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A REMARK TO THE PAPER M. FRODA-SCHECHTER: PRÉORDRES ET ÉQUIVALENCES DANS L'ENSEMBLE DES FAMILLES D'UN ENSEMBLE

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The aim of this remark is to deepen the knowledge about the lattice of all classes $\mathfrak{K}_e(\mathscr{L})$ from the preceeding paper [1]. We use the notation introduced in [1]. Cardinal and ordinal operations with ordinal types are denoted as in [2].

Let G be a partially ordered set. If $X \subseteq G$, $x \in X$, $x \leqq y \Rightarrow y \in X$ then X is an *end* of G. The set of all ends is denoted by $\mathscr{E}(G)$. $\mathscr{E}(G)$ is supposed to be ordered by inclusion, i e. $X, Y \in \mathscr{E}(G)$, $X \leqq Y \equiv X \subseteq Y$. Now, we shall deal with type of $\mathscr{E}(G)$. Let f be an isotonic mapping of G into $\{0, 1\}$, $0 < 1$. The set of all $g \in G$, for which $f(g) = 1$ is an end. On the other hand, if X is an end and $h(x) = 1$ for $x \in X$, $h(x) = 0$ for $x \in X$, then h is an isotonic mapping of G into $\{0, 1\}$. Hence we get immediately that the *ordinal type of* $\mathscr{E}(G)$ *is* 2^γ, where γ is an ordinal type of G and 2 is an ordinal type of $\{0, 1\}$. This result can be also easily obtained from general considerations in [3] (theorem 5.4). Especially, if $\mathscr{P}(E)$ is ordered by means of inclusion, ε the type of an antichain with cardinal number card E, the ordinal type of $\mathscr{E}(\mathscr{P}(E))$ is 2^{2^ε}. In following, we put $\mathscr{E} = \mathscr{E}(\mathscr{P}(E))$.

Put $\mathfrak{K}_e = \{\mathfrak{K}_e(\mathscr{L}) : \mathscr{L} \subseteq \mathscr{P}(E)\}$ and order $\mathfrak{K}_e$ by (D 10) from § 5 in [1]. According to (3.2) in [1], $\mathscr{M}_e(\mathscr{L}) = \{M \in \mathscr{P}(E) : \underset{L \in \mathscr{L}}{\exists} L \subseteq M\}$ is the greatest element in $\mathfrak{K}_e(\mathscr{L})$. Clearly $\mathscr{M}_e(\mathscr{L}) \in \mathscr{E}$. If $\mathscr{L} \in \mathscr{E}$, then $\mathscr{M}_e(\mathscr{L}) = \mathscr{L}$. Thus a mapping f which maps $\mathfrak{K}_e(\mathscr{L})$ on $\mathscr{M}_e(\mathscr{L})$ is an one-to-one mapping of $\mathfrak{K}_e$ on $\mathscr{E}$.

Let $\mathfrak{K}_e(\mathscr{L}_1) \underset{e}{\prec} \mathfrak{K}_e(\mathscr{L}_2)$. Then there exist $\mathscr{L}^1 \in \mathfrak{K}_e(\mathscr{L}_1)$ and $\mathscr{L}^2 \in \mathfrak{K}_e(\mathscr{L}_2)$ such that $\mathscr{L}^1 \underset{e}{\subsetneq} \mathscr{L}^2$. Thus $\mathscr{M}_e(\mathscr{L}_1) = \mathscr{M}_e(\mathscr{L}^1) \supseteq \mathscr{M}_e(\mathscr{L}^2) = \mathscr{M}_e(\mathscr{L}_2)$.

On the contrary if $\mathscr{L}_1, \mathscr{L}_2 \in \mathscr{E}$, $\mathscr{L}_1 \subseteq \mathscr{L}_2$ it is $\mathfrak{K}_e(\mathscr{L}_2) \underset{e}{\prec} \mathfrak{K}_e(\mathscr{L}_1)$. This implies that f *is an antiisomorphism.*

By [2] I, § 7 it is $\breve{\alpha^\beta} = \breve{\alpha}^{\breve{\beta}}$, where $\breve{\alpha}$ is an ordinal type of a set which is antiisomorphic to a set of the type α. Thus

$$\text{(a)} \qquad \breve{2^{2^\varepsilon}} = 2^{2^\varepsilon}.$$

Hence we get

The ordinal type of $\mathfrak{K}_e$ *is* 2^{2^t}.

From (a) it follows that the following assertion may be added to (5.3) in [1].

The set of all classes e-superior is a lattice which is isomorphic to $\mathfrak{K}_e$.

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