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SOLVABILITY OF A FORCED AUTONOMOUS DUFFING’S EQUATION WITH PERIODIC BOUNDARY CONDITIONS IN THE PRESENCE OF DAMPING

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Summary. Let \( g: \mathbb{R} \to \mathbb{R} \) be a continuous function, \( e: [0, 1] \to \mathbb{R} \) a function in \( L^2[0, 1] \) and let \( c \in \mathbb{R}, \ c \neq 0 \) be given. It is proved that Duffing’s equation \( u'' + cu' + g(u) = e(x), \ 0 < x < 1, \ u(0) = u(1), \ u'(0) = u'(1) \) in the presence of the damping term has at least one solution provided there exists an \( R > 0 \) such that \( g(u)u \geq 0 \) for \( |u| \geq R \) and \( \int_0^1 e(x) \, dx = 0 \). It is further proved that if \( g \) is strictly increasing on \( \mathbb{R} \) with \( \lim_{u \to -\infty} g(u) = -\infty, \ \lim_{u \to \infty} g(u) = \infty \) and is Lipschitz continuous with Lipschitz constant \( \alpha < 4\pi^2 + c^2 \), then Duffing’s equation given above has exactly one solution for every \( e \in L^2[0, 1] \).

Keywords: Duffing’s equation, damping

AMS classification: 34B15, 34C25, 47H15

1. INTRODUCTION

Let \( g: \mathbb{R} \to \mathbb{R} \) be a continuous function, \( e: [0, 1] \to \mathbb{R} \) a function in \( L^2[0, 1] \) and let \( c \in \mathbb{R}, \ c \neq 0 \) be given. This paper is devoted to the existence of a solution of the forced autonomous Duffing’s equation

\[
    u'' + cu' + g(u) = e, \quad 0 < x < 1,
\]

\[
    u(0) = u(1), \quad u'(0) = u'(1).
\]

We call the equation in (1.1) “autonomous” since the nonlinear function \( g \) is independent of \( x \). When \( g \) is a function of both the variables \( x \) and \( u \), i.e. \( g: [0, 1] \times \mathbb{R} \to \mathbb{R} \) is a function satisfying Caratheodory’s conditions, the non-autonomous Duffing’s
equation
\[ u'' + cu' + g(x, u) = e, \quad 0 < x < 1, \]
\[ u(0) = u(1), \quad u'(0) = u'(1) \]

(1.2)

has been extensively studied earlier (see e.g. [1], [2], [3], [4], [8], among others). It was shown, for example, by Gupta in [1] that if there exists a \( \rho > 0 \) such that \( g(x, u)u \leq 0 \) for a.e. \( x \in [0, 1] \) and all \( u \in \mathbb{R} \) with \( |u| \geq \rho \) then (1.2) has at least one solution provided \( \int_0^1 e(x) \, dx = 0 \). In the case when there exists a \( \rho > 0 \) such that \( g(x, u)u \geq 0 \) for a.e. \( x \in [0, 1] \) and \( |u| \geq \rho \), it was shown in [3] that (1.2) has at least one solution provided \( \int_0^1 e(x) \, dx = 0 \) and \( \lim \sup_{|u| \to \infty} \frac{g(x, u)}{u} \) is strictly less than \( 4\pi^2 + e^2 \). Now when \( c \neq 0 \), then \( 4\pi^2 + e^2 > 4\pi^2 \), which is the second eigenvalue of the linear eigenvalue problem

\[ -u'' = \lambda u, \]
\[ u(0) = u(1), \quad u'(0) = u'(1). \]

(1.3)

It was remarked in [3] that \( \lambda = 0 \) is the only eigenvalue of the linear eigenvalue problem when \( c \neq 0 \),

\[ u'' + cu' = \lambda u, \]
\[ u(0) = u(1), \quad u'(0) = u'(1) \]

(1.4)

to explain that the nonlinearity in \( g(x, u) \) can resonate beyond the second eigenvalue \( 4\pi^2 \) of the linear eigenvalue problem (1.3). Indeed, the author feels that when \( c \neq 0 \) and \( g(x, u)u \geq 0 \) for a.e. \( x \in [0, 1] \) and \( |u| \geq \rho \), then the boundary value problem (1.2) should have at least one solution when \( \int_0^1 e(x) \, dx = 0 \). But this is not known at this time. The purpose of this paper is to prove this conjecture in the case of the autonomous boundary value problem (1.1) when \( c \neq 0 \). The autonomous problem (1.1) was studied, when \( c \neq 0 \), by Nieto and Rao in [8] in the case when \( g: \mathbb{R} \to \mathbb{R} \) is increasing and \( \lim_{u \to \pm \infty} g(u) = g(\pm \infty) \) exists. But this case was already covered in [1] because then \( g \) is bounded on \( \mathbb{R} \) and accordingly, \( \lim_{|u| \to \infty} \frac{g(u)}{u} = 0 < 4\pi^2 \).

Our methods involve using Mawhin’s version of the Leray-Schauder continuation theorem and Wirtinger type inequalities to get the needed estimates. We also present some uniqueness results for the boundary value problem (1.1).
2. Main results

Let \( X, Y \) denote the Banach spaces \( X = C[0, 1] \) and \( Y = L^1[0, 1] \) with their usual norms. Let \( Y_2 \) be the subspace of \( Y \) spanned by the constant function 1 on \([0, 1]\), i.e.,

\[
Y_2 = \{ u \in Y \mid u(x) \equiv c \text{ for a.e. } x \in [0, 1], \ c \in \mathbb{R} \},
\]

and let \( Y_1 \) be the subspace of \( Y \) such that \( Y = Y_1 \oplus Y_2 \). We note that for \( u \in Y \) we can write

\[
(2.1) \quad u(x) = \left( u(x) - \int_0^1 u(x) \, dx \right) + \left( \int_0^1 u(x) \, dx \right)
\]

for \( x \in [0, 1] \). We define the canonical projection operators \( P: Y \to Y_1 \), \( Q: Y \to Y_2 \) by

\[
(2.2) \quad P(u)(x) = u(x) - \int_0^1 u(x) \, dx,
\]

\[
Q(u) = \int_0^1 u(x) \, dx
\]

for \( u \in Y \). Clearly, \( Q = I - P \), where \( I \) denotes the identity mapping on \( Y \), and the projections \( P \) and \( Q \) are continuous. Now let \( X_2 = X \cap Y_2 \). Clearly \( X_2 \) is a closed subspace of \( X \). Let \( X_1 \) be the closed subspace of \( X \) such that \( X = X_1 \oplus X_2 \). We note that \( P(X) \subset X_1 \), \( Q(X) \subset X_2 \) and the projections \( P|X: X \to X_1 \), \( Q|X: X \to X_2 \) are continuous. In the following, \( X, Y, P, Q \) will refer to the Banach spaces and projections as defined and we will not distinguish between \( P, P|X \) (resp. \( Q, Q|X \)) and rely on the context for proper meaning.

Also for \( u \in X, v \in Y \), let \( (u, v) = \int_0^1 u(x)v(x) \, dx \) denote the duality pairing between \( X \) and \( Y \). We note that for \( u \in X, v \in Y \) such that \( u = Pu + Qu \), \( v = Pv + Qv \) we have

\[
(2.3) \quad (u, v) = (Pu, Pv) + (Qu, Qv).
\]

Let \( c \in \mathbb{R}, c \neq 0 \) be given. Define a linear operator \( L: D(L) \subset X \to Y \) by setting

\[
(2.4) \quad D(L) = \{ u \in X \mid u'(x) \in AC[0, 1], \ u(0) = u(1), \ u'(0) = u'(1) \},
\]

and for \( u \in D(L) \),

\[
(2.5) \quad Lu = u'' + cu'.
\]
(Here $A C^0[0,1]$ denotes the space of real-valued absolutely continuous functions on $[0,1]$. It is easy to see that $L$ is a linear Fredholm mapping with $\ker L = X_2$, $\text{Im} L = Y_1$. Further, the mapping $K : Y_1 \to X_1$, defined for $u \in Y_1$ by

$$ (Ku)(x) = v(x) - \int_0^1 v(x) \, dx, $$

where

$$ v(x) = \int_0^x \int_0^t e^{c(t-\xi)} u(t) \, dt \, d\xi - \frac{e^{-cx} - 1}{c(e^c - 1)} \int_0^1 e^{ct} u(t) \, dt, $$

(note that we have assumed $c \neq 0$), satisfies the following conditions:

(i) for $u \in Y$, we have $KP(u) \in D(L)$, $LK P(u) = P(u)$,

(ii) $(KP(u), P(u)) \geq -\frac{1}{(4\pi^2 + c^2)} \|P(u)\|_{L^2[0,1]}^2$.

Indeed, note that for $v = KP(u) \in D(L)$,

$$ (KP(u), P(u)) = (v, Lv) = -\int_0^1 v'' \, dx \geq -\frac{1}{4\pi^2 + c^2} \|Lv\|_{L^2[0,1]}^2 $$

and so $(KP(u), P(u)) \geq -\frac{1}{(4\pi^2 + c^2)} \|P(u)\|_{L^2[0,1]}^2$ since

$$ \|Lv\|_{L^2[0,1]}^2 = \int_0^1 (v'' + cv')^2 \, dx = \int_0^1 \left[ (v'')^2 + 2cv'v'' + c^2(v')^2 \right] \, dx $$

$$ = \int_0^1 \left[ (v'')^2 + c^2(v')^2 \right] \, dx \geq (4\pi^2 + c^2) \int_0^1 v''^2 \, dx. $$

Let now $g : \mathbb{R} \to \mathbb{R}$ be a given continuous function. Let $N : X \to X \subseteq Y$ be the non-linear mapping defined by

$$ (Nu)(x) = g(u(x)), \quad x \in [0,1] $$

for $u \in X$. It is then easy to see, using Arzèla-Ascoli theorem, that $KPN : X \to X_1$ is continuous and compact.

**Theorem 1.** Let $g : \mathbb{R} \to \mathbb{R}$ be a given continuous function. Let $e, a, A, r, R$ with $a \leq A, r < 0 < R, c \neq 0$ be such that

$$ g(u) \geq A \text{ for } u \geq R, $$

(2.9) and

$$ g(u) \leq a \text{ for } u \leq r. $$

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Then, for every given function \( c(x) \in L^2[0, 1] \) with \( a \leq \int_0^1 c(x) \, dx \leq A \), Duffing’s equation

\[
\begin{align*}
    u'' + cu' + g(u) &= c, & 0 < x < 1, \\
    u(0) &= u(1), & u'(0) = u'(1)
\end{align*}
\]

(2.10)

has at least one solution.

Proof. Define functions \( g_1: \mathbb{R} \to \mathbb{R} \) and \( e_1: [0, 1] \to \mathbb{R} \) by setting

\[
    g_1(u) = g(u) - \frac{A + a}{2},
\]

\[
    e_1(x) = c(x) - \frac{A + a}{2}.
\]

Then \( g_1: \mathbb{R} \to \mathbb{R} \) is a continuous function and \( e_1: [0, 1] \to \mathbb{R} \) is such that \( e_1(x) \in L^2[0, 1] \). Furthermore,

\[
    g_1(u) \geq \frac{1}{2}(A - a) \geq 0 \quad \text{for } u \geq R,
\]

\[
    g_1(u) \leq \frac{1}{2}(a - A) \leq 0 \quad \text{for } u \leq R,
\]

and

\[
    \frac{1}{2}(a - A) \leq \int_0^1 e_1(x) \, dx \leq \frac{1}{2}(A - a).
\]

Duffing’s equation (2.10) is equivalent to the equation

\[
\begin{align*}
    u'' + cu' + g_1(u) &= e_1, & 0 < x < 1, \\
    u(0) &= u(1), & u'(0) = u'(1).
\end{align*}
\]

(2.11)

Now, for \( X = C[0, 1] \) and \( Y = L^1[0, 1] \) we consider the Niemytski operator \( N: X \to Y \) defined for \( u \in X \) by

\[
    (Nu)(x) = g_1(u(x)), \quad x \in [0, 1],
\]

and the linear operator \( L: D(L) \subseteq X \to Y \) defined in (2.4), (2.5).

The equation (2.11) is equivalent to the operator equation

\[
    (2.12) \quad Lu + Nu = e_1
\]

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in $X$. To solve (2.12) it suffices to solve the system of equations

$$Pu + KPNU = KP_{e_1},$$

$$QN_{u} = Q_{e_1}$$

(2.13)

in $X$. Indeed, if $u \in X$ solves (2.13), then $u \in D(L)$ and

$$LP_{u} + LKPNU = Lu + PNU = LKP_{e_1} = P_{e_1},$$

$$QN_{u} = Q_{e_1},$$

which gives, on adding, that $Lu + Nu = e_1$.

Now, (2.13) is clearly equivalent to the single equation

$$Pu + QNU + KPNu = KP_{e_1} + Q_{e_1},$$

(2.14)

which has the form of a compact perturbation of the Fredholm operator $P$ of index zero. We can, therefore, apply the version given in [6, Theorem I, Corollary 1] or [7, Theorem IV.4] or [5] of the Leray-Schauder continuation theorem, which ensures the existence of a solution for (2.14) if the set of all possible solutions of the family of equations

$$Pu + (1 - \lambda)Q_{u} + \lambda QNU + \lambda KPNu = \lambda KP_{e_1} + \lambda Q_{e_1},$$

(2.15)

$\lambda \in [0, 1[$, is a priori bounded independently of $\lambda$. Now (2.15) is equivalent to the system of equations

$$Pu + \lambda KPNu = \lambda KP_{e_1},$$

$$(1 - \lambda)Q_{u} + \lambda QNU = \lambda Q_{e_1}.$$  

(2.16)

Let $u_\lambda \in X$ be a solution of (2.16) for some $\lambda \in [0, 1[$, then $u_\lambda \in D(L)$ and

$$Pu_\lambda + \lambda KPNu_\lambda = \lambda KP_{e_1},$$

$$(1 - \lambda)Q_{u_\lambda} + \lambda QNU_{u_\lambda} = \lambda Q_{e_1}.$$  

(2.17)

It follows that

$$Lu_\lambda + (1 - \lambda)Q_{u_\lambda} + Nu_\lambda = \lambda e_1,$$

i.e.

$$u''_\lambda + cu'_\lambda + (1 - \lambda) \int_{0}^{1} u_\lambda(x) \, dx + g_1(u_\lambda) = \lambda e_1,$$

(2.18)

$$u_\lambda(0) = u_\lambda(1), \quad u'_\lambda(0) = u'_\lambda(1).$$

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Multiplying the equation in (2.17) by $u'_\lambda$ and integrating over $[0, 1]$ we obtain that

$$c \int_0^1 u'^2_\lambda = \lambda \int_0^1 e_1(x)u'_\lambda(x) \, dx,$$

which implies, using the Cauchy-Schwarz inequality, that

$$(2.19) \quad \|c\|\|u'_\lambda\|_{L^2[0,1]} \leq \|e_1\|_{L^2[0,1]}.$$  

Now, we claim that there exists a $\xi \in [0, 1]$ such that $r \leq u_\lambda(\xi) \leq R$. Indeed, suppose that $u_\lambda(x) \geq R$ for every $x \in [0, 1]$, then we get from the second equation in (2.17) and our assumptions on $g_1$ and $e_1$ that

$$(1 - \lambda)R + \lambda \cdot \frac{1}{2}(A - a) \leq (1 - \lambda)Qu_\lambda + \lambda QNu_\lambda = \lambda Qe_1 \leq \lambda \cdot \frac{1}{2}(A - a),$$

so that $(1 - \lambda)R \leq 0$, which is a contradiction since $\lambda \in ]0, 1[$ and $R > 0$. Similarly, $u_\lambda \leq r$ for $x \in [0, 1]$ leads to a contradiction. This proves the claim.

Next it follows that for every $x \in [0, 1]$

$$|u_\lambda(x)| \leq \max(-r, R) + \int_0^1 |u'_\lambda(x)| \, dx$$

$$\leq \max(-r, R) + \|u'_\lambda\|_{L^2[0,1]}$$

$$\leq \max(-r, R) + \frac{1}{|c|}\|e_1\|_{L^2[0,1]} \equiv C.$$

Hence

$$\|u_\lambda\|_{X} \leq C,$$

where $C$ is a constant independent of $\lambda \in ]0, 1[.$

This completes the proof of the theorem.

\[ \square \]

**Corollary 1.** Let $g: \mathbb{R} \to \mathbb{R}$ be a continuous function and let $c \in \mathbb{R}$, $c \neq 0$ be given. Suppose there exists an $R > 0$ such that $g(u)u \geq 0$ for $u \in \mathbb{R}$, $|u| \geq R$.

Then for every $c(x) \in L^2[0, 1]$ with $\int_0^1 c(x) \, dx = 0$, Duffing's equation (2.9) has at least one solution.

**Proof.** The proof follows immediately from Theorem 1 with $a = A = 0$ and $r = -R$.  \[ \square \]
Theorem 2. Let $g : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function with $\lim_{u \to -\infty} g(u) = -\infty$, $\lim_{u \to \infty} g(u) = \infty$ and let $c \in \mathbb{R}$, $c \neq 0$. Suppose that $g$ is a Lipschitz continuous function with a Lipschitz constant $\alpha$, i.e.

$$|g(u) - g(v)| \leq \alpha |u - v|$$

for $u, v \in \mathbb{R}$, with

$$\alpha < 4\pi^2 + c^2$$

for all $u \in \mathbb{R}$.

Then for every $c \in L^2[0, 1]$, the boundary value problem

$$u'' + cu' + g(u) = c(x), \quad 0 < x < 1$$

$$u(0) = u(1), \quad u'(0) = u'(1),$$

has exactly one solution $u$ in $X = C[0, 1]$.

Proof. Under our assumptions, it is easy to see that there exist $a, A, r, R$ with $a \leq A$, $r < 0 < R$ such that

$$g(u) \leq A \text{ for } u \geq R,$$

$$g(u) \leq a \text{ for } u \leq r,$$

and

$$a \leq \int_0^1 c(x) \, dx \leq A.$$

Accordingly, Theorem 1 implies that (2.22) has at least one solution $u$ in $X$.

Let, now, $u_1$, $u_2 \in X$ be two different solutions for (2.22). Then

$$u_1'' - u_2'' + c(u_1' - u_2') + g(u_1) - g(u_2) = 0, \quad 0 < x < 1.$$  

It follows that

$$0 = -\int_0^1 (u_1' - u_2')^2 \, dx + \int_0^1 (g(u_1) - g(u_2))(u_1 - u_2) \, dx$$

$$= -\int_0^1 (u_1' - u_2')^2 \, dx + \int_0^1 |g(u_1) - g(u_2)| |u_1 - u_2| \, dx$$

$$\geq -\frac{1}{4\pi^2 + c^2} \|Lu_1 - Lu_2\|_{L^2[0, 1]}^2 + \frac{1}{\alpha} \int_0^1 |g(u_1) - g(u_2)|^2 \, dx$$

$$= \left(\frac{1}{\alpha} - \frac{1}{4\pi^2 + c^2}\right) \int_0^1 |g(u_1) - g(u_2)|^2 \, dx,$$
in view of (2.23). Using (2.21), we get that
\[ g(u_1(x)) = g(u_2(x)) \]
for a.e. \( x \in [0, 1] \), which implies \( u_1(x) = u_2(x) \) for a.e. \( x \in [0, 1] \), since \( g \) is strictly increasing on \( \mathbb{R} \). Hence \( u_1(x) = u_2(x) \) for every \( x \in [0, 1] \) since \( u_1, u_2 \) are continuous in \([0, 1]\).

This completes the proof of the theorem. \( \square \)

Remark 1. Theorem 2 seems to imply that Duffing's equation (2.22) in the presence of the non-zero damping term \( cu' \) has a unique solution as long as the non-linearity \( g(u) \) does not resonate against too many eigenvalues of the linear eigenvalue problem

\[
-u'' = \lambda u, \quad 0 < x < 1, \\
u(0) = u(1), \quad u'(0) = u'(1).
\]

Also, it indicates that while the presence of even a small amount of damping gives existence, the presence of large enough damping ensures uniqueness.

References


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