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## FINITE NONDENSE POINT SET ANALYSIS

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*Summary.* The paper deals with the decomposition and with the boundary and hull construction of the so-called nondense point set. This problem and its applications have been frequently studied in computational geometry, raster graphics and, in particular, in the image processing (see e.g. [3], [6], [7], [8], [9], [10]). We solve a problem of the point set decomposition by means of certain relations in graph theory.

*Keywords:* nondense point set, decomposition, boundary, hull, stabilized matrix

*AMS classification:* 68U10 (68R10, 05C12)

## 1. POINT SET DECOMPOSITION

Let  $\mathcal{Z}$  be a set of points in plane, i.e.  $\mathcal{Z} = \{A; A_i \in E_2, i = 1, 2, \dots, n, n > 2\}$ . Denote  $d$  the distance of two points from the set  $\mathcal{Z}$  and let  $R = \max |A_i A_j|$ , where  $A_i, A_j \in \mathcal{Z}$ . A non-oriented graph  $G = (V, H, w)$  is created, where  $V$  is the set of points  $A$  (vertices),  $H$  is the set of segments  $A_i A_j$  (edges) and  $w(e) = A_i A_j = d$ ,  $e \in H$ ,  $e = \{A_i A_j\}$ .

Let  $\mathcal{D} = \{d, 0 < d \leq R; G \text{ is a connected graph}\}$ , i.e.  $\mathcal{D}$  is the set of distances of two points from the set  $\mathcal{Z}$ . Elements of the set  $\mathcal{D}$  are less than or equal to  $R = \max |A_i A_j|$  such, that  $G$  is a connected graph. Denote  $d_m(G) = \min \mathcal{D}$ . Obviously  $d(G)$  is the weight of the longest edge of the graph spanning tree, [4].

The graph  $G$  can be defined using an incidence (Boolean) matrix  $\mathcal{G}$ . It is a symmetric  $n$  by  $n$  square matrix ( $n$  is the number of the graph vertices) in which the element

$$g_{ij} = \begin{cases} 1, & \text{if } |A_i A_j| \leq d \text{ vertices are linked up by an edge} \\ 0. & \text{otherwise} \end{cases}$$

Form a matrix

$$\mathcal{S}^k = E + \mathcal{G} + \mathcal{G}^2 + \dots + \mathcal{G}^k,$$

where element  $s_{ij} = s_{ji} = 1$  expressed that there exists a path between the vertices  $A_i, A_j$  whose length (in the sense of graph theory) is equal to or less than  $k, k \geq 1$ ,  $E$  is the unit matrix. Recurrently, it is possible to put

$$\mathcal{S}^k = \mathcal{S}^{k-1} \cdot \mathcal{G} + E,$$

where  $\mathcal{S}^0 = E$  means that each vertex links itself and this way length 0.

**Theorem 1.** *For a finite graph  $\mathcal{G}$  there always exists a number  $k, 0 \leq k \leq u, u = H$  is a cardinal number, such that  $S^k = S^{k+1} = \dots$ . The number  $k$  is the diameter of the graph  $\mathcal{G}$ .*

Addition and multiplication of matrix is boolean.

The matrix  $S$  from Theorem 1 will be called the stabilized matrix of graph  $G, [1]$ .

**Theorem 2.** *If there exists at least one element  $s_{ij} = 0$  in  $\mathcal{S}^k$ , then  $\mathcal{S}^k$  generates a decomposition of the graph  $G$  into components.*

See [1] for details.

**Note.** *If  $d \leq d_m(G)$  then there exist at least two components.*

## 2. OUTER BOUNDARY CONSTRUCTION: ALGORITHM I

A polygon is said to be the boundary of a set of points  $\mathcal{M}$ , if each point of  $\mathcal{M}$  either insides with the interior of the boundary or is a vertex of it.

Let  $\mathcal{M}_t = \{A_i, A_i \in \mathcal{L}, A_i \text{ are the vertices of the component } H_t \text{ of the graph } G\}$ ,  $t = 1, 2, \dots, n$ , where  $n$  is the number of components,  $n \geq 2$ . Denote by  $\ell$  the length of the maximal weight of edge of the graph spanning tree of the component  $H_t$ , i.e.  $\ell = d_m(H_t)$ . We construct the outer boundary as a closed polygon  $P_0 \dots P_n$ :

1. Choose a point  $A_i \in \mathcal{M}_t$  having the smallest  $x$  coordinate and denote it by  $P_0$ .
2. From the points  $A_i \neq P_0$ , where  $P_0 A_i \leq \ell$ , we choose the one—denoting it by  $P_1$ —which is incident with the arm of the greatest oriented angle (orientation opposite to the clockwise direction) the first arm of which is formed by the  $x$ -axis and the other passes through the point  $P_0$ . If there are more than one such points we choose one of those with the smallest  $x$ .
3. Let  $P_r$  be a vertex of the boundary for any natural number  $r > 0$ . From points  $A_i \neq P_r$ , for which  $P_r A_i \leq \ell$  we choose that one denoting it  $P_{r+1}$  which incides

with the arm of the greatest oriented angle (in opposite the clockwise direction)  $P_{r-1}P_rP_{r+1} = \varphi$ ,  $\varphi \leq 2\pi$  (Fig. 1), while

- A** if there are more such points  $A_i$  inciding with this arm we choose that with the smallest distance from vertex  $P_r$
- B1** if segment  $P_rA_i$  intersects a side of the boundary polygon in its inner point, or
- B2** if no point  $A_i \neq P_{r-1}$  in the circle domain with the diameter  $\ell$  exists, we choose point  $P_{r-1}$  to be a boundary polygon vertex, it means  $P_{r+1} = P_{r-1}$ .

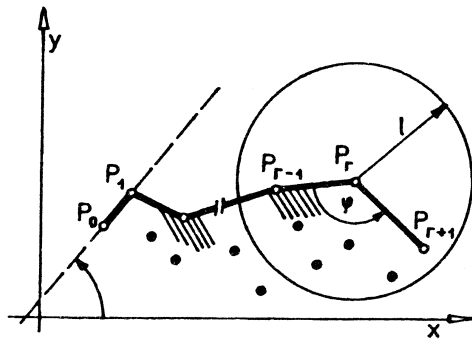


Fig. 1

It follows from the construction that there always exists such a point  $P_{n+1} = P_0$  for which the boundary polygon  $P_0 \dots P_r$  is closed. An example of the boundary of the set  $M_t$  generated by one graph component  $H_t$  for given  $\ell$  is in Fig. 2.

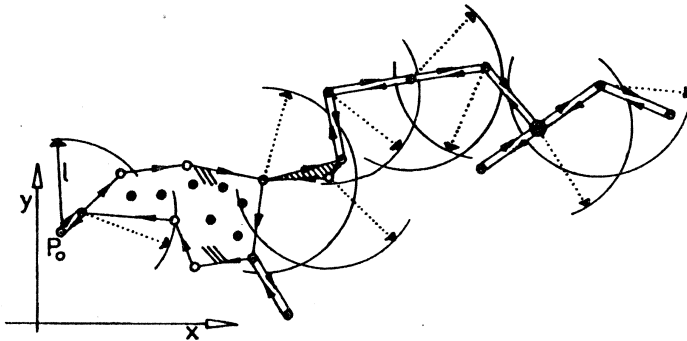


Fig. 2

**Note 1.** Obviously, for an arbitrary motion of the set  $M_t$  in the plane, point  $P_0$  is according to this construction the vertex of the angle  $P_nP_0P_1 = \varphi$ , where  $0 \leq \varphi \leq 2\pi$ . The boundary polygon is one-correspondent at any motion.

**Note 2.** Algorithm I shortly describes a method of boundary construction only. The complexity of computing does not exceed  $O(n^2)$  in any case, because the complexity of the construction algorithm of a vertex  $P_{r+1}$  using the greatest angle is linear. Computation of the angle  $\varphi$  value is connected either with the elementary arithmetic operations or with set operations on a point set. Some algorithms of a boundary construction (for convex hull) e.g. Jarvis's march, Graham's algorithm, Quickhull techniques and other are described in [11].

### 3. DETERMINATION OF TWO COMPONENTS INCIDENCE

Let  $H_1, H_2$  be two components of graph  $G$  and  $\mathcal{M}_1, \mathcal{M}_2$  sets of these components vertices. Let  $\mathcal{V} = \{v; v = A_i A_j; A_i \in \mathcal{M}_1 \text{ and } A_j \in \mathcal{M}_2\}$  and let  $v_d = \min \mathcal{V}$  ( $v_d$  defines the deviation of the  $\mathcal{M}_1$  and  $\mathcal{M}_2$ ).

**Theorem 3.** Let  $\mathcal{M}_1$  and  $\mathcal{M}_2$  be two point sets (in plane) generated by the decomposition of a component graph into components and let  $[A_1, A_2]$ ,  $A_i \in \mathcal{M}_1$ ,  $A_j \in \mathcal{M}_2$  is such a pair of points that  $A_i A_j = v_d$ . Let further  $p_1$  be the boundary of set  $\mathcal{M}_1$ . Then if  $A_i \notin p_1$  and

1. if segment  $A_i A_j$  does not coincide with a point of boundary  $p_1$ , according to the algorithms in part 2,  $\mathcal{M}_2$  coincides with the interior of set  $\mathcal{M}_1$ , which boundary is the outline  $p_1$ .
2. if segment  $A_i A_j$  intersects boundary  $p_1$ ,  $\mathcal{M}_2$  does not coincide with  $\mathcal{M}_1$ .

**Note.** The theorem can be rewritten for the priority of  $\mathcal{M}_2$  to  $\mathcal{M}_1$ .

**Proof.** Assertion 1 is trivial. Let now  $m = [a, A_j]$  be a semiplane (see Fig. 3) and circle domains  $k_1 = (A_i, v_d)$ ,  $k_2 = (A_j, v_d)$ . Form a region  $\mathcal{O} = m \cap k_1 - k_2$  (hatching part). If  $A_i$  is not a vertex of polygon  $p_1$ , there exists, obviously, exactly one point of the neighbouring vertices  $P_i, P_{i+1} \in \mathcal{O}$  such that  $P_i P_{i+1} \leq v_d$  and segment  $P_i P_{i+1}$  coincides with the interior point of segment  $A_i A_j$ . If there existed another pair of points  $P'_i$  and  $P'_{i+1}$  having the same property, according to the boundary construction I, points  $P_i$  and  $P'_i$  would either be vertices of side of polygon  $p_1$ , or one of them lies in region  $\mathcal{O}$ . The same is valid for  $P_{i+1}$  and  $P'_{i+1}$ .  $\square$

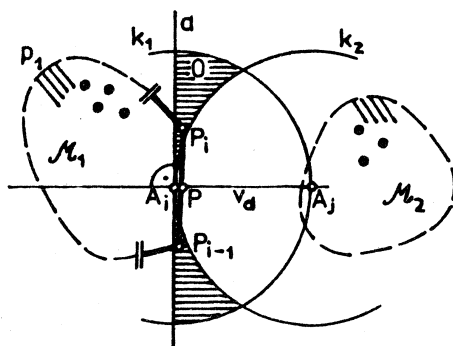


Fig. 3

#### 4. BOUNDARY CONSTRUCTION: ALGORITHMS II

Let  $\mathcal{M}_1$  and  $\mathcal{M}_2$  be two sets of points in the distance of  $v_d$ . Let  $A_i \in \mathcal{M}_1$  and  $A_j \in \mathcal{M}_2$  be two points such that  $A_i A_j \leq v_d$ . Let us choose in  $\mathcal{M}_1$  two points  $K$  and  $L$  for which  $A_i K \leq v_d$ ,  $A_i L \leq v_d$  and angle  $\varphi = A_j A_i K$ ,  $\frac{1}{3}\pi \leq \varphi \leq \pi$  opposite in clockwise direction is the smallest one, the angle  $\psi = A_j A_i L$ ,  $\frac{1}{3}\pi \leq \psi \leq \pi$  in clockwise direction is the smallest one respectively (Fig. 4). It follows from the Theorem 3 that next point of the boundary lies either in the region  $\mathcal{O}$  or in the opposite semiplane to  $[a, A_j]$  (see Fig. 3). Therefore the lower bound of the intervals for angles  $\varphi$  and  $\psi$  is  $\frac{1}{3}\pi$ . If some more points satisfying given conditions incide with the arms  $k$  or  $\ell$ , we choose those  $K$  resp.  $L$ , which are closest distance from point  $A_i$ . If at least one of angles  $\varphi$  or  $\psi$  respectively is equal or greater than  $\frac{1}{2}\pi$  (see part 3), it is possible to regard point  $A_i$  as a vertex of the boundary polygon; then  $K = P_n$  and  $L = P_1$  are boundary vertices and other vertices are constructed using the algorithm in part 2.

If both arms make angles  $\frac{1}{3}\pi \leq \varphi \leq \frac{1}{2}\pi$ ,  $\frac{1}{3}\pi \leq \psi \leq \frac{1}{2}\pi$  and  $KL \leq v_d$  we choose  $K = P_n$  and  $L = P_0$ , and other vertices according the algorithm in part 2.

**Consequence.** *If points  $P_0, \dots, P_n$ , obtained using this algorithms are the same, designation excluding, the boundary is concerned. If it is not true, the boundary obtained using this construction is inner boundary (Fig. 5).*

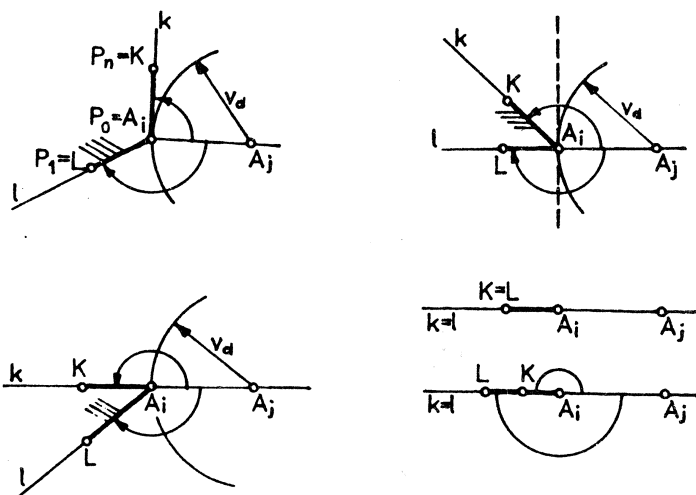


Fig. 4

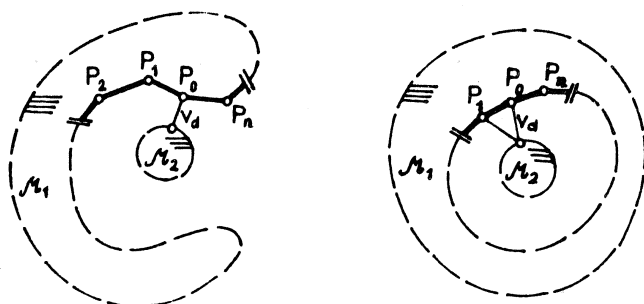


Fig. 5

## 5. BOOLEAN MATRICES APPLICATION TO GRAPH DECOMPOSITION AND APPLICATION

Neighbourhood matrix defining the graph is a Boolean matrix which consists of elements 0 and 1. The occupation of the operation memory of this matrix is not much economizing. For this reason we transformed the Boolean matrix through the binary number system, to reduced matrix, consisting of elements in decadic number system. [5]

Matrix  $\mathcal{G}^k = \mathcal{G}^{k-1} \cdot \mathcal{G} + E$  we formed multiplying competent reduced matrices. Problem of the reduced matrices multiplication has been solved and competent al-

gorithm as well as the program realizing graph component aided decomposition of point sets into groups is detail described in [2].

We multiply matrices as long as the stabilized matrix is obtained, i.e. while  $\mathcal{S}^k = \mathcal{S}^{k+1} = \mathcal{S}^{k+2} = \dots$

At last, we find out which rows of the stabilized matrix  $S^k$  are equal. These equal rows represent single-components of the given graph.

In Fig.6 is the boundary of convex hull (6a), hull for  $d_m(G)$  (6b), the case of boundary for  $d_1 < d_m(G)$  (6c) and the hull for  $d_2 < d_1$ .

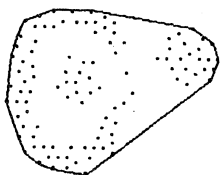


Fig. 6a

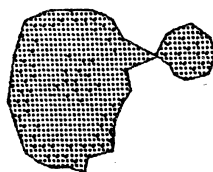


Fig. 6b

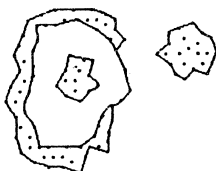


Fig. 6c



Fig. 6d

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### S ú h r n

## ANALÝZA KONEČNÝCH RIEDKYCH BODOVÝCH MNOŽÍN

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V príspevku sa pojednáva o rozklade tzv. riedkej bodovej množiny spolu s konštrukciou obrysu a obalu. Tento problém a jeho aplikácie sú frekventovanou problematikou v počítačovej geometrii a špeciálne v rastrovej grafike, najmä v oblasti spracovania obrazov. (Pozri napr. [3], [6], [7], [8], [9], [10].)

Problém rozkladu bodovej množiny je riešený pomocou istých vzťahov teórie grafov.

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