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MIXED PROBLEMS FOR HYPERBOLIC EQUATIONS WITH A SMALL PARAMETER

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We seek the connection between solutions of mixed problems for the hyperbolic equation $\varepsilon u_{tt} + \beta(t) u_t - Lu = F(x, t)$ and for the parabolic equation $\beta(t) U_t - LU = F(x, t)$. Here Lu is an elliptic operator $\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j}\right) - a(x) u$, $x = (x_1, x_2, \ldots, x_n)$ denotes a point from E_n, $\beta(t) > 0$ for $t \geqq 0$ and ε is a positive parameter.

1. We consider the equation

$$\varepsilon \frac{\partial^2 u}{\partial t^2} + \beta(t) \frac{\partial u}{\partial t} - Lu = F(x, t), \tag{1}$$

where

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j}\right) - a(x)u. \tag{2}$$

$x = (x_1, x_2, \ldots x_n)$ denotes a point from E_n, ε is a positive parameter, the coefficients $a_{ij}(x)$, $a(x)$ are defined in a bounded domain $\Omega \subset E_n$, the right-hand side $F(x, t)$ is defined in the cylinder $\overline{Q} = \overline{\Omega} \times \langle 0, T\rangle$ and the coefficient $\beta(t)$ in the interval $\langle 0, T\rangle$. We assume that it holds

$$a(x) \geqq 0, \quad a_{ij}(x) = a_{ji}(x), \ \sum_{i,j=1}^{n} a_{ij}\zeta_i\zeta_j \geqq \alpha \sum_{i=1}^{n} \zeta_i^2, \quad \alpha = \text{konst} > 0, \tag{3}$$

$$\beta(t) > 0. \tag{4}$$

We shall deal with the following mixed problems:

A_1: The first mixed problem when there are given the initial conditions

$$u(x, 0) = f(x), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x), \quad x \in \overline{\Omega} \tag{5}$$

and the homogeneous[1]) boundary condition of the Dirichlet type

$$u(x, t)\,|_S = 0, \quad S = F(\Omega) \times (0, T\rangle. \tag{6}$$

[1]) A nonhomogeneous condition can be transformed to the homogeneous one.

A_2: The second mixed problem when besides the initial conditions (5) there is given a boundary condition of the Neumann type

$$\frac{du}{dp}\bigg|_S = 0, \quad \text{where} \ \frac{du}{dp}\bigg|_S = \left[\sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \cos(\nu, x_i)\right]\bigg|_S; \tag{7}$$

the numbers $\cos(\nu, x_i)$ are the direction cosines of the inward normal.
A_3: The third boundary value problem when besides the initial conditions (5) there is given the boundary condition

$$\frac{du}{dp}\bigg|_S - h(x, t)u\bigg|_S = 0, \tag{8}$$

where $h(x, t)$ is a positive function defined on S.

We shall look for the connection between the solution[2]) $u(x, t)$ of these problems and the solution $U(x, t)$ of the analogous problem for the reduced equation

$$\beta(t) \frac{\partial U}{\partial t} - LU = F(x, t). \tag{9}$$

The boundary conditions remain the same whereas only one initial condition can be prescribed:

$$U(x, 0) = f(x). \tag{10}$$

These problems will be denoted by A_1^* A_2^*, A_3^* respectively.

Zlámal dealt with this problem in the case of the first mixed problem (see [1], [2], [3]). His reasoning was the following: If $\varepsilon = 0$ then, it is true, the order of the equation (1) does not change but the equation changes its type. From this reason it is not possible to expect that $u(x, t)$ is an analytic function of the parameter ε in the point $\varepsilon = 0$. What can be expected is the appearance of the so called boundary layer terms in the asymptotic formula for $u(x, t)$ (see [4], p. 7 and 8). With respect to the fact that the parameter ε appears at the second derivative u_{tt} only and that $u(x, 0) = U(x, 0)$, Zlámal sought the solution $u(x, t)$ in the form

$$u(x, t) = U(x, t) + \varepsilon H(x, t)\left[1 - e^{-\frac{\nu(t)}{\varepsilon}}\right] + \varepsilon z(x, t, \varepsilon).$$

[2]) Under the solution of a partial differential equation in a domain A we always understand a function which belongs to C^1 in the closed domain $\bar{A}$, to C^2 in the open domain A, satisfies the given boundary condition and the given differential equation in A. Such solutions are sometimes called biregular.

He chose the functions $H(x, t)$ and $\nu(t)$ in such a manner that the function $z(x, t, \varepsilon)$ satisfies the homogeneous initial conditions and that it holds

$$\varepsilon \frac{\partial^2 z}{\partial t^2} + \beta(t) \frac{\partial z}{\partial t} - Lz = 0(1). \tag{11}$$

By means of the Fourier method he managed to get the estimates of the function $z(x, t, \varepsilon)$.

In this paper we use the energy method for getting the estimates and we get the results for all three mixed problems mentioned above.

As in Zlámal's papers we seek the solution $u(x, t)$ of the problem $A_k (k = 1, 2, 3)$ in the form

$$u(x, t) = U(x, t) + \varepsilon k(x) \left[1 - e^{-\frac{\nu(t)}{\varepsilon}}\right] + \varepsilon z(x, t, \varepsilon), \tag{12}$$

where $U(x, t)$ is the solution of the problem A_k^* and the functions $k(x)$ and $\nu(t)$ will be chosen in such a way that (11) holds and that the function $z(x, t, \varepsilon)$ satisfies the homogeneous initial conditions

$$z(x, 0, \varepsilon) = \frac{\partial z}{\partial t}\bigg|_{t=0} = 0 \tag{13}$$

and the boundary conditions

(14) $z(x, t, \varepsilon) \mid_S = 0$ in the case of problem A_1,

(15) $\frac{dz}{dp}\Big|_S = 0$ in the case of problem A_2,

(16) $\frac{dz}{dp}\Big|_S - h(x, t)z \mid_S = 0$ in the case of problem A_3.

(11) is required from the reason that if it is true we can expect $z(x, t, \varepsilon$ to be also 0(1) in a suitable norm. By an easy calculation we find ou that the requirements are fulfilled if we choose

$$k(x) = \frac{\beta(0)g(x) - Lf - F(x, 0)}{\beta^2(0)} \tag{17}$$

$$\nu(t) = \int_0^t \beta(s)\, ds. \tag{18}$$

The function $z(x, t, \varepsilon)$ satisfies the equation

$$\varepsilon \frac{\partial^2 z}{\partial t^2} + \beta(t) \frac{\partial z}{\partial t} - Lz = P(x, t, \varepsilon), \tag{19}$$

where

$$(20) \qquad P(x, t, \varepsilon) = -\frac{\partial^2 U}{\partial t^2} - \beta'(t)k(x)\,e^{-\frac{\nu(t)}{\varepsilon}} + \left[1 - e^{-\frac{\nu(t)}{\varepsilon}}\right] Lk.$$

Further it must hold

$$(21) \qquad k(x) = 0, \quad x \in F(\Omega) \qquad \text{in the case of the problem } A_1,$$

$$(22) \qquad \left.\frac{dk}{dp}\right|_{F(\Omega)} = 0 \qquad \text{in the case of the problem } A_2,$$

$$(23) \qquad \left.\left[\frac{dk}{dp} - h(x,t)\,k(x)\right]\right|_S = 0 \text{ in the case of the problem } A_3.$$

The fulfilment of these relations will be ensured by suitable assumptions about the functions $f(x)$, $g(x)$ and $F(x, t)$.

If we suppose that $k(x) \in L_2(\overline{\Omega})$ and $z(x, t, \varepsilon) \in L_2(\overline{Q})$ we get from (12)

$$(24) \qquad \int_{\overline{Q}} [u(x, t) - U(x, t)]^2 \, dx\, dt \leq 2\varepsilon^2 \left\{ \int_{\overline{Q}} k^2(x) \left[1 - e^{-\frac{\nu(t)}{\varepsilon}}\right]^2 dx\, dt + \int_{\overline{Q}} z^2(x, t, \varepsilon)\, dx\, dt \right\}.$$

If we manage to prove that the integral

$$\int_{\overline{Q}} z^2(x, t, \varepsilon)\, dx\, dt$$

is bounded by a constant independent on ε we have from (24)

$$\| u(x, t) - U(x, t) \|_{L_2(\overline{Q})} = 0\,(\varepsilon),$$

what is the desired result. As a by-product we shall get similar results for the first derivatives. To prove $k(x) \in L_2(\Omega)$ is a matter of some assumptions about the functions $f(x)$, $g(x)$ and $F(x, t)$. The only remaining question is to prove

$$(25) \qquad \int_{\overline{Q}} z^2(x, t, \varepsilon)\, dx\, dt \leq M,$$

where the constant M does not depend on ε. We shall estimate the integral by means of the energy method using the fact that $z(x, t, \varepsilon)$ satisfies (19).

2. In deriving the energy estimates we shall often use the well known Green's formula in a slightly more general formulation (see [5], p. 134):

Lemma 1 (Green's formula)

1. Let E *be a closed bounded domain from* E_{n+1} *with the boundary* $F(\mathsf{E})$ *which is a surface of the class* C^1_σ.[3])
2. Let $\{Q_j\}(j = 1, 2, \ldots, n+1)$ *be functions continuous in* $\bar{\mathsf{E}}$ *and belonging to* C^1 *in* E.
3. Let the integral (possibly as an unproper)

$$(26)\qquad \int_{\bar{\mathsf{E}}} \left| \sum_{j=1}^{n+1} \frac{\partial Q_j}{\partial x_j} \right| \mathrm{d}x.$$

exist.
Then it holds

$$(27)\qquad \int_{\bar{\mathsf{E}}} \sum_{j=1}^{n+1} \frac{\partial Q_j}{\partial x_j}\, \mathrm{d}x + \int_{F(\mathsf{E})} \sum_{j=1}^{n+1} Q_j \cos(\nu, x_j)\, \mathrm{d}\sigma = 0,$$

where the numbers $\cos(\nu, x_j)$ *are the direction cosines of the inward normal to the boundary* $F(\mathsf{E})$ *in the point* $(x_1, x_2, \ldots, x_{n+1})$.

We now begin with deriving the energy estimates.

Lemma 2: *Let the following assupmtions be satisfied.*

1. $z(x, t, \varepsilon)$ *belongs to* C^2 *in the cylinder* $Q = \Omega \times (0, T)$, *to* C^1 *in* $\bar{Q}$, *it satisfies (19), (13) and one of the boundary conditions (14), (15), (16). The boundary of the domain* Ω *belongs to* C^1_σ.
2. *The coefficients* $a_{ij}(x)$ *and* $a(x)$ *belong to* C^1 *in* $\bar{\Omega}$, $\beta(t)$ *is continuous in* $\langle 0, T\rangle$, *the function* $h(x, t)$ *is positive and belongs to* C^1 *on* $\bar{S}$ *and (3) and (4) are fulfilled.*
3. *The right-hand side* $P(x, t, \varepsilon)$ *is continuous in* $\bar{Q}$.

Then it holds

$$(28)\qquad \int_{\bar{Q}} z^2(x, t, \varepsilon)\, \mathrm{d}x\, \mathrm{d}t \leqq M_1 \int_{\bar{Q}} P^2(x, t, \varepsilon)\, \mathrm{d}x\, \mathrm{d}t,$$

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[3]) The definition of the class C^1_σ is introduced, i.e., in [5], p. 132. We only point out that a circular cylinder belongs to C^1_σ.

$$(29)\qquad \int\limits_{\bar{Q}} \left(\frac{\partial z}{\partial x_i}\right)^2 \mathrm{d}x\,\mathrm{d}t \leqq M_1 \int\limits_{\bar{Q}} P^2\,\mathrm{d}x\,\mathrm{d}t \quad (i = 1, 2, \ldots, n),$$

$$(30)\qquad \int\limits_{\bar{Q}} \left(\frac{\partial z}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t \leqq M_1 \int\limits_{\bar{Q}} P^2\,\mathrm{d}x\,\mathrm{d}t,$$

where the constant M_1 does not depend on ε.

Proof. We multiply (19) by $\mathrm{e}^{-rt}\dfrac{\partial z}{\partial t}$ where r is a nonnegative constant which will be determined later. We have

$$(31)\qquad \varepsilon\,\mathrm{e}^{-rt}\frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial t^2} + \mathrm{e}^{-rt}\beta(t)\left(\frac{\partial z}{\partial t}\right)^2 - \left[\mathrm{e}^{-rt}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial z}{\partial x_j}\right) - \right.$$

$$\left. - \mathrm{e}^{-rt} az\right]\frac{\partial z}{\partial t} = \mathrm{e}^{-rt}P\frac{\partial z}{\partial t}.$$

A straightforward calculation yields

$$(32)\qquad \mathrm{e}^{-rt}\left\{\varepsilon\frac{\partial z}{\partial t}\frac{\partial^2 z}{\partial t^2} - \frac{\partial z}{\partial t}\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial z}{\partial x_j}\right) + az\frac{\partial z}{\partial t}\right\} =$$

$$= \frac{1}{2}\frac{\partial}{\partial t}\left\{\mathrm{e}^{-rt}\left[\sum_{i,j=1}^{n}a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j} + az^2 + \varepsilon\left(\frac{\partial z}{\partial t}\right)^2\right]\right\}$$

$$-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\mathrm{e}^{-rt}a_{ij}\frac{\partial z}{\partial t}\frac{\partial z}{\partial x_j}\right) + \frac{r}{2}\mathrm{e}^{-rt}\left[\sum_{i,j=1}^{n}a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j} + az^2 + \varepsilon\left(\frac{\partial z}{\partial t}\right)^2\right].$$

(Affer putting into (31) we get

$$(33)\qquad -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(\mathrm{e}^{-rt}a_{ij}\frac{\partial z}{\partial t}\frac{\partial z}{\partial x_j}\right) + \frac{1}{2}\frac{\partial}{\partial t}\left\{\mathrm{e}^{-rt}\left[\sum_{i,j=1}^{n}a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j} + \right.\right.$$

$$\left.\left. + az^2 + \varepsilon\left(\frac{\partial z}{\partial t}\right)^2\right]\right\} = \mathrm{e}^{-rt}\left\{P\frac{\partial z}{\partial t} - \beta(t)\left(\frac{\partial z}{\partial t}\right)^2 - \right.$$

$$-\frac{r}{2}\left[\sum_{i,j=1}^{n} a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}+az^2+\varepsilon\left(\frac{\partial z}{\partial t}\right)^2\right]\Bigg\}.$$

The left-hand side of (33) has a form of the divergence and the right-hand side is continuous in $\bar{Q}$. We can perform the integration over the cylinder $\bar{Q}_\xi = \Omega \times \langle 0, \xi\rangle$, $0 \leqq \xi \leqq T$ and apply the Green's formula to the left-hand side. If we denote $S_\xi = F(\Omega) \times (0, \xi)$ and $\Omega_\xi = \bar{Q} \cap \{t = \xi\}$ then taking into account (13) we get

$$(34)\qquad \int_{\bar{S}_\xi} e^{-rt}\frac{\partial z}{\partial t}\frac{dz}{dp}\,d\sigma + \frac{1}{2}\int_{\Omega_\xi} e^{-rt}\sum_{i,j=1}^{n} a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}\,dx +$$
$$+\frac{1}{2}\int_{\Omega_\xi} e^{-rt}\left[az^2+\varepsilon\left(\frac{\partial z}{\partial t}\right)^2\right]dx + \int_{\bar{Q}_\xi} e^{-rt}\beta(t)\left(\frac{\partial z}{\partial t}\right)^2 dx\,dt +$$
$$+\frac{r}{2}\int_{\bar{Q}_\xi} e^{-rt}\left[\sum_{i,j=1}^{n} a_{ij}\frac{\partial z}{\partial x_i}\frac{\partial z}{\partial x_j}+az^2+\varepsilon\left(\frac{\partial z}{\partial t}\right)^2\right]dx\,dt =$$
$$=\int_{\bar{Q}_\xi} e^{-rt}P\frac{\partial z}{\partial t}\,dx\,dt.$$

If we leave out the third and the fifth integral on the left-hand side of the equation (34) (they are nonnegative with respect to (3)) and if we majorize the right-hand side we get, using (3) and (4),

$$(35)\qquad \int_{\bar{S}_\xi} e^{-rt}\frac{\partial z}{\partial t}\frac{dz}{dp}\,d\sigma + \frac{\alpha}{2}\int_{\Omega_\xi} e^{-rt}\sum_{i=1}^{n}\left(\frac{\partial z}{\partial x_i}\right)^2 dx +$$
$$+\beta_0\int_{\bar{Q}_\xi} e^{-rt}\left(\frac{\partial z}{\partial t}\right)^2 dx\,dt \leqq \left|\int_{\bar{Q}_\xi} e^{-rt}P\frac{\partial z}{\partial t}\,dx\,dt\right|$$

where $\beta_0 = \min\limits_t \beta(t)$.

In the case that $z(x, t, \varepsilon)$ fulfils the boundary condition (14) or (15) the first integral on the left-hand side of (35) is equal to zero. Choosing $r = 0$ we obtain

$$(36)\quad \frac{\alpha}{2}\int\limits_{\Omega_\xi}\sum_{i=1}^{n}\left(\frac{\partial z}{\partial x_i}\right)^2 \mathrm{d}x + \beta_0\int\limits_{\bar{Q}_\xi}\left(\frac{\partial z}{\partial t}\right)^2 \mathrm{d}x\,\mathrm{d}t \leq \left|\int\limits_{\bar{Q}_\xi} P\,\frac{\partial z}{\partial t}\,\mathrm{d}x\,\mathrm{d}t\right|.$$

Consider now the case when $z(x, t, \varepsilon)$ satisfies the boundary condition (16). We have

$$\mathrm{e}^{-rt}\frac{\partial z}{\partial t}\frac{\mathrm{d}z}{\mathrm{d}p} = \mathrm{e}^{-rt}h(x, t)z\frac{\partial z}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\mathrm{e}^{-rt}hz^2) + \frac{r}{2}\mathrm{e}^{-rt}hz^2 - \frac{1}{2}\mathrm{e}^{-rt}\frac{\partial h}{\partial t}z^2.$$

Hence we get, with respect to (13),

$$(37)\quad \frac{1}{2}\int\limits_{\bar{S}_\xi}\mathrm{e}^{-rt}\frac{\partial z}{\partial t}\frac{\mathrm{d}z}{\mathrm{d}p}\,\mathrm{d}\sigma = \frac{1}{2}\int\limits_{F(\Omega_\xi)}\mathrm{e}^{-r\xi}hz^2\,\mathrm{d}\omega + \frac{r}{2}\int\limits_{\bar{S}_\xi}\mathrm{e}^{-rt}hz^2\,\mathrm{d}\sigma - \frac{1}{2}\int\limits_{\bar{S}_\xi}\mathrm{e}^{-rt}\frac{\partial h}{\partial t}z^2\,\mathrm{d}\sigma.$$

Denote

$$h_0 = \min_{\bar{S}} h(x, t),\quad h_1 = \max_{\bar{S}}\left|\frac{\partial h}{\partial t}\right|.$$

After setting from (37) into (35) we have

$$(38)\quad \frac{h_0}{2}\int\limits_{F(\Omega_\xi)}\mathrm{e}^{-r\xi}z^2\,\mathrm{d}\omega + \frac{rh_0}{2}\int\limits_{\bar{S}_\xi}\mathrm{e}^{-rt}z^2\,\mathrm{d}\sigma + \frac{\alpha}{2}\int\limits_{\Omega_\xi}\mathrm{e}^{-rt}\sum_{i=1}^{n}\left(\frac{\partial z}{\partial x_i}\right)^2\mathrm{d}x + \beta_0\int\limits_{\bar{Q}_\xi}\mathrm{e}^{-rt}\left(\frac{\partial z}{\partial t}\right)^2\mathrm{d}x\,\mathrm{d}t \leq \left|\int\limits_{\bar{Q}_\xi}\mathrm{e}^{-rt}P\frac{\partial z}{\partial t}\,\mathrm{d}x\,\mathrm{d}t\right| + \frac{h_1}{2}\int\limits_{\bar{S}_\xi}\mathrm{e}^{-rt}z^2\,\mathrm{d}\sigma.$$

Choose

$$(39)\quad r = \frac{2h_1}{h_0}.$$

Leaving out the first integral on the left-hand side of (38) (it is nonnegative) we obtain

$$(40)\qquad \frac{h_1}{2}\int\limits_{\bar S_\xi} e^{-rt} z^2\, d\sigma + \frac{\alpha}{2}\int\limits_{\bar\Omega_\xi} e^{-rt}\sum_{i=1}^{n}\left(\frac{\partial z}{\partial x_i}\right)^2 dx + $$
$$+\beta_0\int\limits_{\bar Q_\xi} e^{-rt}\left(\frac{\partial z}{\partial t}\right)^2 dx\, dt \leqq \left|\int\limits_{\bar Q_\xi} e^{-rt}P\frac{\partial z}{\partial t}\, dx\, dt\right|,$$

where r is defined by (39).
Again leaving out the first integral on the left-hand side of (40) (it is nonnegative) we get, with respect to the inequality $e^{-rT} \leqq e^{-rt} \leqq 1$, $t \in \langle 0, T\rangle$,

$$(41)\qquad \frac{\alpha}{2}\int\limits_{\bar\Omega_\xi}\sum_{i=1}^{n}\left(\frac{\partial z}{\partial x_i}\right)^2 dx + \beta_0\int\limits_{\bar Q_\xi}\left(\frac{\partial z}{\partial t}\right)^2 dx\, dt \leqq e^{\frac{2h_1T}{h_0}}\left|\int\limits_{\bar Q_\xi} P\frac{\partial z}{\partial t}\, dx\, dt\right|.$$

Comparing (36), (41) and using Schwartz inequality we have in all cases an inequality of the energy type

$$(42)\qquad \frac{\alpha}{2}\int\limits_{\Omega_\xi}\sum_{i=1}^{n}\left(\frac{\partial z}{\partial x_i}\right)^2 dx + \beta_0\int\limits_{\bar Q_\xi}\left(\frac{\partial z}{\partial t}\right)^2 dx\, dt \leqq$$
$$\leqq K\left(\int\limits_{\bar Q_\xi} P^2\, dx\, dt\right)^{\frac{1}{2}}\left(\int\limits_{\bar Q_\xi}\left(\frac{\partial z}{\partial t}\right)^2 dx\, dt\right)^{\frac{1}{2}}$$

where the constant K is independent on ε.
Hence

$$(43)\qquad \int\limits_{\bar Q_\xi}\left(\frac{\partial z}{\partial t}\right)^2 dx\, dt \leqq \frac{K^2}{\beta_0^2}\int\limits_{\bar Q_\xi} P^2\, dx\, dt.$$

From this inequality it follows (30). From (42) and (43) it follows

$$(44)\qquad \int\limits_{\Omega_\xi}\sum_{i=1}^{n}\left(\frac{\partial z}{\partial x_i}\right)^2 dx \leqq \frac{2K^2}{\alpha\beta_0}\int\limits_{\bar Q_\xi} P^2\, dx\, dt.$$

Integrating this inequality over the interval $\langle 0, \xi\rangle$ we get the inequality (29). It remains to prove (28). For an arbitrary s it holds

$$(45) \qquad \frac{\partial}{\partial t}(e^{-st}z^2) = -se^{-st}z^2 + 2e^{-st}z\frac{\partial z}{\partial t}.$$

Hence and from (13) it follows

$$\int_{\Omega_\xi} e^{-st}z^2\,dx + s\int_{\bar{Q}_\xi} e^{-st}z^2\,dx\,dt \leqq \int_{\bar{Q}_\xi} e^{-st}z^2\,dx\,dt + \int_{\bar{Q}_\xi} e^{-st}\left(\frac{\partial z}{\partial t}\right)^2 dx\,dt.$$

Choosing $s = 2$ we obtain

$$\int_{\bar{Q}_\xi} z^2\,dx\,dt \leqq e^{2\xi}\int_{\bar{Q}_\xi}\left(\frac{\partial z}{\partial t}\right)^2 dx\,dt.$$

From this inequality and from (43) it follows (28) and Lemma 2 is proved.

3. Now we can easily find the connection between the solution $u(x, t)$ of the mixed problem $A_k(k = 1, 2, 3)$ and the solution $U(x, t)$ of the corresponding problem A_k^*.

Theorem. *Let the following assumptions be satisfied:*

1. *$u(x, t)$ is the solution*[4]*) of one the problems $A_k(k = 1, 2, 3)$ in the domain $Q = \Omega \times (0, T)$ and the boundary $F(\Omega)$ belongs to the class C^1_σ.*
2. *$U(x, t)$ is the solution of the corresponding problem A_k^* and belongs to $C^2(\bar{Q})$.*
3. *$a_{ij}(x) \in C^3(\bar{\Omega})$, $a(x) \in C^2(\bar{\Omega})$, $\beta(t) \in C^1(\langle 0, T\rangle)$, $F(x, t)$ is continuous in $\bar{Q}$, $F(x, 0) \in C^2(\bar{\Omega})$ and (3) and (4) are fulfilled.*
4. *$f(x) \in C^4(\bar{\Omega})$, $g(x) \in C^2(\bar{\Omega})$.*
5. *In the case of the problem A_2 it holds*

$$\beta(0)\frac{dg}{dp} = \frac{dLf}{dp} + \frac{dF(x, 0)}{dp}, \quad x \in F(\Omega) \quad \text{(see the condition (22))}$$

and in the case of the problem A_3 the function $h(x, t)$ is positive, belongs to $C^1(\bar{S})$ and it holds

$$\beta(0)\frac{dg}{dp} - \frac{dLf}{dp} - \frac{dF(x, 0)}{dp} = h(x, t)\,[\beta(0)g - Lf - F(x, 0)],$$

$x \in S$ (see (23)).

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[4]) See the footnote 2).

Then we have[5])

$$(46)\qquad \| u(x, t) - U(x, t) \|_{L_2(\bar{Q})} = 0(\varepsilon),$$

$$(47)\qquad \left\| \frac{\partial u}{\partial x_i} - \frac{\partial U}{\partial x_i} \right\|_{L_2(\bar{Q})} = 0(\varepsilon) \quad (i = 1, 2, \ldots, n),$$

$$(48)\qquad \left\| \frac{\partial u}{\partial t} - \frac{\partial U}{\partial t} - \beta(t)k(x)\, e^{-\frac{\nu(t)}{\varepsilon}} \right\|_{L_2(\bar{Q})} = 0(\varepsilon).$$

Proof. From (20) it follows

$$\int_{\bar{Q}} P^2(x, t, \varepsilon)\, dx\, dt \leqq 3 \int_{\bar{Q}} \left(\frac{\partial^2 U}{\partial t^2} \right)^2 dx\, dt + 3 \int_{\bar{Q}} \beta'^2(t) k^2(x)\, dx\, dt + $$
$$+ 3 \int_{\bar{Q}} (Lk)^2\, dx\, dt.$$

By means of Lemma 2 we get

$$\int_{\bar{Q}} z^2(x, t, \varepsilon)\, dx\, dt \leqq M_1,$$
$$\int_{\bar{Q}} \left(\frac{\partial z}{\partial t} \right)^2 dx\, dt \leqq M_1,$$
$$\int_{\bar{Q}} \left(\frac{\partial z}{\partial x_i} \right)^2 dx\, dt \leqq M_1,$$

where the constant M_1 is independent of ε. Hence from (12) the assertions (46), (47), (48) follow immediately.

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[5]) In the case of the problem A_1 the condition (21) is satisfied on the basis of the assumptions 1. and 2. of the Theorem.

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