Archivum Mathematicum

Milan Sekanina

On the power of ordered sets

*Archivum Mathematicum*, Vol. 1 (1965), No. 2, 75--82

Persistent URL: http://dml.cz/dmlcz/104583

# ON THE POWER OF ORDERED SETS

MILAN SEKANINA, BRNO

(Received March 6, 1965)

## 1.

Under the notion "an ordered set" we understand a set e. g. $A$ on which a reflexive, antisymmetric and transitive relation is defined. If we denote this relation by the symbol $\leqq$, we write detailed $(A, \leqq)$. In several parts of this paper we shall deal with several ordered sets at the same time. We shall use for them — if there does not occur the danger of mistake — the same symbol. In opposite case the symbol will be provided with an index (e. g. $\leqq_1$). The ordered set will be said to fulfil the condition of decreasing chains, when for every decreasing sequence $x_1 \geqq x_2 \geqq \ldots \geqq x_n \geqq \ldots$ there exists $m$ so that $x_m = x_{m+1} = \ldots$. We write then $(A, \leqq) \in \mathscr{K}$ (or simply $A \in \mathscr{K}$). A set of minimal elements of the set $A$ we denote by $m(A)$. We shall say that $A$ fulfils the condition of minimality when there exists $m \in m(A)$ for every $a \in A$ such that $m \leqq a$. In this case we write $A \in \mathscr{M}$. Let $A$, $B$ be sets (they do not need to be ordered). $A^B$ is a system of all mappings of a set $B$ into $A$. Let $f, g \in A^B$. We put $n(f, g) = \{x : x \in B, f(x) \neq g(x)\}$.

The one-to-one mapping $f$ of a set $(A, \leqq)$ on $(B, \leqq)$ is called a similar mapping, if $x \leqq y \equiv f(x) \leqq f(y)$. The set $A$ is said to be similar to $B$ and we write $A \simeq B$. The category of ordered sets, where morphisms are similar mappings, is denoted by $\mathscr{U}$. The category of sets with one binary relation is denoted by $\mathscr{B}$. Morphisms are isomorphic mappings.

The aim of this paper is to present the definition of a certain operation in $\mathscr{U}$, which is a modification of the ordinal power of ordered sets. The ordinal power of ordered sets ${}^B A$ has been defined by G. Birkhoff in [1], [2] and M. M. Day in [3] (the definitions, presented in these papers, are formaly different; in what follows we shall define ${}^B A$ according to [2]). ${}^B A$ is not in general case an ordered set. According to [3], p. 23, the theorem 4.17, the following statement holds: $(D)$ *${}^B A$ is an ordered set just when $A$ is an antichain or $B \in \mathscr{K}$.*

In the paragraph 2 there is defined an operation $\exp_A B$ which in case, when all presumptions from $(D)$ are fulfilled, is equal to ${}^B A$. If $A$ and $B$ are totally ordered, then $\exp_A B$ is equal to general power of Hausdorff ([4] p. 150).

Ordinal and cardinal operations with ordered sets are denoted like in [2] with the difference that no symbol for a cardinal power is introduced (definition $A^B$ see above)

2.

**Lemma 1.** $A \in \mathscr{K} \Rightarrow A \in \mathscr{M}$.

Evident.

**Lemma 2.** *Let $H$ be an ordered set, $H_i$ for $i \in H$ an ordered set. Then $\langle i, a\rangle \in m(\sum_{i \in H} H_i)$ is equivalent to the validity of one of these statements.*

1. $a \in m(H_i)$ *and* $i \in m(H)$
2. $a \in m(H_i)$ *and* $j < i \Rightarrow H_j = \emptyset$.

$\sum_{i \in H} H_i$ *denotes a lexicographic sum.*

Proof. Let $\langle i, a\rangle \in m\left(\sum_{i \in H} H_i\right)$. Let $a \in H_i$. If there existed $b < a$, $b \in H_i$, then $\langle i, b\rangle < \langle i, a\rangle$ in $\sum_{i \in H} H_i$ what is impossible. Let there exist $j < i$. Let us admit that $c \in H_j$. Then $\langle j, c\rangle < \langle i, a\rangle$ what is again a contradiction to the presumption.

Let there hold 1 or 2. Then from $\langle k, b\rangle \in \sum_{i \in H} H_i$, $\langle k, b\rangle < \langle i, a\rangle$ there follows either $k < i$ and then $b \in H_k = \emptyset$, or $k = i$ and $b < a$, so $a$ non $\in m(H_i)$. Both is in contradiction with the presumption.

**Consequence of the lemma 2.**

$$A, B \in \mathscr{M} \Rightarrow A + B,\ A \oplus B,\ A \bigcirc B \in \mathscr{M}.$$

**Lemma 3.**

*Let $A, B \in \mathscr{M}$. Then $A \,.\, B \in \mathscr{M}$.*

Proof. Let $a \in m(A)$, $b \in m(B)$. Then evidently $\langle a, b\rangle \in m(A \,.\, B)$. If $\langle c, d\rangle \in A \,.\, B$, so there exists $a \in m(A)$, $b \in m(B)$ such that $a \leqq c$, $b \leqq d$. Then $\langle a, b\rangle \leqq \langle c, d\rangle$.

There hold even these evident statements.

**Lemma 4.**

*Let $A = B + C$. Then*

$$A \in \mathscr{M} \equiv B,\ C \in \mathscr{M}.$$

**Lemma 5.**

*Let $A = B \oplus C$. Then*

$$A \in \mathscr{M} \equiv \begin{cases} B \neq \emptyset \Rightarrow B \in \mathscr{M}. \\ B = \emptyset \Rightarrow C \in \mathscr{M}. \end{cases}$$

**Definition 1.** Let $f, g \in A^B$, $A, B \in \mathscr{U}$. Let us put $f \leqq g \equiv n(f, g) \in \mathscr{M}$ and for $m \in m(n(f, g))$ there is $f(m) < g(m)$.

**Theorem 1.** *$(A^B, \leqq)$ is an ordered set.*

Proof. 1. Reflexivity is evident.

2. Let $f \leqq g$, $g \leqq f$. Then necessarily $n(f, g) = \emptyset$, so $f = g$.

3. Let $f \leqq g$, $g \leqq h$. Let $b \in n(f, h)$. Then $b \in n(f, g) \cup n(g, h)$. There exists $m \in m(n(f, g))$ or $m \in m(n(g, h))$ such that $m \leqq b$. In what follows we shall investigate the first case. The second case can be investigated analogously. Let us admit that there exists $m_1 \leqq m$ such that $g(m_1) \neq \neq h(m_1)$. Then there exists $m_2 \leqq m_1$, $m_2 \in m(n(g, h))$. It must be $f(m_2) \leqq g(m_2) < h(m_2)$. Simultaneously $m_2 \in m(n(f, h))$. If there does not exist $m_1$ with the above mentioned property, there is $m \in m(n(f, h))$ and $f(m) < g(m) = h(m)$. Thus $n(f, h) \in \mathscr{M}$ and $f \leqq h$.

**Definition 2.** *Let us put* $\exp_A B = (A^B, \leqq)$.

**Theorem 2.** *Let $A$ be an antichain or $B \in \mathscr{K}$. Then* $\exp_A B = {}^B A$.

Proof. $A$ being an antichain, ${}^B A$ and $\exp_A B$ are antichains.

Let $B \in \mathscr{K}$. Let $f, g \in A^B$. Let $f \leqq g$ in ${}^B A$. $B \in \mathscr{K} \Rightarrow n(f, g) \in \mathscr{K} \Rightarrow \Rightarrow n(f, g) \in \mathscr{M}$. According to the definition ${}^B A$ we have $m \in m(n(f, g)) \Rightarrow \Rightarrow f(m) < g(m)$, thus $f \leqq g$ in $\exp_A B$. Let $f \leqq g$ in $\exp_A B$. Then for every $x \in B$ for which $f(x) \neq g(x)$ there exists $y \in m(n(f, g))$ such that $y \leqq x$ and $f(y) < g(y)$, thus $f \leqq g$ in ${}^B A$.

**Theorem 3.** $\operatorname{Exp}_A (B + C) \cong \exp_A B \,.\, \exp_A C$.

Proof. Let $f \in \exp_A (B + C)$. Let $f_B, f_C$ (similar in the following explication) be partial mappings induced by the mapping $f$ of the set $B$ into $A$, eventually $C$ into $A$. Then $f \to \langle f_B, f_C \rangle$ is a one-to-one mapping $\exp_A (B + C)$ on $\exp_A B \,.\, \exp_A C$. We shall show that it is a similar mapping.

a) Let $f, g \in \exp_A (B + C)$, $f \leqq g$. In general it holds

(1) $n(f, g) = n(f_B, g_B) + n(f_C, g_C)$ and

$$m(n(f, g)) = m(n(f_B, g_B)) + m(n(f_C, g_C)).$$

Thus

$$x \in m(n(f_B, g_B)) \Rightarrow x \in m(n(f, g)) \Rightarrow f(x) < g(x) \Rightarrow f_B(x) < g_B(x).$$

According to the lemma 4 there is $n(f_B, g_B) \in \mathscr{M}$. Hence $f_B \leqq g_B$ in $\exp_A B$. In a similar way one can prove $f_C \leqq g_C$ in $\exp_A C$. Thus $\langle f_B, f_C \rangle \leqq \langle g_B, g_C \rangle$.

b) Let $\langle f_B, f_C \rangle \leqq \langle g_B, g_C \rangle$. From (1) there follows $x \in m(n(f, g)) \Rightarrow \Rightarrow f(x) < g(x)$. As according to the lemma 4 $n(f, g) \in \mathscr{M}$, it is $f \leqq g$.

**Theorem 4.** $\exp_A (B \oplus C) \cong \exp_A B \bigcirc \exp_A C$.

Proof. We prove that also in this case a mapping $f \to \langle f_B, f_C \rangle$ is a similar mapping. Let $f, g \in \exp_A (B \oplus C)$.

It is $n(f, g) = n(f_B, g_B) \oplus n(f_C, g_C)$.

a) Let $f \leqq g$.

$a_1$) Let $n(f_B, g_B) \neq \emptyset$. According to the lemma 5 there is $n(f_B, g_B) \in \mathcal{M}$. For $x \in m(n(f_B, g_B))$ there is $f_B(x) = f(x) < g(x) = g_B(x)$. Consequently $f_B < g_B$ and therefore $\langle f_B, f_C \rangle < \langle g_B, g_C \rangle$.

$a_2$) Let $n(f_B, g_B) = \emptyset$. Then $n(f_C, g_C) \in \mathcal{M}$ and similarly as in $a_1$) there is $f_C \leqq g_C$. Thus $\langle f_B, f_C \rangle \leqq \langle g_B, g_C \rangle$.

b) Let $\langle f_B, f_C \rangle \leqq \langle g_B, g_C \rangle$. According to the lemma 5 there is $n(f, g) \in \mathcal{M}$. Let $m \in m(n(f, g))$.

$b_1$) Let $f_B < g_B$. Then $m \in m(n(f_B, g_B))$ and $f(m) < g(m)$.

$b_2$) Let $f_B = g_B$, $f_C \leqq g_C$. Then $m \in m(n(f_C, g_C))$ and $f(m) < g(m)$. Thus $f \leqq g$.

**Theorem 5.** $\exp_C (A \bigcirc B) \cong \exp_{\exp_C B} A$.

Proof. Let $f \in \exp_C (A \bigcirc B)$. Let $f^* \in \exp_{\exp_C B} A$ be such an element for which, for $a \in A$, $f^*_a$ is a mapping of $B$ into $C$ defined by means of this equation

$$f^*_a(b) = f(a, b)$$

for every $b \in B$.

It is easy to find out that $f \to f^*$ is a one-to-one mapping of the set $\exp_C (A \bigcirc B)$ on $\exp_{\exp_C B} A$. We shall show that the mapping is a similar one.

a) Let $f, g \in \exp_C (A \bigcirc B)$, $f \leqq g$. Let $a \in n(f^*, g^*)$, thus $f^*_a \neq g^*_a$, that is, there exists $b \in B$ such that $f^*_a(b) \neq g^*_a(b)$ thus $f(a, b) \neq g(a, b)$. Let $\langle a_1, b_1 \rangle \in m(n(f, g))$, $\langle a_1, b_1 \rangle \leqq \langle a, b \rangle$. It is $f(a_1, b_1) < g(a_1, b_1)$. Let us admit that there exists $a_2 < a_1$ such that $f^*_{a_2} \neq g^*_{a_2}$. Then there exists $b_2 \in B$ such that $f(a_2, b_2) \neq g(a_2, b_2)$ and at the same time $\langle a_2, b_2 \rangle < \langle a_1, b_1 \rangle$ which is impossible. Thus $a_1 \in m(n(f^*, g^*))$. Let $f^*_{a_1}(b_3) \neq g^*_{a_1}(b_3)$. Then $f(a_1, b_3) \neq g(a_1, b_3)$ and therefore there exists $a_4$, $b_4$ such that $\langle a_4, b_4 \rangle \leqq \langle a_1, b_3 \rangle$, $\langle a_4, b_4 \rangle \in m(n(f, g))$ and $f(a_4, b_4) < g(a_4, b_4)$. For the reasons mentioned a while ago, there is $a_4 = a_1$. Thus $b_4 \in m(n(f^*_{a_1}, g^*_{a_1}))$ $b_4 \leqq b_3$. Consequently $n(f^*_{a_1}, g^*_{a_1}) \in \mathcal{M}$ and $f^*_{a_1} < g^*_{a_1}$. Accordingly $n(f^*, g^*) \in \mathcal{M}$ and $f^* \leqq g^*$.

b) Let $f^* \leqq g^*$. Let $\langle a, b \rangle \in n(f, g)$. Thus $f(a, b) \neq g(a, b)$ which gives $f^*_a \neq g^*_a$. There exists $a_1 \leqq a$, $a_1 \in m(n(f^*, g^*))$ such that $f^*_{a_1} < g^*_{a_1}$. Let for $b_1 \in B$ there be $f^*_{a_1}(b_1) \neq g^*_{a_1}(b_1)$. Then there exists $b_2 \in m(n(f^*_{a_1}, g^*_{a_1}))$ such that $b_2 \leqq b_1$ $f^*_{a_1}(b_2) < g^*_{a_1}(b_2)$, i. e. $f(a_1, b_2) < g(a_1, b_2)$. Let us show that $\langle a_1, b_2 \rangle \in m(n(f, g))$. Let $\langle a', b' \rangle \leqq \langle a_1, b_2 \rangle$, $\langle a', b' \rangle \in n(f, g)$, then $f(a', b') \neq g(a', b')$, i. e. $f^*_a \neq g^*_a \Rightarrow a' = a_1$. But then $b' = b_2$.

b$_1$) Let $a_1 < a$. Then $\langle a_1, b_2\rangle < \langle a, b\rangle$.

b$_2$) Let $a_1 = a$. Then it is possible to put $b$ instead of $b_1$ and again $\langle a_1, b_2\rangle \leqq \langle a, b\rangle$.

Consequently $n(f, g) \in \mathscr{M}$ and $f \leqq g$.

For purposes of the following paragraph we pronounce this evident statement.

**Theorem 6.** *Let $B$ be an antichain. Let $f, g \in \exp_A B$. Then*

$$f \leqq g \equiv f(x) \leqq g(x) \quad \text{for every} \quad x \in B.$$

### 3.

Let $(A, \leqq)$, $(B, \leqq_1)$, $A \subset B$ and $x, y \in A$, $x \leqq y \Rightarrow x \leqq_1 y$. Then we say that $(B, \leqq_1)$ is a prolongation of $(A, \leqq)$. We write $(A, \leqq)\, \pi(B, \leqq_1)$. If it is even $x, y \in A \Rightarrow (x \leqq y \equiv x \leqq_1 y)$ we say that $(A, \leqq)$ is isomorphly embedded in $(B, \leqq_1)$ and we write $(A, \leqq)\, \iota(B, \leqq_1)$ or briefly $A\, \iota\, B$.

Let $(B, \leqq)$, $A\, \iota\, B$. We say that $A$ is coinicial with $B$, when for every $b \in B$ there exists $a \in A$ that $a \leqq b$. We write $A\varkappa B$.

Let $(A, \leqq)\, \iota\, (B, \leqq)$. Let $x \in A$, $y \in B$, $y \leqq x \Rightarrow y \in A$. Then $A$ is an ideal of $(B, \leqq)$.

We say that $(B, \leqq_1)$ is an unsubstantial prolongation of $(A, \leqq)$ when $(A, \leqq)\, \pi(B, \leqq_1)$ and there exists an ideal $A_1$ in $A$, $A_1 \iota B$, $A_1 \varkappa B$. We write $A\sigma B$.

The following statement is valid.

**Lemma 6.** *Let $A \in \mathscr{M}$. Then $mA\varkappa A$, $mA\sigma A$, $mA$ is an ideal of the set $A$.*

Proof is evident. Let us notice only that if it is not said anything else, in what follows, we suppose for the subset $A$ of the ordered set $B$ such an ordering that $A\iota B$.

**Lemma 7.** *Let $A\sigma B$, $A \in \mathscr{M}$. Then $B \in \mathscr{M}$ and $mA \supset mB$.*

Proof. Let $A_1$ be an ideal from $A$, $A_1\iota B$ and $A_1\varkappa B$. Let $b \in B$. There exists $a \in A_1$, $a \leqq_1 b$ and further there exist $a_1 \in mA$, $a_1 \leqq a$ and $a_1 \in A_1$. Then also $a_1 \in mA_1$ and because of $A_1\varkappa B$, also $a_1 \in mB$. Let further $b \in mB$. Then the above constructed $a_1$ is equal to $b$ and accordingly $mB \subset mA$.

**Lemma 8.** *$A\varkappa B$, $A \in \mathscr{M} \Rightarrow B \in \mathscr{M}$ and $mA = mB$.*

Proof. $A\varkappa B$, $A \in \mathscr{M} \Rightarrow A\sigma B$ and the statement follows from the lemma 7.

**Lemma 9.** *$A$ is an ideal of $B$, $B \in \mathcal{M} \Rightarrow A \in \mathcal{M}$ and $mA \subset mB$.*

Proof is analogous as in the lemma 7.

Let $\Phi$ be a twoplace functor mapping $\mathcal{U} \times \mathcal{U}$ into $\mathcal{U}$ with following properties:

$A$ 1 $\Phi(A, B) = (A^B, \leqq)$ for a certain $\leqq$.

$A$ 2 Let $\varphi$: $A \rightarrow A_1$, $\psi$: $B \rightarrow B_1$. Then $\Phi(\varphi, \psi)$ is defined in such a way:
$[\Phi(\varphi, \psi)(f)](\psi(b)) = \varphi(f(b))$ where $f \in A^B$.

$A$ 3 (Axiom of the initial condition). Let $B$ be an antichain, $(A, \leqq) \in \in \mathcal{U}$. Then in $\Phi(A, B) = (A^B, \leqq_1)$ there is $f \leqq_1 g \equiv f(b) \leqq g(b)$ for $b \in B$.

$A$ 4 (Axiom of relative mappings). Let an ordered set $A$ be isomorphly embedded in $B$. Let $C$ be an ordered set. Let for $f, g \in \Phi(C, A)$, $f^*, g^* \in \Phi(C, B)$ there hold: $x \in n(f, g) \Rightarrow f(x) = f^*(x)$, $g(x) = g^*(x)$.

Then there holds

a) If it is $n(f, g) \varkappa n(f^*, g^*)$ then $f \leqq g \Rightarrow f^* \leqq g^*$.

b) If $n(f, g)$ is an ideal of $n(f^*, g^*)$, then $f^* \leqq g^* \Rightarrow f \leqq g$.

A 5 (Axiom of relative orderings). Let $(A, \leqq)$, $(A, \leqq_1) \in \mathcal{U}$. Let $f, g \in C^A$ where $C \in \mathcal{U}$. Let $(n(f, g), \leqq_1)$ $\sigma(n(f, g), \leqq)$. Then $f \leqq g$ in $\Phi(C, (A, \leqq_1)) \Rightarrow f \leqq g$ in $\Phi(C, (A, \leqq))$.

**Theorem 7.** *For $\Phi(A, B) = \exp_A B$ are the axioms* A1, A3—A5 *fulfilled and $\Phi(\varphi, \psi)$ is a similar mapping.*

Proof. Validity of A1 and statement on $\Phi(\varphi, \psi)$ are obvious. A3 follows from the theorem 6.

Ad A4

a) Let $n(f, g) \varkappa n(f^*, g^*)$ and $f \leqq g$. Then $n(f, g) \in \mathcal{M}$ and according to the lemma 8 $n(f^*, g^*) \in \mathcal{M}$ and $m(n(f, g)) = m(n(f^*, g^*))$. Thus $f^* \leqq g^*$.

Ad A4

b) Let $n(f, g)$ be an ideal of $n(f^*, g^*)$ and $f^* \leqq g^*$. According to the lemma 9 $n(f, g) \in \mathcal{M}$ and $m(n(f, g) \subset m(n(f^*, g^*))$. Thus $f \leqq g$.

Ad A5

From $(n(f, g), \leqq_1)$ $\sigma(n(f, g), \leqq)$ and $f \leqq g$ in $\exp_C (A, \leqq_1)$ there follows both $(n(f, g), \leqq_1) \in \mathcal{M}$ and, according to the lemma 7, $(n(f, g), \leqq) \in \mathcal{M}$ and $m(n(f, g), \leqq_1) \supset m(n(f, g), \leqq)$. Thus $f \leqq g$ in $\exp_C (A, \leqq)$.

**Theorem 8.** *Let $\mathcal{U} \times \mathcal{U} \rightarrow \mathcal{U}$ be replaced in formulations* A1—A5 *for $\mathcal{U} \times \mathcal{U} \rightarrow \mathcal{B}$ and symbols $\leqq$, $\leqq_1$ for $\Phi(A, B)$ signify binary relations. Then $\Phi(A, B) = {}^B A$ fulfils* A1—A5.

The proof is evident from the definition of ${}^B A$.

**Theorem 9.** *Let $\Phi(A, B)$ be a functor on $\mathscr{U}$ into $\mathscr{U} \times \mathscr{U}$ fulfilling* A1—A5. *Then* $\Phi(A, B) = \exp_A B$.

Proof. Let us denote the ordering in $\Phi(A, B)$ as $\leqq$, in $\exp_A B$ as $\leqq_1$. First we prove that $(A^B, \leqq_1)\ \pi(A^B, \leqq)$. Let $f \leqq_1 g$. Let us put $N = m(n(f, g))$, $N \iota B$. There is $f_N \leqq g_N$ in $\exp_A N$. According to A3 there is $f_N \leqq g_N$ in $\Phi(A, N)$ and according to A4 a) and the lemma 6, there is $f \leqq g$ in $\Phi(A, B)$.

Let us suppose that there exists $f$ and $g$ in $A^B$ such that $f \leqq g$ and $f$ non $\leqq_1 g$. Thus it is $f < g$. For this reason either $S = n(f, g)$ non $\in \mathscr{M}$ or $S \in \mathscr{M}$ and there exists $x \in mS$ such that $f(x)$ non $< g(x)$. The second case can be immediately excluded, because according to A4 b) there is $f_{mS} \leqq g_{mS}$ in $\Phi(A, mS)$ and then according to A3 $f(x) < g(x)$ what is a contradiction.

Thus let be $S$ non $\in \mathscr{M}$. By A 4b) $f_S < g_S$ in $\Phi(A, S)$. Let $T \subset S$ be a set of those $x \in S$ under which there exists no minimal element. Then $x \in T$, $y \in S - T \Rightarrow x$ non $\geqq y$. Consequently $T \oplus (S - T)$ is the unsubstantial prolongation of $S$ because $T$ is a demanded ideal of $S$ coinicial with $T \oplus (S - T)$ and $T \iota T \oplus (S - T)$. According to A5 there is $f_S < g_S$ in $\Phi(A, T \oplus (S - T))$. According to A4 b) there is $f_T \leqq g_T$ in $\Phi(A, T)$. Let us put $V = T \bigcirc Z$, where $Z$ is a set of all integers in natural ordering and let us identify $t \in T$ with $\langle t, 0\rangle \in T \bigcirc Z$. Let us define $f'_V$ and $g'_V$ in such a way

$$f'_V(t, 2i) = f(t), f'_V(t, 2i + 1) = g(t)$$

$$g'_V(t, 2i) = g(t), g'_V(t, 2i + 1) = f(t).$$

According to A4 a) there is $f'_V < g'_V$ in $\Phi(A, V)$. Let $\varphi$ be a mapping $V$ onto $V$ for which $\varphi(t, i) = \langle t, i + 1\rangle$. Then $\varphi$ is a similar mapping $V$ onto $V$ and $\Phi(\varepsilon, \varphi)(f'_V) = g'_V$, $\Phi(\varepsilon, \varphi)(g'_V) = f'_V$, where $\varepsilon$ denotes an identical mapping on $A$. According to A2 there is $f'_V > g'_V$, which is a contradiction. In such a way is the theorem proved.

The introduced system of axioms A1—A5 characterizes in a certain way $\exp_A B$ among possible modifications of ordinal power. Let us introduce, for interest only, those modifications that come into consideration in the first line. Let $\Phi_1(A, B)$ be defined as ${}^B A$ in case that ${}^B A$ is an ordered set (see theorem (*D*)), otherwise we put $\Phi_1(A, B) = (A^B, \leqq)$ where $\leqq$ is an ordering into antichain. It is easy to see that for this functor there holds a statement analogous to the theorems 3—5. But, there is not fulfilled the conditions of "embedding" given in A4 a). There naturally arises the question how strong a condition "of embedding" is to be demanded. For one of the weakest formulations is possible to take the following condition:

(P) Let $\Phi(A, B) \in \mathscr{U}$ for $A, B \in \mathscr{U}$. Let $B \iota B_1$. Let $f, g \in \Phi(A, B_1)$, $f(x) = g(x)$ for $x \in B_1 - B$. Then $f \leqq g$ in $\Phi(A, B_1) \equiv f_B \leqq g_B$ in $\Phi(A, B)$.

The most natural modification of the operation ${}^B A$ fulfilling (P) is the operation $\Phi_2$ defined in this way: Let $A, B \in \mathscr{U}$. Then $\Phi_2(A, B) = (A^B, \leqq)$ where $\leqq$ is defined as follows: $f, g \in A^B, f \leqq g \equiv n(f, g) \in \mathscr{K}$ and $m \in m(n(f, g)) \Rightarrow f(m) < g(m)$.

It is easy to find that for $\Phi_2$ there hold theorems analogous to theorems 1—4. On the contrary the statement of the theorem 5 is not valid as the following example will prove.

Let $A = \{1, 2\}$, $B = \{\ldots, -n, \ldots, 0\}$, $n$ a positive integer, the ordering being equal to arithmetic ordering of integers.

Let $f^*, g^* \in \Phi_2(\Phi_2(A, B), A)$ be these mappings $f_1^*(-n) = f_2^*(-n) = 2$ for $n$ non negative, $g_1^*(-n) = 2$ for $n$ positive, $g_1^*(0) = 1$, $g_2^*(-n)$ arbitrary. Thus there exist $2^{\aleph_0}$ of functions $g^*$. At the same time $f_1^* > g_1^*$ in $\Phi_2(A, B)$, thus $f^* > g^*$.

Let $h, k \in \Phi_2(A, A \bigcirc B)$, $h < k$. Then $n(h, k) \subset A \bigcirc B$, $n(h, k)$ fulfils the condition of decreasing chains, consequently $n(h, k)$ is a finite subset in $A \bigcirc B$. In $A \bigcirc B$ there are $\aleph_0$ finite subsets. For any finite subset $S$ (for a fixed $k$) there exist finite many $h$ such that $n(h, k) = S$. Thus there exist, for a given $k$, at most $\aleph_0$ functions $h$ for which $h < k$.

Accordingly $\Phi_2(A, A \bigcirc B)$ non $\simeq \Phi_2(\Phi_2(A, B), A)$.

## REFERENCES


[1] G. Birkhoff, *Generalized Arithmetic*, Duke Math. J. 9 (1942) 283—302.

[2] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloquium Publ. 25, revised edition. New York 1948.

[3] M. M. Day, *Arithmetic of Ordered Systems*, Transactions of the Amer. Math. Soc. 58 (1945) 1—43.

[4] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914.