Jaromír Vosmanský

The monotonicity of extremants of integrals of the differential equation $y'' + q(t)y = 0$

Archivum Mathematicum, Vol. 2 (1966), No. 3, 105--111

Persistent URL: http://dml.cz/dmlcz/104613

THE MONOTONICITY OF EXTREMANTS OF INTEGRALS OF THE DIFFERENTIAL EQUATION $y'' + q(t)y = 0$

J. VOSMANSKÝ

Received June 12, 1966

In papers [1] and [2] L. LORCH and P. SZEGO have deduced a simple, sufficient condition for the monotonicity of the sequence of zeros of an arbitrary integral of the differential equation $y'' + q(t)y = 0$. The subject of this paper is to deduce similar conditions for the monotonicity of the sequence of extremants for an arbitrary solution of the same differential equation.

By "extremant" of the function $y(t) \in C_2$ we understand any number t' in which the function $y(t)$ acquires an extreme value.

A function $F(t)$ will be said to be of class $M_n(a, b)$ or monotone of order n in (a, b) (see [1]), if it has $n(n \geqq 0)$ continuous derivates $F^{(0)}$, F', F'', ..., $F^{(n)}$ complying with the relation

$$(-1)^j F^{(j)}(t) \geqq 0 \text{ for } t \in (a, b), j = 0, 1, \ldots, n.$$

If the previous inequality is fulfilled for $j = 0, 1, \ldots$, then the function $F(t)$ will be said to be completely monotone in (a, b) and we shall denote it by $M_\infty(a, b)$. M_n will stand, as abreviation, for $M_n(0, \infty)$.

Futhermore, let $\{t_k\}$ denote the sequence and $\Delta^n t_k$ the n-th differences of the sequence $\{t_k\}$, so that

$$\Delta^0 t_k = t_k, \quad \Delta t_k = t_{k+1} - t_k, \ldots \Delta^n t_k = \Delta^{n-1} t_{k+1} - \Delta^{n-1} t_k,$$
$$k = 1, 2, \ldots \qquad n = 1, 2, \ldots$$

The sequence $\{t_k\}$ will be said to be monotone of order n, if

$$(-1)^j \Delta^j t_k \geqq 0, \; k = 0, 1, 2, \ldots, j = 1, 2, \ldots, n.$$

In the case $n = \infty$ the sequence $\{t_k\}$ will be said to be completely monotone.

Lemma 1. *Let $q(t) > 0$ for $t \in (a, b)$ and $q' \in M_n(a, b)$. Then, for $p < n + 1 - i$, there holds*

$$\left\{\frac{q^{(i)}}{q}\right\}^{(p)} = \sum_{k=1}^{N} \varepsilon_k C_k \frac{q^{(\nu_{1,k})}}{q} \cdots \frac{q^{(\nu_{s,k})}}{q}, \tag{1}$$

where $0 \leqq \nu_{j,k} \leqq p+i$, $j=1, 2, \ldots, e$, $\sum_{j=1}^{e} \nu_{j,k} = p+i$ *for* $k=1, \ldots, N$. C_k *and* N *are suitable positive integers and*

$$\varepsilon_k = (-1)^{p+i-1} \, sign \left\{ \frac{q^{(\nu_{1,k})}}{q} \cdots \frac{q^{(\nu_{e,k})}}{q} \right\},$$

so that all non-zero terms of the expression (1) have the same sign. At the same time, sign $f = 0$, *if* $f(t) \equiv 0$, *sign* $f = +1$, *if* $f(t) \geqq 0$ *and sign* $f = -1$, *if* $f(t) \leqq 0$ *for* $t \in (a, b)$.

Proof. We use the induction with regard to p for $i = 1, 2, \ldots, n+1-p$

a) For $p = 0$ one has, according to the supposition,

$$\frac{q'}{q} \geqq 0, \quad \frac{q''}{q} \leqq 0, \ldots, \frac{q^{(n+1)}}{q} (-1)^n \geqq 0$$

so that

$$\frac{q^{(i)}}{q} = \varepsilon \frac{q^{(i)}}{q}, \quad \text{where } \varepsilon = (-1)^{i+1} \, \text{sign} \left\{ \frac{q^{(i)}}{q} \right\},$$

which evidently applies.

b) Let us suppose that the assertion for $p = 1$ holds. Then

$$\left[\frac{q^{(i)}}{q}\right]^{(p+1)} = \sum_k \varepsilon_k C_k \left\{ \left[\frac{q^{(\nu_{1,k})}}{q}\right]' \frac{q^{(\nu_{2,k})}}{q} \cdots \frac{q^{(\nu_{e,k})}}{q} + \right.$$

$$+ \frac{q^{(\nu_{1,k})}}{q} \left[\frac{q^{(\nu_{2,k})}}{q}\right]' \cdots \frac{q^{(\nu_{e,k})}}{q} + \ldots +$$

$$\left. + \frac{q^{(\nu_{1,k})}}{q} \cdots \frac{q^{(\nu_{e-1,k})}}{q} \left[\frac{q^{(\nu_{e,k})}}{q}\right]' \right\}.$$

Since

$$\left[\frac{q^{(\nu_{j,k})}}{q}\right]' = \frac{q^{(\nu_{j,k}+1)} q - q^{(\nu_{j,k})} q'}{q^2} = \frac{q^{(\nu_{j,k}+1)}}{q} - \frac{q^{(\nu_{j,k})}}{q} \frac{q'}{q}$$

and

$$\text{sign} \frac{q^{(\nu_{j,k}+1)}}{q} = -\text{sign} \frac{q^{(\nu_{j,k})}}{q} \quad \text{and} \quad \text{sign} \frac{q'}{q} = 1,$$

there holds

$$\text{sign} \left[\frac{q^{(\nu_{j,k})}}{q}\right]' = -\text{sign} \frac{q^{(\nu_{j,k})}}{q}.$$

Hence we get

$$\operatorname{sign}\left[\frac{q^{(i)}}{q}\right]^{(p+1)} = -\operatorname{sign}\left[\frac{q^{(i)}}{q}\right]^{(p)}.$$

Consequently

(2) $$\left[\frac{q^{(i)}}{q}\right]^{(p+1)} = \sum_k \bar{\varepsilon}_k \bar{C}_k \frac{q^{(\mu_{1,k})}}{q} \cdots \frac{q^{(\mu_{m,k})}}{q},$$

where all the terms on the right-hand side of (2) have the same sign and $0 \leqq \mu_{j,k} \leqq i + p + 1$ for $j = 1, 2, \ldots, m$, $\sum_{j=1}^{m} \mu_{j,k} = p + i + 1$ for all k.

Thereby Lemma 1 is proved.

Lemma 2. *Let $q(t) > 0$ for $t \in (a, b)$ and $q' \in M_n(a, b)$. Then*

$$(-1)^{i+1} \frac{q^{(i)}}{q} \in M_{n-i+1}(a, b) \quad \text{for} \quad i = 1, 2, \ldots, n + 1.$$

Proof. From the suppositions of the lemma it follows that $\frac{q^{(i)}}{q}$ has continuous derivatives up to and including the order $n - i + 1$ and there holds $(-1)^{i-1} \frac{q^{(i)}}{q} \geqq 0$. By Lemma 1, $\left[\frac{q^{(i)}}{q}\right]^{(p)}$ may be expressed in the form (1), where all terms on the right-hand side have the same signs and the sign of every terms equals

$$\operatorname{sign}\left[\varepsilon_k \frac{q^{(\nu_{1,k})}}{q} \cdots \frac{q^{(\nu_{s,k})}}{q}\right] = \varepsilon_k \operatorname{sign}\left[\frac{q^{(\nu_{1,k})}}{q} \cdots \frac{q^{(\nu_{s,k})}}{q}\right] =$$
$$= (-1)^{p+i-1}\left\{\operatorname{sign}\left[\frac{q^{(\nu_{1,k})}}{q} \cdots \frac{q^{(\nu_{s,k})}}{q}\right]\right\}^2 = (-1)^{p+i-1}.$$

Thus

$$\left[\frac{q^{(i)}}{q}\right]^{(p)} (-1)^{i+p-1} \geqq 0 \quad \text{for} \quad p \leqq n - i + 1$$

and the lemma is proved.

Lemma 3. *Let $q(t) > 0$ for $t \in (a, b)$ and $q' \in M_n(a, b)$, $(n \geqq 2)$. Set*

$$Q(t) = q - \frac{3}{4}\left(\frac{q'}{q}\right)^2 + \frac{1}{2}\frac{q''}{q}.$$

Then $Q'(t) \in M_{n-2}(a, b)$ and in the case $b = \infty$ there holds $Q(\infty) = q(\infty)$:

Proof.

$$Q'(t) = q' - \frac{3}{2}\left(\frac{q'}{q}\right)\left(\frac{q'}{q}\right)' + \frac{1}{2}\left(\frac{q''}{q}\right)' . \tag{3}$$

At the same time $q' \geqq 0, \left(\frac{q'}{q}\right)\left(\frac{q'}{q}\right)' \leqq 0, \left(\frac{q''}{q}\right)' \geqq 0$ and therefore $Q'(t) \geqq 0$.

Now we are going to show that $\left[-\left(\frac{q'}{q}\right)'\left(\frac{q'}{q}\right)\right] \in M_{n-1}(a, b)$. The p-th derivative of this expression can be written in the form

$$\left[\left(\frac{q'}{q}\right)'\left(\frac{q'}{q}\right)\right]^{(p)} = \sum_{i=0}^{p} \binom{p}{i} \left(\frac{q'}{q}\right)^{(i+1)} \left(\frac{q'}{q}\right)^{(p-i)} .$$

Sign $\left(\frac{q'}{q}\right)^{(k)} = (-1)^k$, so that, sign $\left(\frac{q'}{q}\right)^{(i+1)} \left(\frac{q'}{q}\right)^{(p-i)} = (-1)^{p+1}$, cohence sign $\left[-\left(\frac{q'}{q}\right)'\left(\frac{q'}{q}\right)\right]^{(p)} = (-1)^p$ and therefore $\left[-\left(\frac{q'}{q}\right)'\left(\frac{q'}{q}\right)\right] \in M_{n-1}(a, b)$.

From Lemma 2 we have that $\left(\frac{q''}{q}\right)' \in M_{n-2}(a, b)$. All the members on the right-hand side of (3) belong to $M_{n-2}(a, b)$ and therefore $Q'(t) \in M_{n-2}(a, b)$.

To prove $Q(\infty) = q(\infty)$ we consider two cases:

a) Let $q'(\infty) = 0$. Since $q' \in M_n(a, \infty)$, it is obvious, that

$$q''(\infty) = 0, \ldots, q^{(n-1)}(\infty) = 0$$

and therefore $Q(\infty) = q(\infty)$,

b) Let $q'(\infty) = c > 0$. Since $q' \in M_n(a, \infty)$, then $c < \infty$, $q(\infty) = \infty$, $q''(\infty) = 0$ holds and therefore $Q(\infty) = q(\infty)$.

In the proofs of Lemmas 1,2 and 3 we have assumed, for simplicity, that $q^{(i)} \not\equiv 0$ for any $i \leqq n$. It is, of course, evident that the Lemmas remain valid even in case of $q^{(i)} \equiv 0$ for $i_0 < i \leqq n$.

Now let us denote $M_k = \int_{t_k}^{t_{k+1}} \left| \frac{y'}{\sqrt{q}} \right|^{\lambda} dt$ for $\lambda > -1$, where $\{t_k\}$ signifies the sequence of extremants of the integral $y(t)$ of the differential equation

$$y'' + q(t)y = 0 \tag{4}$$

Now let $\{\bar{t}_k\}$ denote the sequence of extremants of a different integral $\bar{y}(t)$ of the same differential equation. Then the following theorem holds.

Theorem. *Let* $q(t) > 0$ *for* $t \in (0, \infty)$ *and* $q' \in M_n$, *so that* $(-1)^j q^{(j+1)} \geqq 0$ *for* $t \in (0, \infty)$; $j = 0, 1, \ldots, n$.
Then one has

$$(5) \qquad (-1)^i \Delta^i M_k \geqq 0 \qquad (i = 0, 1, \ldots, n-2; \quad k = 1, 2, \ldots)$$

and, in particular

$$(6) \qquad (-1)^i \Delta^{i+1} t_k \geqq 0 \qquad (i = 0, 1, \ldots, n-2; \quad k = 1, 2, \ldots)$$

consequently, the sequence of the differences of successive extremants of any solution of the differential equation (4) is in the interval $(0, \infty)$, *monotone of order* $n - 2$.
Furthermore, if one has $t_1 > \bar{t}_1$, *then*

$$(7) \qquad (-1)^i \Delta^i (t_k - \bar{t}_k) \geqq 0 \qquad (i = 0, 1, \ldots, n-2; \quad k = 1, 2, \ldots)$$

Proof. $y(t)$ being an arbitrary solution of the differential equation (4), the function $Y = \dfrac{y'}{\sqrt{q}}$ satisfies the equation

$$(8) \qquad Y'' + Q(t) Y = 0,$$

where $Q(t) = q - \dfrac{3}{4}\left(\dfrac{q'}{q}\right)^2 + \dfrac{1}{2}\dfrac{q''}{q}$ and conversely; if $\bar{Y}$ is the solution of the differential equation (8), there exist such a solution $\bar{y}$ of the differential equation (4) that $\bar{Y} = \dfrac{\bar{y}'}{\sqrt{q}}$ holds.

According to Lemma 3 one has $Q'(t) \in M_{n-2}$ and $Q(\infty) > 0$. P. Hartman has shown ([1]) that, under these assumptions, there exist integrals Y_1, Y_2 of equation (8) such that, for $W = Y_1^2 + Y_2^2$, there holds $W \in M_{n-2}$. Then, by [3], for $N_k = \int_{T_k}^{T_{k+1}} | Y(t) |^{\lambda} \, dt$ $(k = 1, 2, \ldots)$, where $\{T_k\}$ signifies the sequence of zeros of the integral $Y(t)$ of equation (8), there applies

$$(-1)^i \Delta^i N_k \geqq 0 \qquad (i = 0, 1, \ldots, n-2; \quad k = 1, 2, \ldots)$$

and, if $T_1 > \bar{T}_1$, then

$$(-1)^i \Delta^i (T_k - \bar{T}_k) \geqq 0 \qquad (i = 0, 1, \ldots, n-2; \quad k = 1, 2, \ldots)$$

Since $Y \sqrt{q} = y'$, we have $\{T_k\} = \{t_k\}$, where $\{t_k\}$ denotes the sequence of extremants of arbitrary integral y of the differential equation (4) and $N_k = \int\limits_{t_k}^{t_{k+1}} \left| \frac{y'}{\sqrt{q}} \right|^\lambda dt = M_k$ and the theorem is proved.

If $q(t)$ is completely monotone in $(0, \infty)$, then

$$(9) \qquad (-1)^i q^{(i)}(t) > 0 \quad \text{for} \quad t \in (0, \infty), \quad i = 1, 2, \ldots,$$

unless q is a constant (see [2]). From the preceding theorem and Theorem 2.1 from [2] the corollary follows.

Corollary. *If* $q(t)$ *is not a constant and* $q'(t)$ *is completely monotone in the interval* $(0, \infty)$, *then*

$$(10) \qquad (-1)^i \Delta^i M_k > 0 \text{ for all } i, k$$

and for that reason the sequence of extremants of an arbitrary integral of equation (4) is completely monotone. The inequalities (7) may then be written in a stronger form:

$$(11) \qquad (-1)^i \Delta^i (t_k - \bar{t}_k) > 0 \text{ for all } i, k$$

Remark. The above results may be extended on the interval $\langle 0, \infty)$. If $q(0) = 0$, $q' \in M_n$, then zero cannot be an extremant of any integral y of equation (4). Consequently, the expression M_k always has a meaning. In this case, if $y'(0) = 0$, then $y''(0) = 0$, and $y'''(0) \neq 0$ holds and it is possible to define

$$(12) \qquad M_0 = \int\limits_0^{t_1} \left| \frac{y'}{\sqrt{q}} \right|^\lambda dt \quad \text{for} \quad \lambda > -\frac{2}{3}$$

and to take $k = 0$ in all the inequalities introduced above.

The mentioned results may be applied to a number of important differential equations.

The general solution of Bessel's differential equation

$$(13) \qquad y'' + \left\{ 1 - \frac{\nu^2 - \frac{1}{4}}{t} \right\} y = 0 \qquad (t > 0)$$

can be written in the form

$$(14) \qquad y(t) = \left(\frac{1}{2} \pi x \right)^{\frac{1}{2}} \{ c_1 J_\nu(t) + c_2 Y_\nu(t) \},$$

where $J_\nu(t)$, $Y_\nu(t)$ are the Bessel functions. In this case

$$q(t) = 1 - \frac{\alpha^2}{t}, \quad \text{where} \quad \alpha^2 = \nu^2 - \frac{1}{4}, \quad \alpha > 0.$$

For $|\nu| > \frac{1}{2}$ one has $q(t) > 0$ for $t \in (\alpha, \infty)$,

$$q^{(n)} = (-1)^{n+1}\alpha^2(n+1)!\frac{1}{t^{n+2}}, \quad n = 1, 2, \ldots$$

Evidently $q'(t) \in M_\infty(\alpha, \infty)$

$$Q(t) = q - \frac{3}{4}\left(\frac{q'}{q}\right)^2 + \frac{1}{2}\frac{q''}{q} = 1 - \frac{\alpha^2}{t^2} - 3\frac{\alpha^2}{(t^2 - \alpha^2)^2},$$

whence it can immediatly be seen, that $Q'(t) \in M_\infty(\alpha, \infty)$.

If $y(t)$ is an arbitrary solution of equation (13), then the sequence of the differences of successive extremants of this solution is, in the interval (α, ∞), completely monotone. Especially the extremants of the functions

$$t^{\frac{1}{2}} J_\nu(t), \quad t^{\frac{1}{2}} Y_\nu(t)$$

form a completely monotone sequence in this interval.

REFERENCES

[1] Hartman Philip; *On Differential Equations and the Functions* $J_\mu^2 + Y_\mu^2$. (Amer. Journ. of Math. Vol. 83, 1961, str. 154—187)

[2] Lorch Lee and Szego Peter; *Higher Monotonicity Properties of Certain Sturm-Liouville Functions* (Acta Math. 109, 1963, str. 55—73)

[3] Lorch Lee and Szego Peter; *Higher Monotonicity Properties of Certain Sturm-Liouville Functions II* (Bull. de l'Academie Polonaise des Sci. Ser. Math. Astr. et Phys. Vol. XI, 1963, str. 455—457).