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# ON GENERALIZATIONS OF A THEOREM ON RECURSIVE SETS

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In this paper, we shall discuss some possibilities of generalization of the following well-known theorem:

**Theorem 1.** *If a set together with its complement is recursively enumerable, then it is recursive.*

The generalization to a finite number of sets is trivial: If there are given finitely many recursively enumerable (r.e.), mutually disjoint sets, the union of which is the whole set of natural numbers, then any of these sets is recursive. If we call every system of mutually disjoint nonempty sets, the union of which is — in our considerations — the set of all nonnegative integers, a *decomposition* ([B]), we then have the statement: a finite decomposition into r.e. sets is the decomposition into recursive sets.

An arbitrary decomposition into r.e. sets, evidently, does not possess this property: It suffices to consider a decomposition, one element of which is some non-recursive r.e. set and the remaining elements of which are one-element sets only. But on the other hand, even the very fact that a given decomposition is a decomposition into recursive sets, need not always express the property adequate to the case, the generalization of which we are seeking. There exist, namely, decompositions into recursive sets which are defined in such a way that this does not provide sufficient means for the effective construction of an algorithm which would decide, for an arbitrary number $x$, whether $x$ belongs to a given set of the decomposition. The sense of this statement is made precise in the theorem 2 below.

Let us call *recursive* such *decomposition* $\overline{N}$, for which the relation $R$ of the corresponding equivalence ($R(x, y) \equiv x$ and $y$ belong to the same element $\bar{a} \in \overline{N}$) is recursive. As it is easily seen, the decomposition is recursive if and only if some of its characteristic function $f$ (i.e. function for which $f(x) = f(y) \equiv R(x, y)$) is recursive. Evidently, each element of a recursive decomposition is a recursive set. However:

**Theorem 2.** *There exists a non-recursive decomposition with recursive elements only.*

Proof. As an example we can take a decomposition $\overline{N}$ into intervals, constructed by an arbitrary non-recursive set $M$, $0 \notin M$, in this way:

$R(x, y) \equiv \sim \bigvee z(z \in N \wedge (x < z \leqq y \vee y < z \leqq x))$. From the non-recursivity of $M$ there follows its infiniteness and thus finiteness and recursivity of elements of the decomposition $\overline{N}$. If $\overline{N}$ were recursive, it would be recursive also the function $f(x)$, defined by the scheme: $f(0) = 1$, $f(x + 1) = 1$ if $R(x + 1, x)$, $f(x + 1) = 0$ in the opposite case. This function is, however, the characteristic function of the set $M$.

Our generalization of the theorem 1 will consist in trying to find out, under which conditions the decomposition into r.e. sets is a recursive decomposition, rather then in asking when it is a decomposition into recursive sets only; the last case will be however discussed too. The theorem 2 shows that even the question, when the decomposition into recursive sets is recursive, is a non-trivial one. The example mentioned in its proof leads to the conjecture that the constructed decomposition is not recursive for the reason only, that the investigated elements— although they are recursive — cannot be effectively determined. This conjecture is correct when we take the effective determination of the system of sets as the recursivity of a set of its Gödel numbers. As the following theorem shows, it suffices to suppose merely recursive enumerability instead of recursivity.

Before its formulation and proof let us mention that notions and symbols, which are not defined here, can the reader find e.g. in [D] (including notation; especially the functions $U, J, K, L$ and the predicate $T$ have the standard significance there explained). Let us stipulate, in addition, that under the Gödel number (GN) of a recursive (recursively enumerable) set we shall understand the GN of its characteristic function (of a function which generates it). The GN of a function is of course — in the terminology of [D] — the GN of a corresponding Turing machine. A recursive set considered as recursive has thus in general another GN than the same set considered as r.e. When speaking of a set $E$ of Gödel numbers (GNs) of sets from some system $\mathscr{S}$, we assume that for any set from $\mathscr{S}$ there is in $E$ at least one of its GN. The function determined by the GN $e$ is denoted by $[e]$.

**Theorem 3.** *Let the elements of a decomposition $\overline{N}$ be all recursive sets and let a set of GNs of elements of the decomposition $\overline{N}$ be r.e. Then $\overline{N}$ is recursive.*

Proof. Let the set $E$ of the mentioned GNs be generated by the recursive function $\eta(t)$. The total function

$$f(x) = \mu t(U(\mu y T(\eta(t), x, y)) = 0)$$

which assigns to each $x$ the least $t$ such that $\eta(t)$ is GN of the characteristic function of the $\bar{a} \in \overline{N}$, to which $x$ belongs, is a characteristic function of the decomposition $\overline{N}$. According to the construction $f$ is — with regard to the supposed recursivity of $\eta(t)$ and the obvious effectivity of the involved minimum-operations — recursive. (A minimum operation $\mu$ is said to be effective when it is applied to a regular function.)

It should be noticed that the condition of recursive enumerability of the set $E$ is not necessary for the recursivity of the decomposition: There exist recursive decompositions such that suitable sets of GN*s* of their elements are not r.e. This follows immediately from the results of [R].

If it were possible to replace the requirement of recursivity of elements of the decomposition $\bar{N}$ in the theorem 3 by the requirement of their mere recursive enumerability, we should get a strong strenghtening of the theorem 1. It is shown, however, that this is not the case (theorem 4), because even the supposition of recursivity of the set of GN*s* does not always ensure the recursivity of the decomposition. This can be ensured under additional conditions; two of them will be mentioned (theorem 5, 8).

**Theorem 4.** *There exists a decomposition $\bar{N}$ such that each of its elements is a r.e. set and a suitable set of* GN*s of elements of $\bar{N}$ is recursive, whereas the decomposition $\bar{N}$ itself is not recursive.*

Proof. Let $P(y, i)$ be a recursive predicate such that $\bigvee yP(y, i)$ is no more recursive. Let us now define the function $f(i, x)$ in the following way:

$$\begin{aligned} f(2i, x) &= J(i, 2x) \quad \text{if} \quad \bigwedge y(y < x \rightarrow \sim P(y, i)) \\ &= J(i, x \dot{-} \mu y P(y, i)) \quad \text{in the opposite case} \\ f(2i+1, x) &= J(i, 2x+1) \quad \text{if} \quad \bigwedge y(y < x \rightarrow \sim P(y, i)) \\ &= J(i, x \dot{-} \mu y P(y, i)) \quad \text{in the opposite case.} \end{aligned}$$

This function is, according to the construction, recursive and has these properties: for any $i$ it is

$$f(2i, N) = J(i, 2N) \quad \text{or} = J(i, N)$$

in dependence on whether

$$\sim \bigvee yP(y, i) \quad \text{or} \quad \bigvee yP(y, i);$$

$$f(2i+1, N) = J(i, 2N+1) \quad \text{or} = J(i, N)$$

in dependence on whether

$$\sim \bigvee yP(y, i) \quad \text{or} \quad \bigvee yP(y, i)$$

(here under the symbol $J(2i, 2N+1)$ one understands the set $\{J(2i, 2n+1)\}_{n=0}^{\infty}$; analogously for similar further symbols). Thus for arbitrary $i_1, i_2$ it is either $f(i_1, N) \cap f(i_2, N) = \emptyset$ or $f(i_1, N) = f(i_2, N)$ (the last case can occur only for $|i_1 - i_2| \leqq 1$). On the other hand for all $i$ it is $f(2i, N) \cup f(2i+1, N) = J(i, N)$ so that to an arbitrary $z \in N$ there exists at least one pair of numbers $i, x$ such that either $z = f(2i, x)$ or $z = f(2i+1, x)$. One can see from it that the system of sets $\{f(i, N)\}_{i=0}^{\infty}$ forms a decomposition. This decomposition is not recursive. If it were, the relation $R(x, y)$ of corresponding equivalence would be of the same property, and thus so would be the predicate $Q(i) \equiv R(J(i, 0), J(i, 1))$. The predicate $Q(i)$ is however equivalent to the predicate $f(2i, N) = f(2i+1, N) = J(i, N)$ and this again to the predicate $\vee y P(y, i)$ which is not recursive according to the supposition. Let us now denote the GN of the recursive function $f(i, x)$ by $e$: $f(i, x) = [e](i, x)$. Consider the functions $f_i(x)$ of the variable $x$, which generate the elements of $N$; for a given $i_0$ it is $f_{i_0}(x) = f(i_0, x)$. According to the iteration theorem it is $f_i(x) = [\pi'(i)]\ (x)$ for some recursive function $\pi'$ (the dependence of $\pi'$ on $e$ is here not expressed). $\pi'(i)$ enumerates a list of GN*s* of some Turing machines. A system of machines with the same activity can be however enumerated by another function, say $\pi(i)$, which is strictly increasing (we can e.g. add to every machine with the number $\pi'(i)$ some quadruples that do not change its action so that the GN $\pi(i)$ of the resulting machine exceeds all $\pi(j)$ for $j < i$). This yields the recursivity of a suitable set of GN*s* of elements of the decomposition $\overline{N}$ and put the end to the proof.

*Note.* Each element of $\overline{N}$ has in $E$ at most two GN*s*.

The following remark to the proof of the theorem 4 concerns the character of sets $f(i, N)$: It is immediately seen that any of these sets is not only r.e., but even recursive. This fact could mislead to the statement that there exists a non-recursive decomposition with recursive elements and with a r.e. set of GN*s* of the elements, which would be in contradiction to the theorem 3. This apparent paradox is, of course, of terminological origine: although the sets $f(i, N)$ are recursive, we are given their GN*s* only as numbers of sets recursively enumerable.

Now, let us mention some positive results. First, we shall show that if we exclude the possibility of giving the same element of a decomposition in two or more different ways (and the non-effectivity of establishing equality between such two elements made crux of the counterexample constructed in the preceeding proof), it is possible to generalize the theorem 3 in the above mentioned way. More exactly speaking, it holds:

**Theorem 5.** *Let the elements of the decomposition $\overline{N}$ be r.e. sets only and let there be a r.e. set $E$ of* GN*s of elements of the decomposition $\overline{N}$ such that:* (1) *to any element $\bar{a} \in \overline{N}$ there exists in the set $E$ at most (and thus just) one its* GN. *Then the decomposition $\overline{N}$ is recursive.*

Proof. Let the set $E$ be generated by the recursive function $\eta(t)$. We remark that the supposition (1) does not exclude that $\eta(t)$ generates the set $E$ with repetitions; this can be, as it is well-known, assumed if $E$ is infinite. Let us suppose it. For an arbitrary $t$ then the function $g(t, x) = U\mu y T(\eta(t), x, y)$ generates one of the sets $\bar{a} \in \bar{N}$ and this set cannot be generated according to our supposition by any other function $g(t', x)$ for $t' \neq t$. Thus the function

$$f(x) = K(\mu u(g(K(u), L(u)) = x))$$

is a characteristic function of $\bar{N}$. With regard to the recursivity of the function $\eta$ and the effectivity of used minimum-operations, this is, according to the construction, a recursive function. If the set $E$ is finite, then such is also the decomposition $\bar{N}$ and the theorem can be proved by induction using the theorem 1.

Let us note that the additional supposition (1) is evidently not a necessary for the recursivity of $\bar{N}$.

Now, we shall prove:

**Theorem 6.** *Assume that suppositions from the theorem* 5 *hold with the only exception: instead of* (1) *we suppose only* (2): *to any element* $\bar{a} \in \bar{N}$ *there is in the set E a finite many of its* GN*s. Then any element* $\bar{a} \in \bar{N}$ *is recursive.*

*Remarks.* 1. According to the note following the proof of the theorem 4 we cannot assert in this case the recursivity of $\bar{N}$. 2. There occurs a question whether in the theorem 6 the condition (2) may be omitted.

Proof. Let us define the function $f(x)$ as in the proof of the preceding theorem, where all used symbols have the same meaning as it is there defined. The function $f(x)$ is, now, a recursive characteristic function of some decomposition $\bar{N}'$, which is a refinement of the decomposition $\bar{N}$. Every $\bar{a} \in \bar{N}$ is therefore the union of a finite many of elements of $\bar{N}'$, which is recursive. Hence, $\bar{a}$ is recursive too.

In the classical theory of decompositions and their applications an important role is plaid by the notion of the *choice set* of a given decomposition $\bar{N}$, as the set of numbers, precisely one drawn from each of sets belonging to $\bar{N}$. Using this notion, we formulate a further sufficient condition for the recursivity of a decomposition. One can prove without any difficulty the following statement:

**Theorem 7.** *If* $\bar{N}$ *is a recursive decomposition, then its suitable choice set is recursive too.*

Now, we prove:

**Theorem 8.** *Let all elements of the decomposition* $\bar{N}$ *be r.e. sets, let a suitable set of their* GN*s have the same property and* (3) *let be r.e. also a suitable choice set of it. Then the decomposition* $\bar{N}$ *is recursive.* (*Cf* [*M*, *theorem* 1.4]).

Proof. Let the set $E$ of GN*s* be generated by the function $\eta$, the choice set $F$ by the function $\tau$. Similarly like in the proof of the theorem 5 we shall suppose that $\overline{N}$ and thereby $E$ and $F$ are infinite, the last two sets being generated by corresponding functions without repetitions. Let us define the predicate $R$: $R(x, y) \equiv [\eta(x)](N) = [\eta(y)](N)$. This predicate is evidently an equivalence relation and thus corresponds to a decomposition $\overline{N}^*$ on $N$. We shall show that under the supposition (3), $\overline{N}^*$ is recursive. From it, according to the theorem 7, there follows the recursivity of a suitable choice set $F^*$ of the decomposition $\overline{N}^*$, thus, also its recursive enumerability. If we denote by $\tau^*$ the recursive function generating the set $F$, the function $\eta(\tau^*(x))$ will then generate a set $E^* \subset E$, that contains just one GN to any element of $\bar{a} \in \overline{N}$. In this way, the proof will be finished, according to the theorem 5. Now, we see that numbers $x$ and $y$ contained in the same element $\bar{a} \in \overline{N}$ are characterized by the property that in the sets $[\eta(x)](N)$ and $[\eta(y)](N)$ there must lie just one element of the set $F$. Hence, it is possible to take as a characteristic function of $\overline{N}^*$ the following:

$$f(x) = \tau(\mu t(\mu u([\eta(x)](u) = \tau(t))))$$

which is, according to the construction — with regard to the recursivity of the function $\eta$, $\tau$ and effectivity of applied minimum-operations — recursive.

The suppositions of the theorem 5 and 8 are evidently satisfied by the supposition of the theorem 1, thus, they represent its generalizations.

Let us note to the just proved theorem that the requirement (3) cannot replace the condition laid on the set $E$, and in this way ensure itself the recursivity of $\overline{N}$. We can consider e.g. the decomposition from the proof of the theorem 2 with the r.e. set $M$, playing simultaneously the role of the choice set. Moreover, we prove this stronger result:

**Theorem 9.** *For every pair of cardinal numbers $p, q$, $p + q = \aleph_0$, there is a decomposition $\overline{N}$ with a recursive choice set, consisting of $p$ recursive and $q$ r.e. non-recursive sets.*

**Corollary.** There exists a decomposition with r.e. non-recursive sets only.

Proof. Let $\overline{N}$ be an arbitrary recursive decomposition into two sets $\bar{a}, \bar{b}$, $\bar{a}$ having $p$ elements, $\bar{b}$ having $q$ elements. Let $A$ be a r.e. non-recursive set, $0 \in A$. Then, $J(i, N)$, $i \in \bar{a}$, are $p$ recursive sets, $J(i, A)$, $i \in \bar{b}$, are $q$ r.e. non-recursive sets. The complement $C$ of the union of all these sets is clearly infinite. Suppose that $q$ is infinite, too. Then, there is a one-to-one function $\varphi$ that maps $\bar{b}$ onto $C$($\varphi$ is of course non-recursive). The system consisting of the sets $J(i, N)$ for $i \in \bar{a}$ and $J(i, A) \cup \{\varphi(i)\}$ for $i \in \bar{b}$ forms the desired decomposition $\overline{N}$. The set $J(N, 0)$ serves as the choice set. If $q$ is finite, then $p$ is infinite and the argument is analogous.

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