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ON SOME TOPOLOGIES ON PRODUCTS OF ORDERED SETS

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1. FORMULATION OF THE PROBLEM

Several definitions of topologies on ordered sets can be found in the literature ([1], [2], [4]). One of them is due to O. Frink [3], who defined a certain type of ideal in ordered sets and obtained his topology by taking the family of all completely (meet-) irreducible ideals and dual ideals as a sub-basis for the open sets. Frink found out that the topology of a cardinal product of a finite number of ordered sets defined in this manner, is identical with the product topology if every factor of the product has been topologized following his definition.

We shall generalize Frink's notion of ideal in ordered sets by substituting in his definition, finite sets by sets of cardinality $<\mathfrak{m}$ where $\mathfrak{m}$ is a given infinite cardinal number. Then we take the family of all completely irreducible generalized ideals and dual ideals as an open sub-basis. This topology is equal to the product topology if the number of factors is less than $\mathfrak{m}$ and if every factor is $\mathfrak{m}$-directed, i.e. every subset of the factor of cardinality less than $\mathfrak{m}$ has an upper and a lower bound. In particular we can take an arbitrary system of ordered sets with a greatest and a least element and define the family of all completely irreducible normal ideals and normal dual ideals to be an open sub-basis. Then the product topology is identical with the topology defined on the cardinal product of these sets in the described fashion, using the family of all completely irreducible normal ideals and normal dual ideals of the product as an open sub-basis.

By an ordered set we mean a partially ordered set, by a product of ordered sets $P_\nu (\nu \in N)$ their cardinal product which we denote by $\prod_{\nu \in N} P_\nu$. Every $x \in \prod_{\nu \in N} P_\nu$ is a function $x(.)$ with the property $x(\nu) \in P_\nu$ for every $\nu \in N$; we set $\mathrm{pr}_\nu x = x(\nu)$, so that pr_ν is a function with the domain

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$\prod_{\nu \in N} P_\nu$ and the range P_ν. The symbols $\mathrm{pr}_\nu A$ and $\mathrm{pr}_\nu^{-1} B$ have the obvious meanings: If $A \subset \prod_{\nu \in N} P_\nu$ then $\mathrm{pr}_\nu A = \{\mathrm{pr}_\nu x \mid x \in A\}$; if $B \subset P_\nu$ then $\mathrm{pr}_\nu^{-1} B = \{x \mid x \in \prod_{\nu \in N} P_\nu,\ \mathrm{pr}_\nu x \in B\}$. The following statements are obvious:

1.1 *If* $B \subset P_\nu$ *then* $\mathrm{pr}_\nu^{-1} B = \prod_{\mu \in N} C_\mu$ *where* $C_\nu = B$ *and* $C_\mu = P_\mu$ *for every* $\mu \neq \nu$.

1.2 *If* $A_\nu \subset P_\nu$ *for every* $\nu \in N$ *and* $A = \prod_{\nu \in N} A_\nu$ *then* $A = \bigcap_{\nu \in N} \mathrm{pr}_\nu^{-1} A_\nu$.

1.3 *If* $\nu_0 \in N$ *and* $A_{\nu_0}^\mu \subset P_{\nu_0}$ *for every* $\mu \in M$, *then* $\mathrm{pr}_{\nu_0} (\bigcap_{\mu \in M} \mathrm{pr}_{\nu_0}^{-1} A_{\nu_0}^\mu) = \bigcap_{\mu \in M} A_{\nu_0}^\mu$.

If (P_ν, τ_ν) are topological spaces for every $\nu \in N$ (P_ν is a set and τ_ν a topology on P_ν), we denote by $\prod_{\nu \in N} (P_\nu, \tau_\nu)$ the cartesian product of the sets P_ν with the usual product topology.

The cardinal number of a set M is symbolized by $|M|$. And we write "iff" instead of "if and only if".

2. SOME PROPERTIES OF THE OPERATORS * AND +

G. Birkhoff ([1], p. 58) defined the antitone operators * and + on the family of all subsets of an ordered set. In the sequel we need some properties of these operators. In the whole paragraph, N is an arbitrary non-empty set, P_ν an arbitrary non-empty ordered set, for every $\nu \in N$.

2.1 Lemma *If* $A \subset \prod_{\nu \in N} P_\nu$ *then* $\mathrm{pr}_\nu(A^*) \subset (\mathrm{pr}_\nu A)^*$, *and* $\mathrm{pr}_\nu(A^+) \subset (\mathrm{pr}_\nu A)^+$, *for every* $\nu \in N$.

Proof. If $x_\nu \in \mathrm{pr}_\nu(A^*)$ then there exists an element $x \in A^*$ with the property $\mathrm{pr}_\nu x = x_\nu$. Hence $t \leqq x$ and $\mathrm{pr}_\nu t \leqq \mathrm{pr}_\nu x = x_\nu$, for every $t \in A$. This implies $x_\nu \in (\mathrm{pr}_\nu A)^*$. The proof of the second inclusion is analogous.

2.2 Lemma. *If the set* A, $A \subset \prod_{\nu \in N} P_\nu$, *has an upper [lower] bound, then* $(\mathrm{pr}_\nu A)^* \subset \mathrm{pr}_\nu(A^*)$ $[(\mathrm{pr}_\nu A)^+ \subset \mathrm{pr}_\nu(A^+)]$, *for every* $\nu \in N$.

Proof. If $x_\nu \in (\mathrm{pr}_\nu A)^*$ then $x_\nu \geqq t_\nu$ for every $t_\nu \in \mathrm{pr}_\nu A$. We take an arbitrary element $p \in A^*$ and set $q(\mu) = p(\mu)$ for every $\mu \neq \nu$ and $q(\nu) = x_\nu$. Therefore we have $q \in A^*$ and $x_\nu = q(\nu) = \mathrm{pr}_\nu q \in \mathrm{pr}_\nu(A^*)$. The second formula has a similar proof.

2.3 Lemma. *If* $\emptyset \neq A \subset \prod_{\nu \in N} P_\nu$ *then* $(\mathrm{pr}_\nu A)^{*+} \subset \mathrm{pr}_\nu(A^{*+})$, *for every* $\nu \in N$.

Proof. If A has no upper bound, we have $A^* = \emptyset$, hence $A^{*+} = \prod_{\nu \in N} P_\nu$ and $\mathrm{pr}_\nu(A^{*+}) = P_\nu \supset (\mathrm{pr}_\nu A)^{*+}$.

If A has an upper bound, 2.1 and 2.2 imply $(\mathrm{pr}_\nu A)^* = \mathrm{pr}_\nu(A^*)$. For every $x \in A$ and $y \in A^*$ we have $x \leqq y$; it follows that $x \in A^{*+}$ and that A^* has a lower bound. Hence, with 2.2 we obtain $(\mathrm{pr}_\nu A)^{*+} = (\mathrm{pr}_\nu(A^*))^+ \subset \mathrm{pr}_\nu(A^{*+})$.

3. $\mathfrak{m}$-IDEALS

3.1 Definition. Let $\mathfrak{m}$ be an arbitrary infinite cardinal number, P an ordered set. A subset $I \subset P$ is called an $\mathfrak{m}$-*ideal* of P iff for every subset M, $\emptyset \neq M \subset I$ with $|M| < \mathfrak{m}$ the inclusion $M^{*+} \subset I$ holds.

3.2 Remark. The ideals of Frink are precisely the $\aleph_0$-ideals. A subset I of the ordered set P is called *normal ideal* iff it is an $\mathfrak{m}$-ideal of P for every cardinal number $\mathfrak{m}$.

Some results of this paragraph are deduced under the following assumption about notation:

(α) Let $\mathfrak{m}$ be an infinite cardinal number, N a non-empty set, P_ν a non-empty ordered set for every $\nu \in N$.

3.3 Lemma. *Let* (α) *hold. If* $I \subset \prod_{\nu \in N} P_\nu$ *is an* $\mathfrak{m}$-*ideal of* $\prod_{\nu \in N} P_\nu$, *then* $\mathrm{pr}_\nu I$ *is an* $\mathfrak{m}$-*ideal of* P_ν *for every* $\nu \in N$.

Proof. Let M_ν be a set with the properties $\emptyset \neq M_\nu \subset \mathrm{pr}_\nu I$, $|M_\nu| < \mathfrak{m}$. For every $x \in M_\nu$ there exists an element $y^x \in I$ with the property $\mathrm{pr}_\nu y^x = x$. We set $A = \{y^x \mid x \in M_\nu\}$, and have $\emptyset \neq A \subset I$, $|A| \leqq |M_\nu| < \mathfrak{m}$. Therefore $A^{*+} \subset I$ and $\mathrm{pr}_\nu(A^{*+}) \subset \mathrm{pr}_\nu I$. By 2.3 we find $(M_\nu)^{*+} = (\mathrm{pr}_\nu A)^{*+} \subset \mathrm{pr}_\nu(A^{*+}) \subset \mathrm{pr}_\nu I$.

3.4 Lemma. *Let* (α) *hold and* $|N| < \mathfrak{m}$. *If* $I \subset \prod_{\nu \in N} P_\nu$ *is an* $\mathfrak{m}$-*ideal of* $\prod_{\nu \in N} P_\nu$, *then* $I = \prod_{\nu \in N} \mathrm{pr}_\nu I$.

Proof. It is clear that $I \subset \prod_{\nu \in N} \mathrm{pr}_\nu I$. Let $x \in \prod_{\nu \in N} \mathrm{pr}_\nu I$. For every $\nu \in N$ there exists an element $x^\nu \in I$ with $\mathrm{pr}_\nu x^\nu = \mathrm{pr}_\nu x$. We put $M = \{x^\nu \mid \nu \in N\}$. Then $\emptyset \neq M \subset I$, $|M| \leqq |N| < \mathfrak{m}$, and by assumption $M^{*+} \subset I$. For every $y \in M^*$ and every $\nu \in N$ we have $x^\nu \leqq y$, hence $\mathrm{pr}_\nu x = \mathrm{pr}_\nu x^\nu \leqq \mathrm{pr}_\nu y$. This implies $x \leqq y$ for every $y \in M^*$, i.e. $x \in M^{*+} \subset I$. So we have shown $\prod_{\nu \in N} \mathrm{pr}_\nu I \subset I$ which proves the lemma.

3.5 Definition. Let $\mathfrak{m}$ be an arbitrary infinite cardinal number, P an ordered set. The set P is called $\mathfrak{m}$-*directed from above* iff every non-empty subset $M \subset P$ with the property $|M| < \mathfrak{m}$ has an upper bound in P.

Most results of this paper are deduced under the following assumption about notation:

(β) $\mathfrak{m}$ is an infinite cardinal number, N a non-empty set with $|N| < \mathfrak{m}$, and P_ν a non-empty ordered set which is $\mathfrak{m}$-directed from above for every $\nu \in N$.

It is easy to see that the product $\prod_{\nu \in N} P_\nu$ is $\mathfrak{m}$-directed from above under the assumption (β).

3.6 Lemma. *Let (β) hold. If $I_\nu \subset P_\nu$ is an $\mathfrak{m}$-ideal of P_ν for every $\nu \in N$, then $\prod_{\nu \in N} I_\nu$ is an $\mathfrak{m}$-ideal of $\prod_{\nu \in N} P_\nu$.*

Proof. Let $\emptyset \neq M \subset \prod_{\nu \in N} I_\nu$, $|M| < \mathfrak{m}$. Then $\mathrm{pr}_\nu M \subset I_\nu$, $|\mathrm{pr}_\nu M| \leqq \leqq |M| < \mathfrak{m}$ for every $\nu \in N$. Hence, we have $(\mathrm{pr}_\nu M)^{*+} \subset I_\nu$ for every $\nu \in N$. Since the set M has an upper bound it satisfies $(\mathrm{pr}_\nu M)^* \subset \mathrm{pr}_\nu(M^*)$ by 2.2. Because the operator $^+$ is antitone, and because of 2.1 we have $\mathrm{pr}_\nu(M^{*+}) = \mathrm{pr}_\nu((M^*)^+) \subset (\mathrm{pr}_\nu M^*)^+ \subset (\mathrm{pr}_\nu M)^{*+} \subset I_\nu$, for every $\nu \in N$. This implies $M^{*+} \subset \prod_{\nu \in N} \mathrm{pr}_\nu(M^{*+}) \subset \prod_{\nu \in N} I_\nu$.

For an $\mathfrak{m}$-ideal $I_\nu \subset P_\nu$ we can write $\mathrm{pr}_\nu^{-1} I_\nu$ in the form of a cardinal product, following 1.1; since all factors of this product are $\mathfrak{m}$-ideals we have

3.7 Corollary. *Let (β) hold. Let $I_\nu \subset P_\nu$ be an $\mathfrak{m}$-ideal of P_ν. Then $\mathrm{pr}_\nu^{-1} I_\nu$ is an $\mathfrak{m}$-ideal of $\prod_{\nu \in N} P_\nu$.*

From 3.3, 3.4 and 3.6 we obtain

3.8 Theorem. *Let $\mathfrak{m}$ be an infinite cardinal number, N a non-empty set with $|N| < \mathfrak{m}$, P_ν a non-empty ordered set which is $\mathfrak{m}$-directed from above for every $\nu \in N$. Then for a set $I \subset \prod_{\nu \in N} P_\nu$ the following statements are equivalent:*

(A) *I is an $\mathfrak{m}$-ideal of $\prod_{\nu \in N} P_\nu$,*

(B) *$I = \prod_{\nu \in N} \mathrm{pr}_\nu I$ and every $\mathrm{pr}_\nu I$ is an $\mathfrak{m}$-ideal of P_ν.*

4. COMPLETELY IRREDUCIBLE $\mathfrak{m}$-IDEALS

4.1 Definition. Let $\mathfrak{m}$ be an arbitrary infinite cardinal number, P an ordered set, $I \subset P$ an $\mathfrak{m}$-ideal of P. This ideal is called *completely irreducible* iff for every family $I^\mu (\mu \in M,\ M \neq \emptyset)$ of $\mathfrak{m}$-ideals with $I = \bigcap_{\mu \in M} I^\mu$ there exists an index $\mu_0 \in M$ such that $I^{\mu_0} = I$.

4.2 Theorem. *Let $\mathfrak{m}$ be an arbitrary infinite cardinal number, N a non-empty set with $|N| < \mathfrak{m}$, P_ν a non-empty ordered set which is $\mathfrak{m}$-directed from above for every $\nu \in N$. Then for an $\mathfrak{m}$-ideal $I \subset \prod_{\nu \in N} P_\nu$ the following statements are equivalent:*

(A) *I is completely irreducible,*

(B) *There exists an index $\nu_0 \in N$ and a completely irreducible* $\mathfrak{m}$*-ideal* I_{ν_0} *of* P_{ν_0} *such that* $I = \overset{-1}{\mathrm{pr}}_{\nu_0} I_{\nu_0}$.

Proof. a) If (A) holds, then $I = \prod_{\nu \in N} \mathrm{pr}_\nu I = \bigcap_{\nu \in N} \mathrm{pr}_\nu^{-1}(\mathrm{pr}_\nu I)$, by 3.4 and 1.2. Then 3.3 and 3.7 imply that every set $\mathrm{pr}_\nu^{-1}(\mathrm{pr}_\nu I)$ is an $\mathfrak{m}$-ideal of $\prod_{\nu \in N} P_\nu$. By the irreducibility of I the existence of an index $\nu_0 \in N$ is assured, with the property $I = \overset{-1}{\mathrm{pr}}_{\nu_0}(\mathrm{pr}_{\nu_0} I)$. We set $\mathrm{pr}_{\nu_0} I = I_{\nu_0}$. Let $I^\mu_{\nu_0}(\mu \in M,\ M \neq \emptyset)$ be a family of $\mathfrak{m}$-ideals of P_{ν_0} with $I_{\nu_0} = \bigcap_{\mu \in M} I^\mu_{\nu_0}$. Clearly $I = \overset{-1}{\mathrm{pr}}_{\nu_0}(\bigcap_{\mu \in M} I^\mu_{\nu_0}) = \bigcap_{\mu \in M} \overset{-1}{\mathrm{pr}}_{\nu_0} I^\mu_{\nu_0}$ and every set $\overset{-1}{\mathrm{pr}}_{\nu_0} I^\mu_{\nu_0}$ is an $\mathfrak{m}$-ideal of $\prod_{\nu \in N} P_\nu$, by 3.7. This implies the existence of an index $\mu_0 \in M$ such that $I = \overset{-1}{\mathrm{pr}}_{\nu_0} I^{\mu_0}_{\nu_0}$. Hence $I_{\nu_0} = \mathrm{pr}_{\nu_0} I = \mathrm{pr}_{\nu_0}(\overset{-1}{\mathrm{pr}}_{\nu_0} I^{\mu_0}_{\nu_0}) = I^{\mu_0}_{\nu_0}$, and we have proved that I_{ν_0} is completely irreducible, i.e. (A) implies (B).

b) Let (B) hold and let $I^\mu(\mu \in M,\ M \neq 0)$ be a family of $\mathfrak{m}$-ideals with $I = \bigcap_{\nu \in M} I^\mu$. We have $I^\mu \supset I = \overset{-1}{\mathrm{pr}}_{\nu_0} I_{\nu_0}$, and therefore $I^\mu = \overset{-1}{\mathrm{pr}}_{\nu_0}(\mathrm{pr}_{\nu_0} I^\mu)$ for every $\mu \in M$. Following 1.3 we obtain $I_{\nu_0} = \mathrm{pr}_{\nu_0}(\overset{-1}{\mathrm{pr}}_{\nu_0} I_{\nu_0}) = \mathrm{pr}_{\nu_0} I = \mathrm{pr}_{\nu_0}(\bigcap_{\mu \in M} I^\mu) = \mathrm{pr}_{\nu_0}(\bigcap_{\mu \in M} \overset{-1}{\mathrm{pr}}_{\nu_0}(\mathrm{pr}_{\nu_0} I^\mu)) = \bigcap_{\mu \in M} \mathrm{pr}_{\nu_0} I^\mu$. Every set $\mathrm{pr}_{\nu_0} I^\mu$ is an $\mathfrak{m}$-ideal of P_{ν_0} by 3.3. The irreducibility of I_{ν_0} implies the existence of an index $\mu_0 \in M$ for which $I_{\nu_0} = \mathrm{pr}_{\nu_0} I^{\mu_0}$. Hence $I = \overset{-1}{\mathrm{pr}}_{\nu_0} I_{\nu_0} = \overset{-1}{\mathrm{pr}}_{\nu_0}(\mathrm{pr}_{\nu_0} I^{\mu_0}) \supset I^{\mu_0}$. Obviously, $I \subset I^{\mu_0}$. Hence $I = I^{\mu_0}$ and I is completely irreducible, i.e. (B) implies (A).

4.3 Remark. The notion of $\mathfrak{m}$-ideal and of set $\mathfrak{m}$-directed from above can be dualized. The new notions will be called *dual* $\mathfrak{m}$*-ideal* and $\mathfrak{m}$*-directed set from below.* A set is called $\mathfrak{m}$*-directed* iff it is $\mathfrak{m}$-directed from above and from below. It is clear that a dual theorem can be formulated to every of our theorems.

By dualizing the notion of normal ideal we obtain the notion of *normal dual ideal.*

5. $\mathfrak{m}$-IDEAL TOPOLOGIES IN ORDERED SETS

5.1 Definition. Let $\mathfrak{m}$ be an infinite cardinal number, P an ordered set. Let $(P, \tau_{\mathfrak{m}}(P))$ be the topological space in which the topology is defined by taking the family consisting of all completely irreducible $\mathfrak{m}$-ideals and of all completely irreducible dual $\mathfrak{m}$-ideals of P as a subbasis for the open sets. Then $\tau_{\mathfrak{m}}(P)$ is called the $\mathfrak{m}$*-ideal topology* on P.

5.2 Main Theorem. *Let* $\mathfrak{m}$ *be an infinite cardinal number, N a non-empty* *set with* $|N| < \mathfrak{m}$, P_ν *a non-empty ordered set which is* $\mathfrak{m}$*-directed for every* $\nu \in N$. *Then*

$$\prod_{\nu \in N} (P_\nu, \tau_{\mathfrak{m}}(P_\nu)) = (\prod_{\nu \in N} P_\nu, \tau_{\mathfrak{m}}(\prod_{\nu \in N} P_\nu)).$$

Proof. Both topologies are defined on the set $\prod_{\nu \in N} P_\nu$. We shall show that both have the same open sub-bases. According to the definition and to 4.2 an open sub-basis $\mathfrak{S}$ of the topology $\tau_{\mathfrak{m}}(\prod_{\nu \in N} P_\nu)$ is composed of all sets of the form $\mathrm{pr}_\nu^{-1} I_\nu$ where $\nu \in N$, and I_ν is an arbitrary completely irreducible $\mathfrak{m}$-ideal or a completely irreducible dual $\mathfrak{m}$-ideal of P_ν.

By definition, an open sub-basis of the space $\prod_{\nu \in N} (P_\nu, \tau_{\mathfrak{m}}(P_\nu))$ consists of all sets of the form $\prod_{\nu \in N} J_\nu$ where $J_\nu = P_\nu$ for all but a finite number of indices $\nu \in N$; for these indices, J_ν is a completely irreducible $\mathfrak{m}$-ideal or a completely irreducible dual $\mathfrak{m}$-ideal of P_ν. It is easy to see that every set of this form can be written as an intersection of a finite number of elements of $\mathfrak{S}$ and that every element of $\mathfrak{S}$ has this form. It follows that $\mathfrak{S}$ is itself a sub-basis for the open sets of the topological space $\prod_{\nu \in N} (P_\nu, \tau_{\mathfrak{m}}(P_\nu))$. Since both topologies have the same sub-basis, they are identical.

5.3 Definition. Let P be an ordered set. We denote by $(P, \tau(P))$ the topological space in which the topology is defined by taking the family consisting of all completely irreducible normal ideals and of all completely irreducible normal dual ideals of P as an open sub-basis. Then $\tau(P)$ is called the *normal ideal topology* on P.

5.4 Corollary. *Let* N *be a non-empty set,* P_ν *a non-empty ordered set with a greatest and a least element for every* $\nu \in N$. *Then*

$$\prod_{\nu \in N} (P_\nu, \tau(P_\nu)) = (\prod_{\nu \in N} P_\nu, \tau(\prod_{\nu \in N} P_\nu)).$$

Proof. We take an infinite cardinal number $\mathfrak{m}$ with $\mathfrak{m} > |\prod_{\nu \in N} P_\nu|$, $\mathfrak{m} > |N|$. Then $\mathfrak{m} > |P_\nu|$ for every $\nu \in N$. The set P_ν has a greatest and a least element, hence it is $\mathfrak{m}$-directed. Therefore the equality in 5.2 holds. But $\tau_{\mathfrak{m}}(\prod_{\nu \in N} P_\nu) = \tau(\prod_{\nu \in N} P_\nu)$ and $\tau_{\mathfrak{m}}(P_\nu) = \tau(P_\nu)$ for every $\nu \in N$.

6. PROBLEMS

The following questions are among those raised by our results.

6.1 Is it possible to construct for every pair of infinite cardinal numbers $\mathfrak{m} < \mathfrak{n}$ an ordered set P such that $\tau_{\mathfrak{m}}(P) \neq \tau_{\mathfrak{n}}(P)$?

6.2 Is it possible to construct for every cardinal number $\mathfrak{m} > \aleph_1$ such an $\mathfrak{m}$-directed set P, that for every pair of infinite cardinal numbers $\mathfrak{p} < \mathfrak{n} < \mathfrak{m}$ the inequality $\tau_{\mathfrak{p}}(P) \neq \tau_{\mathfrak{n}}(P)$ holds?

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