k_2 = n+1$, $k_3 = 2n$ we get $[E\xi^{(n+1)/2}]^2 \leqslant E\xi E\xi^n$.

Substituting the last inequality into (2.7) we obtain $E\xi^{n+1} \leqslant E\xi E\xi^n + \left(\frac{n+1}{2}\right)^2 R_\xi E\xi^n \leqslant E\xi^n\left(E\xi + \left(\frac{n+1}{2}\right)^2 R_\xi\right)$, which proves the property (iv). □

Let us consider the functional

$$U_\xi = U(F) = \frac{R_\xi \cdot E\xi}{D\xi} = \sup_{g \in H_1(\xi)} \frac{E\xi \cdot Dg(\xi)}{D\xi \cdot E[g'(\xi)]^2\xi}.$$

Theorem 2 gives that $U_\xi \geqslant 1$ for any distribution function F and $U_{a\xi} = U_\xi$, $a \neq 0$.

In the following theorem we characterize the Gamma distribution in terms of the functional U_ξ.

Theorem 3. *Let ξ be a nonnegative, nondegenerate random variable such that $R_\xi < \infty$. Then the functional U_ξ assumes its minimal value equal to 1 iff ξ is a Gamma distibuted random variable.*

Proof. To verify sufficiency we have only to note that for the random variable ξ having a Gamma distribution with parameters $(\alpha, 1)$, Corollary 1 implies that the equality $U_\xi = 1$ holds.

To prove necessity we first note that the equality $U_\xi = 1$ implies $R_\xi = \mathrm{D}\xi/\mathrm{E}\xi = \beta = \text{const}$. Now let us denote $\eta = \xi/\beta$.

As $R_\eta = R_{\xi/\beta} = \frac{1}{\beta} \cdot R_\xi = 1$ and $U_\eta = U_{\xi/\beta} = U_\xi = 1$ we have $1 = U_\eta = \frac{R_\eta \cdot \mathrm{E}\eta}{\mathrm{D}\eta} = \frac{\mathrm{E}\eta}{\mathrm{D}\eta}$. Consequently,

$$\mathrm{E}\eta = \mathrm{D}\eta. \tag{2.8}$$

Let us consider $h_n(x) = x^n \in H_1$. Applying the properties (i), (iv) from Theorem 2 for $h_n(x)$ with $n = 1, 2, \ldots$ we obtain $\mathrm{D}h_n(\eta) \leqslant \mathrm{E}[h_n'(\eta)]^2\eta < \infty$, i.e.

$$\mathrm{E}\eta^{2n} \leqslant (\mathrm{E}\eta^n)^2 + n^2\mathrm{E}\eta^{2n-1} \quad n = 1, 2, \ldots \tag{2.9}$$

Now, setting $g_n(x, \lambda) = x + \lambda x^n \in H_1$, $n = 1, 2, \ldots$ and using Theorem 2 and the properties (i), (iv) for $n = 1, 2, \ldots$ in the same way we get the inequality $\mathrm{D}g_n(\eta, \lambda) \leqslant \mathrm{E}[g_n'(\eta, \lambda)]^2\eta < \infty$.

Some calculations yield $\mathrm{E}(\eta + \lambda\eta^n)^2 - [\mathrm{E}(\eta + \lambda\eta^n)]^2 \leqslant \mathrm{E}(1 + n\lambda\eta^{n-1})^2\eta$, that is

$$\{\mathrm{E}\eta^{2n} - (\mathrm{E}\eta^n)^2 - n^2\mathrm{E}\eta^{2n-1}\}\lambda^2 + 2\{\mathrm{E}\eta^{n+1} - \mathrm{E}\eta\mathrm{E}\eta^n - n\mathrm{E}\eta^n\}\lambda + \mathrm{D}\eta - \mathrm{E}\eta \leqslant 0.$$

To complete the proof we have to note that $a\lambda^2 + b\lambda \leqslant 0$ for all λ iff $b = 0$, $a \leqslant 0$. Then from (2.8) and (2.9) we derive $\mathrm{E}\eta^{n+1} = (\mathrm{E}\eta + n)\mathrm{E}\eta^n$, $n = 1, 2, \ldots$. By induction, we finally get

$$\mathrm{E}\eta^{n+1} = \mathrm{E}\eta(\mathrm{E}\eta + 1)\ldots(\mathrm{E}\eta + n).$$

If we denote $\mathrm{E}\eta = \alpha$ then obviously all moments of η coincide with the corresponding moments of a Gamma distributed random variable with parameters $(\alpha, 1)$. In accordance with the theory (the problem of moments), if $\limsup_{n\to\infty} \frac{\sqrt[n]{\nu_n}}{n} < \infty$ then the moments ν_n determine the probability distribution uniquely. In our case it is easy to calculate that $\limsup_{n\to\infty} \frac{\sqrt[n]{\mathrm{E}\eta^n}}{n} = \mathrm{e}^{-1} < \infty$.

This proves that the moments of η determine its distribution uniquely, namely $\eta \sim \Gamma(\alpha, 1)$ and consequently $\xi \sim \Gamma(\alpha, \beta)$. $\square$

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