Bohumil Šmarda
Topologies in $\ell$-groups

*Archivum Mathematicum*, Vol. 3 (1967), No. 2, 69--81

Persistent URL: [http://dml.cz/dmlcz/104632](http://dml.cz/dmlcz/104632)

**Terms of use:**

© Masaryk University, 1967

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use.*

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* [http://project.dml.cz](http://project.dml.cz)
This paper deals with the considerations of lattice-ordered groups which are topological groups and topological lattices at the same time (briefly with topological 1-groups). Topology of 1-groups is defined altogether by means of the complete system of neighbourhoods of zero $\Sigma$, what is possible with regard to homogeneity of 1-groups.

First of all, it deals with the investigation of topological properties of 1-groups in connection with their algebraic structure. A number of results about topological groups (they are not necessary topological lattices) is transferred here, as is given by Pontrjagin in [8]. Up to this time, considerations of similar kind for topological 1-groups have been carried through only for special topologies, as it is e.g. the order topology (see [1]), or the interval topology (see papers [3], [4], [5], [6], [7], [11]).

Further on, some special kinds of topological 1-groups are investigated. They are determined either by the algebraic character of 1-groups or by the special topology of investigated 1-groups. In the first case, it deals with 1-groups with realization, and in the second case with topologies defined by means of a filter in the lattice of polars of 1-group. The results in this part of this paper come out of the ideas of theory of disjunctivity and out of the properties of 1-groups with realization which are mentioned in papers [9], [10].

**Preliminary Notes and Definitions**

0.1: A topological space is the space in the meaning of Bourbaki (i.e. $\emptyset = \emptyset$, $A \subseteq A$, $\overline{A} = \overline{A}$, $A \cup B = \overline{A \cup B}$) — see [2]. This space is not a T$_0$-space which is usually defined by this condition: To arbitrary two different points of space there exists a neighbourhood at least of one of them that does not include the second point.

0.2: Lattice operations be indicated $\lor$, $\land$. Let $\Lambda$ be a $\land$-semilattice with zero. A filter $x$ in $\Lambda$ is a non-empty subset of the set $\Lambda$ with the following properties: 1. $0 \notin x$, 2. $a, b \in x \Rightarrow a \land b \in x$, 3. $a \in x$, $c \in \Lambda$, $c \geq a \Rightarrow c \in x$. Each chain of filters (ordered by inclusion) is upper bounded and then accordingly to Zorn’s Lemma, each filter is included in a maximal filter in $\Lambda$. Maximal filters are called ultrafilters in $\Lambda$. Further on, if $Q$ is a lattice, then the set of all filters in $Q$, and the
set of all ultrafilters in \( Q \), be designated \( \mathcal{F}(Q) \), and \( \mathcal{U}(Q) \), respectively. A dual object to a filter be called antifilter, a maximal antifilter be called ultraantifilter. The set of all ultraantifilters in \( Q \) be designated \( \mathcal{U}(Q) \).

0.3: An 1-subgroup of an 1-group \( G \) is a subgroup in \( G \) which is a sublattice of a lattice \( G \) as well. A convex 1-subgroup in \( G \) is an 1-subgroup \( H \) in \( G \) with the following property: \( a, b \in H, \ g \in G, \ a > g > b \Rightarrow g \in H \). A convex 1-subgroup which is a normal divisor in \( G \) as well is called 1-ideal in \( G \).

0.4: Elements \( a, b \in G \) are disjunctive iff \( |a \wedge b| = 0 \); notation \( a \parallel b \). An element \( g \in G \) is a weak unit element in \( G \), if the only disjunctive element to \( g \) in \( G \) is zero. The needful terms as for the disjunctivity are given in introduction of [9]. Especially, the disjunctive complement of a set \( A \subset G \) is the set \( A' = \{ g \in G : g \parallel a \text{ for arbitrary } a \in A \} \). Further on, \( A'' = (A')' \) and it holds \( A''' = A' \). The set \( A \subset G \) for which it holds \( A'' = A \) is called a polar in \( G \).

0.5: The complement of a set \( Q \subset G \) be indicated \( cQ \). Further we shall indicate \( g^+ = g \lor 0, \ g^- = g \land 0, |g| = g \lor -g \). If the elements \( a, b \) are incomparable, we write \( a \parallel b \). If a set has no incomparable elements, then it is a fully ordered set.

0.6: We say that an 1-group \( G \) has a realization if it is isomorphic with a subdirect sum of fully ordered groups \( G_x, \ x \in M \) (notation \( (G_x : x \in M) \)). We shall assume that \( G_x \neq 0 \), for any \( x \in M \). An 1-group with realization is briefly called an r-group. Further let us introduce this indication for \( f \in G, \ P \subset G, \ A \subset M : Z(f) = \{ x \in M : f(x) = 0 \}, cZ(f) = M \setminus Z(f) = \{ x \in M : f(x) \neq 0 \}, Z(P) = \{ x \in M : f(x) = 0 \text{ for arbitrary } x \in A \} \). A realization \( (G_x : x \in M) \) of an 1-group \( G \) be called completely regular if for any element \( f \in G \) and any element \( x \in Z(f) \) there exists an element \( g \in G \) so that \( x \in cZ(g) \subseteq Z(f) \). Every r-group always has a completely regular realization, see [10].

0.7: Let \( G \) be an 1-group and \( x \) an arbitrary ultrafilter in the set \( \Gamma \) of all polars in \( G \); let us indicate \( Ux \) the union of all polars in \( x \). Then the system \( \mathcal{R} = \{ G/Ux : x \in \mathcal{U}(\Gamma) \}, \ Ux \neq G \} \) defines a realization of the 1-group \( G \), which is called the \( \Gamma \)-realization of \( G \). Similarly the system \( \mathcal{S} = \{ G/Ux : x \in \mathcal{U}(\Pi') \} \) defines a realization of \( G \), which is called the \( \Pi' \)-realization of \( G \). The introduction of these terms is given in [10].

\[1] \) This set \( A \) is sometimes called a component in \( G \).
1.

**Definition:** A topological space $G$ is called a *topological 1-group* if $G$ has the following properties:

1. $G$ is an 1-group.
2. The group operation (notation $+$) and lattice operations are continuous in $G$, it means to hold:

Let the symbol $\circ$ indicates any of the following symbols: $+$, $-$, $\lor$, $\land$ and let $a, b \in G$. Then for each neighbourhood $W$ for which it holds $a \circ b \in W$ there exist neighbourhoods $U$, $V$ such that $a \in U$, $b \in V$ and for arbitrary elements $x \in U$, $y \in V$ it holds $x \circ y \in W$.

Let $G$ be a topological 1-group. Everywhere on, we shall indicate with $\Sigma_\ast$ the complete system of neighbourhoods in $G$ and with $\Sigma_0$ the complete system of neighbourhoods of zero in $G$. In the case it does not come to misunderstanding, we shall write briefly $\Sigma_\ast$ and $\Sigma$. With regard to homogeneity of 1-group $G$, $\Sigma$ determines fully $\Sigma_\ast$. A topological 1-group $G$ with the complete system of neighbourhoods of zero $\Sigma$ be indicated $(G, \Sigma)$.

1.1: Let $G$ be an 1-group which is a topological group and one of operations $\lor$, $\land$ is continuous. Then the second operation is also continuous.

Proof: Let the operation $\land$ be continuous and let $W \in \Sigma_\ast$ be a neighbourhood of the point $a \lor b$. Then $-(a \lor b) = -a \land -b \in -W$. Accordingly to [8], p. 104, def. 22 b) there exists a neighbourhood $W_0 \in \Sigma_\ast$ such that $-a \land -b \in W_0 \subset -W$. According to supposition there exist neighbourhoods $U_0$, $U$, $V$, $V_0 \in \Sigma_\ast$ such that $a \in U_0 \subset -U$, $b \in V_0 \subset -V$ and $U \land V \subset W_0$. Hence $-U_0 \land -V_0 \subset U \land V \subset W_0 \subset \subset -W$. Together $U_0 \lor V_0 \subset W$, such that the operation $\lor$ is continuous. The second part is proved dually (with the change $\lor$ and $\land$).

The following theorems are principal for further considerations:

1.2. **Theorem:** Let $(G, \Sigma)$ be a topological 1-group. Then $\Sigma$ fulfils the following conditions:

1. The intersection of two arbitrary sets of $\Sigma$ contains a set of $\Sigma$.
2. For any set $U \in \Sigma$ there exists a set $V \in \Sigma$ such that $V - V \subset U$.
3. For any set $U \in \Sigma$ and any element $u \in U$ there exists a set $V \in \Sigma$ such that $V + u \subset U$.
4. For any set $U \in \Sigma$ and any element $g \in G$ there exists a set $V \in \Sigma$ such that $-g + V + g \subset U$.
5. For any set $U \in \Sigma$ and any element $g \in G$ there exists a set $V \in \Sigma$ such that $(V - g^+) \lor (V + g^-) \subset U$.

**Note:** In the condition 5., it is possible to write equivalently $(V - g^-) \land (V + g^+) \subset U$.

Proof: The validity of conditions 1.—4. is proved in [8], p. 107, B.
We shall prove the condition 5: Let us choose \( U \in \Sigma, \ g \in G. \) The set \( U + (g \lor 0) \) is a neighbourhood of the point \( g \lor 0 \) and as the operation \( \lor \) is continuous, there exist neighbourhoods \( V', V'' \in \Sigma^*, \) such that \( g \in V', 0 \in V'' \) and \( V' \lor V'' \subset U + (g \lor 0). \) According to the condition 4., there exists a neighbourhood \( V_0 \in \Sigma \) such that \( V_0 + g \subset V' \) and consequently \( (V_0 + g) \lor V'' \subset U + (g \lor 0). \) Hence \( (V_0 + g + g^+) \lor (V'' + \neg g^+) = [(V_0 + g) \lor V''] - g^+ \subset U, \) i.e. \( (V_0 + g^-) \lor (V'' - g^+) \subset U. \)

Finally in accordance with condition 1., there exists a neighbourhood \( V \in \Sigma \) such that \( V \subset V_0 \cap V'' \) and therefore it holds \( (V + g^-) \lor (V - g^+) \subset U. \)

1.3. Theorem: Let \( G \) be an 1-group. Let \( \Sigma \) be a system of subsets of \( G \) fulfilling the conditions 1.—5. of Theorem 1.2. Then \( (G, \Sigma) \) is a topological 1-group. Topology in \( G \) defined by means of \( \Sigma \) is determined uniquely.

Proof: According to [8], p. 108, Th. 9, \((G, \Sigma)\) is a topological group. Topology in \( G \) defined by means of \( \Sigma \) is uniquely determined. The continuity of lattice operations with regard to this topology is remaining to be proved: Let \( g, h \in G. \) \( U \) be a neighbourhood of the point \( g \lor h. \) It holds \( g \lor h = [(g - h) \lor O] + h = (g - h)^+ + h. \) Let us indicate \( f = g - h. \) Then in accordance with [8], p. 104, def. 22 a), there exist neighbourhoods \( U_1, U_2 \) such that \( f^+ \in U_1, h \in U_2 \) and \( U_1 + U_2 \subset U. \) Further there exists a neighbourhood \( U_3 \in \Sigma \) such that \( U_3 + f^+ \subset U_1. \) Regarding the condition 5, of 1.2. there exists a neighbourhood \( V \in \Sigma \) such that \( (V + f^-) \lor (V - f^+) \subset U_3. \) Hence \((V + f^- + f^+ + h) \lor (V - f^+ + f^+ + h) = [(V + f^-) \lor (V - f^+)] + f^+ + f^- + h \subset U_3 + f^+ + h \subset U_1 + h \subset U_1 + U_2 \subset U \) and consequently \((V + f + h) \lor (V + h) \subset U. \) Together \((V + g) \lor (V + h) \subset U, \) what means the operation \( \lor \) in \( G \) to be continuous. The continuity of operation \( \wedge \) follows from 1.1. and the proof of the Theorem is finished.

1.4: Let \((G, \Sigma)\) be a topological 1-group. Then it holds:

1. The topological space \( G \) is discrete if, and only if, \( \{O\} \in \Sigma. \)
2. The topological space \( G \) is a Kuratowski space if, and only if, the intersection of all neighbourhoods of \( \Sigma \) is zero in \( G. \)
3. The topological space \( G \) is regular.
4. If \( G \) is a Kuratowski space, then \( G \) is completely regular.

Proof: 1. It results from [8], p. 106, A. 2. With regard to the homogeneity of \( G \) it holds, the intersection of all neighbourhoods of \( \Sigma \) is zero in \( G \) if, and only if, all points of \( G \) are closed. 3. Let \( U \in \Sigma. \) Then in accordance with the condition 2. of 1.2. there exists \( V \in \Sigma \) such that \( V - V \subset U. \) We shall prove that \( V \subset U. \) In fact if it is \( p \in V, \) then an arbitrary neighbourhood of the point \( p \) has a common point with \( V. \) Consequently for the neighbourhood \( p + V \) of the point \( p \) it holds \((p + V) \cap V \neq \emptyset, \) i.e. there exist elements \( a, b \in V \) such that \( p + a = b. \) Hence \( p = b - a \in V - V \subset U. \) In conclusion, for arbitrary \( U \in \Sigma \)
there exists $V \in \Sigma$ such that $V \subset U$ and consequently with regard to the homogeneity of 1-group $G$, $G$ is a regular topological space. 4. It follows from [8], p. 114, Th. 10.

2.

Let us give now some examples of topological 1-groups.

2.1: Let $G$ be a complete 1-group. Then $G$ is a topological 1-group with respect to the order topology. The space $G$ is, moreover, Hausdorff and completely regular.

Proof: It follows from the corollary of Th. 18, [1], p. 320.

2.2: A fully ordered additive group of real numbers $R$ with the order topology is a topological 1-group.

2.3: Let $R^n$ be an $n$-dimensional Euclidean space. Let a partial order on $R^n$ be introduced in the following way:

\[(x_1, x_2, \ldots, x_n) \preceq (y_1, y_2, \ldots, y_n), \text{ where } x_i, y_i \in R, \ i = 1, 2, \ldots, n\]

\[\iff x_i \preceq y_i, \text{ for } i = 1, 2, \ldots, n.\]

Then $R^n$ is a topological 1-group with respect to the order topology.

The 1-groups in the examples 2.2 and 2.3 are complete 1-groups and so they have the properties described in 2.1.

2.4: The interval topology in an 1-group $G$ is usually defined by means of the subbasis of closed sets. This subbasis is formed by closed intervals \( \{g \in G : g \geq a\} \), \( \{g \in G : g \leq a\} \), for all \( a \in G \). The interval topology in an 1-group $G$ is a Kuratowski one. If $G$ is a topological group with respect to the interval topology, then its space is regular and also Hausdorff.

Northam has given in [7] an example of the additive group of continuous functions on the closed interval $[0, 1]$, the space of which is not Hausdorff with respect to the interval topology. Choe in [5], Conrad in [3], Jakubik in [6], Wolk in [11] have been investigating the classes of 1-groups with the request so that these groups may form topological groups with respect to the interval topology. In all cases, these 1-groups had to be fully ordered and it arose a presumption that an 1-group which is a topological group with respect to the interval topology is fully ordered. This presumption has been disproved by Holland in [4] with the case of non-fully ordered topological 1-group:

Let $G$ be the set of all $\sigma$-automorphisms $f$ of the fully ordered set $R$ of real numbers that fulfil this condition:

\[(x - 1)f = xf - 1, \text{ for all } x \in R.\]

The group operation in $G$ is defined by composition of $\sigma$-automorphisms and the partial order in this way:

\[f \geq g \iff fx \geq gx, \text{ for all } x \in R.\]
2.5: Let $R$ be the fully ordered additive group of real numbers. The topology in $R$ is defined by means of the system $\Sigma$, which consists of the sets $U_r = \bigcup \{ x \in R : m - r < x < m + r \}$, $m = 0, \pm 1, \pm 2, \ldots$, for all rational numbers $r$, $0 < r < 1$.

Then $R$ is a topological group with the complete system of neighbourhoods of zero $\Sigma$. It is neither a topological 1-group nor a Kuratowski space.

Note: If we define the system $\Sigma$ in the example 2.7 with the sets $U_r = \{ x \in R : -r < x < r \}$, for all rational numbers $r$, $0 < r < 1$, then it holds for $R$ the same as in 2.7 and $R$ is a Kuratowski space.

3.

Let us go on in investigation of topological 1-groups, whose complete system of neighbourhoods of zero is formed with subgroups.

3.1: Let $(G, \Sigma)$ be a topological 1-group and let $U \in \Sigma$ be a subgroup in $G$. Then $U$ is a closed set.

Proof: Let $g \in U$. Since $U + g$ is a neighbourhood of the point $g$, the sets $U + g$ and $U$ have a common point. Then there exist elements $f, e U$ such that $f + g = h$. Hence $g = -f + h \in U$ i.e. $g \in U$. Thus $\overline{U} \subset U$ and evidently $U \subset \overline{U}$, so that $\overline{U} = U$.

Corollary: Let $(G, \Sigma)$ be a topological 1-group and let any neighbourhood $U \in \Sigma$ be a subgroup in $G$. Then it holds:
1. Each open set in $G$ is the union of closed sets.
2. Each closed set in $G$ is the intersection of open sets.

Proof: Results follow immediately from 3.1.

3.2: Let $(G, \Sigma)$ be a topological 1-group, let any neighbourhood $U \in \Sigma$ be a closed set in $G$. Further on, let the topological space $G$ be a $T_0$-space. Then $G$ is a Kuratowski space.

Proof: Let $G$ be a $T_0$-space and let us assume that $\bigcap_{U \in \Sigma} U \neq 0$.

It means, there exists an element $0 \neq g \in G$ such that $g \in U$ for every neighbourhood $U \in \Sigma$ and $g$ is also element of every neighbourhood of zero (which is not necessary an element of $\Sigma$).

Then there exists a neighbourhood $W$ of the point $g$ such that $0 \notin W$. Further there exists a neighbourhood $V \in \Sigma$ such that $V + g \subset W$. The neighbourhood $V + g$ is a closed set and thus $P = G \setminus (V + g)$ is an open set, $0 \in P$, $g \notin P$. There exists evidently a neighbourhood $Q \in \Sigma$ such that $0 \in Q, g \notin Q$, what is a contradiction. Thus, $\bigcap_{U \in \Sigma} U = 0$ and $G$ is according to 1.4 a Kuratowski space.
Corollary: Let \((G, \Sigma)\) be a topological 1-group and let any neighbourhood \(U \in \Sigma\) be a subgroup in \(G\). Let further \(G\) be a \(T_0\)-space. Then \(G\) is a Kuratowski space.

Proof: It follows from 3.1 and 3.2.

3.3: Let \((G, \Sigma) \neq 0\) be a topological 1-group and let a neighbourhood \(U \in \Sigma\) exists, which is a proper subgroup in \(G\). Then \(G\) is a disconnected topological space.

Proof: Let us assume \(G\) be a connected topological space and let \(U\) be a subgroup in \(G\). According to 3.1 \(U\) is a closed set in \(G\). And so is the set \(G \setminus U\) closed in \(G\) and it holds \(U \cup (G \setminus U) = G\). That is possible if, and only if, \(G \setminus U = \emptyset\) i.e. \(G = U\) and this is a contradiction.

3.4: Let \((G, \Sigma)\) be a topological 1-group and let any neighbourhood \(U \in \Sigma\) be a subgroup in \(G\). Further let \(G\) be a \(T_0\)-space. Then \(G\) is a totally disconnected topological space.

Proof: Let \(E\) be a maximal connected set containing zero in \(G\). Then \(E \cap U \neq \emptyset\) for any neighbourhood \(U \in \Sigma\). Both sets \(E\) and \(U\) are closed and \(U\) is open, too. Therefore \(E \cap U\) and \((G \setminus U) \cap E\) are closed sets in \(G\). Thus, \((G \setminus U) \cap E = \emptyset\) and hence \(E \subseteq U\) for any neighbourhood \(U \in \Sigma\). Further \(E \subseteq \bigcap_{U \in \Sigma} U = \{0\}\) and in accordance with 3.2 and [8], p. 137, B) the space \(G\) is totally disconnected.

Note: All previous statements of this paragraph and the statement 1.4 hold in topological group, too.

3.5: Theorem: Let \((G, \Sigma)\) be a topological 1-group. Then the intersection of all neighbourhoods from \(\Sigma\) is a normal 1-subgroup in \(G\).

Proof: In the proof, the theorem 1.2 is used (conditions 2.—5.). Let \(I = \bigcap_{U \in \Sigma} U \supseteq \{0\}\). To each \(U \in \Sigma\) and to each \(a \in U\) there exists a neighbourhood \(V \in \Sigma\) such that \(V + a \subseteq U\). Particularly for any neighbourhood \(U \in \Sigma\) and for any element \(j \in I\) is \(I + j \subseteq U\) i.e. \(I + j \subseteq \bigcap_{U \in \Sigma} U = I\). Hence \(I + I \subseteq I\). Further, to any neighbourhood \(U \in \Sigma\) there exists a neighbourhood \(V \in \Sigma\) such that \(-V \subseteq U\) but \(I \subseteq V\) i.e. \(-I \subseteq -V \subseteq U\) and hence \(-I \subseteq \bigcap_{U \in \Sigma} U = I\). Thus \(I\) is a subgroup in \(G\).

For any neighbourhood \(U \in \Sigma\) and any element \(g \in G\) there exists a neighbourhood \(V \in \Sigma\) such that \(-g + V + g \subseteq U\). Thus, \(-g + I + g \subseteq -g + V + g \subseteq U\) for any \(U \in \Sigma\). It means to be \(-g + I + g \subseteq I\).

To the end, it follows from the continuity of lattice operations, neighbourhoods \(V, W \in \Sigma\) exist to any neighbourhood \(U \in \Sigma\) such that \(V \vee V \subseteq U, W \wedge W \subseteq U\). Then \(I \vee I \subseteq \bigcap_{U \in \Sigma} U = I\) and \(I \wedge I \subseteq I\) so that \(I\) is a normal 1-subgroup in \(G\).
Corollary: Let \((G, \Sigma)\) be a topological 1-group and let any neighbourhood in \(\Sigma\) be a subgroup in \(G\). Then the intersection of all neighbourhoods from \(\Sigma\) is a closed normal 1-subgroup in \(G\).

If it is, moreover, any neighbourhood from \(\Sigma\) a convex subset in \(G\), then the intersection of all neighbourhoods from \(\Sigma\) is an 1-ideal in \(G\).

Proof: Results follow from 3.1, 3.5 and from this fact, that the intersection of convex sets is a convex set.

4. In this paragraph, topological 1-groups are investigated, whose complete system of neighbourhoods of zero is formed by convex 1-subgroups.

Definition: A system \(\mathcal{C}\) of convex 1-subgroups of an 1-group \(G\) be called an 0-system, if following properties are fulfilled:

1. The intersection of arbitrary two elements of \(\mathcal{C}\) contains an element of \(\mathcal{C}\).
2. The system \(\mathcal{C}\) contains with each element both all elements conjugated with it.

An 0-system be called a zero 0-system, if the intersection of all its elements is zero in \(G\).

4.1: An 0-system \(\mathcal{C}\) in an 1-group \(G\) is a zero one, iff \((G, \mathcal{C})\) is a Kuratowski space.

Proof: Evident from 1.4.

4.2: A right (left) decomposition of an 1-group \(G\) modulo a convex 1-subgroup \(H\) is a congruence in the lattice \(G\).

Proof: We carry out only for the right decomposition \(G/H\).

It is sufficient to prove, for arbitrary elements \(a, b \in G, h_1, h_2 \in H\) it holds \((h_1 + a) \lor (h_2 + b) = (a \lor b) \in H\) and \((h_1 + a) \land (h_2 + b) = (a \land b) \in H\). It holds \((h_1 + a) \lor (h_2 + b) \leq [(h_1 \lor h_2) + a] \lor [(h_1 \lor h_2) + b] = (h_1 \lor h_2) + (a \lor b)\) and hence \((h_1 + a) \lor (h_2 + b) = (a \lor b) \leq (h_1 \lor h_2)\). In an analogous way, it holds \((h_1 + a) \lor (h_2 + b) \geq [(h_1 \land h_2) + a] \lor [(h_1 \land h_2) + b] = (h_1 \land h_2) + (a \lor b)\) and hence \((h_1 + a) \lor (h_2 + b) = (a \lor b) \geq (h_1 \land h_2)\) and with regard to a convexity of \(H\) it is \((h_1 + a) \lor (h_2 + b) = (a \lor b) \in H\). Similarly we can prove \((h_1 + a) \land (h_2 + b) = (a \land b) \in H\).

4.3: Let \(K\) be a convex 1-subgroup in an 1-group \(G\). Then for any element \(g \in G\) it holds \((K + g^-) \lor (K - g^+) \subset K\).

Proof: According to 4.2 there is \((K + g^-) \lor (K - g^+) \subset K + + (g^- \lor -g^+) = K - (g^+ \land -g^-) = K\), because \(g^+ \land -g^- = 0\).

Corollary: Any set of convex 1-subgroups of an 1-group \(G\) fulfils the condition 5. of the Theorem 1.2.
4.4: Theorem: Each 0-system of an $1$-group $G$ is a complete system of neighbourhoods of zero.

Proof: Let $L$ be an 0-system of an 1-group $G$. We are going to prove, $L$ fulfils the conditions of the Theorem 1.2. The condition 1. is fulfilled evidently. In the conditions 2. and 3., it is sufficient to put $V = U$ and their validity is guaranteed by the fact, it is dealt with 1-subgroups in $G$. The condition 4. is fulfilled for $V = g + U$, (it follows from the definition of an 0-system). At the end, the condition 5. is fulfilled for $V = U$ with regard to the corollary 4.3. Thus, according to the theorem 1.3, $(G, L)$ is a topological 1-group.

Each filter $x$ in a lattice of 1-ideals of an 1-group $G$ forms an 0-system. A topology defined in $G$ in this way is not a discrete one, because the filter $x$ does not contain zero in $G$. If, moreover, the intersection of 1-ideals belonging to the filter $x$ is zero, then it is dealt with a Kuratowski space. Let us investigate similar questions for an ultrafilter in the lattice of convex 1-subgroups of an 1-group $G$. Everywhere in this and in the next paragraph, let us denote $K$ the lattice of convex 1-subgroups of an 1-group $G$.

Definition: A filter $x \in \mathcal{F}(K)$ be called a normal filter if $x$ contains with each element of $x$ all with it conjugated elements. An ultrafilter $x \in \mathcal{U}(K)$, which is a normal filter, be called a normal ultrafilter.

Each normal filter in $K$ forms an 0-system and its corresponding topology in $G$ is evidently not a discrete one.

5.1: An ultrafilter $x \in \mathcal{U}(K)$ is normal, iff it holds:

To arbitrary elements $g \in G$ and $x \in x$ there exists an element $c \in x,$
(N) $c \neq 0$ such that $-g + c + g \in x$.

Proof: Let $x$ be a normal ultrafilter. Then for arbitrary elements $x \in x$ and $g \in G$ there exists an element $\lambda \in x$ such that $-g + x + g = \lambda$. Let us choose an element $d \in x \cap \lambda$. Then it holds $g + d - g = c$, for some element $c \in x$. Thus, $-g + c + g \in x$ and the condition of the statement is fulfilled.

Conversely, let $x \in \mathcal{U}(K)$ be not a normal filter. Then there exists $x \in x$ such that $-g + x + g \notin x$ for suitable element $g \in G$. As $x$ is an ultrafilter, there exists $\lambda \in x$ such that $\lambda \cap (-g + x + g) = 0$. Hence $(x \cap \lambda) \cap \left[-g + (x \cap \lambda) + g\right] = 0$ and $x$ does not fulfil (N).

5.2: Let it be $x \in \mathcal{U}(K)$. Then the ultrafilter $x$ forms an 0-system in $G$, iff the condition (N) is fulfilled.

Proof: 1. If (N) is fulfilled, then $x$ is a normal ultrafilter according to 5.1, and so is an 0-system.

2. An ultrafilter $x \in \mathcal{U}(K)$ forms an 0-system. Then it contains
with each element all elements conjugated with it, thus it is a normal ultrafilter and according to 5.1 the condition (N) is fulfilled.

**Corollary:** Let \( x \in \mathfrak{A}(K) \) fulfills the condition (N). Then \( (G, x) \) is a topological 1-group and the topological space \( G \) is not a discrete one.

Proof follows from 4.4.

6. Let us state further the conditions guaranteeing the topological space of a topological 1-group, for which the system \( \Sigma \) is a normal ultrafilter in \( K \), to be a Kuratowski space.

6.1: 1. Let \( Q \) be any lattice with zero. Then the infimum of elements of an ultrafilter \( x \) in \( Q \) is zero if, and only if, \( x \) is not principal filter.

2. Let \( Q \) be any lattice. Then an ultrafilter \( x \) in \( Q \) is principal filter if, and only if, \( x \) is generated by a minimal element in \( Q \setminus \{0\} \).

Proof: Let \( x \in \mathfrak{A}(Q) \), which is not principal. If it is \( x \in Q \), \( x \leq \lambda \) for all elements \( \lambda \in x \), then it is \( x \notin x \). Thus, there exists an element \( \tau \in x \) such that \( \tau \cap x = 0 \). As \( x \leq 0 \), it is \( x = 0 \) i.e. \( \bigwedge_{\delta \in x} \delta = 0 \).

Conversely, let \( \bigwedge_{\delta \in x} \delta = 0 \) be for some ultrafilter \( x \) in \( Q \). Then \( x \) is not principal, because for each principal filter \( y \) it holds \( \bigcap_{\delta \in y} \delta = \mu \neq \{0\} \), where \( \mu \) is the least element in \( y \).

The proof of the second part is evident.

6.2: In the lattice \( K \) of convex 1-subgroups of an 1-group \( G \) atoms are exactly these convex 1-subgroups that are isomorphic with a subgroup of a fully ordered additive group \( R \) of real numbers.

Proof: If \( J \in K \) is isomorphic with a group \( R_1 \subset R \), \( \varphi \) is the corresponding isomorphism and \( J_0 \subseteq J \), \( J_0 \in K \), then \( J_0 \varphi \) is a convex 1-subgroup in \( R_1 \). Thus, \( J_0 \varphi = 0 \) or \( J_0 \varphi = R_1 \), and hence \( J_0 = 0 \) or \( J_0 = J \). Hence \( J \) is an atom in \( K \).

If \( J \) is an atom in \( K \), then it is a fully ordered group. If it is not namely a fully ordered group, then there exist elements \( a, b \in J \), \( a \neq 0 \neq b \), \( a \neq b \). Thus, \( b \in a' \cap J \), \( a \in a'' \cap J \), i.e. \( a' \cap J \in K \), \( 0 \neq a' \cap J \subseteq \) and \( J \) is not an atom in \( K \), what is a contradiction. Further \( J \) is an Archimedean group. If namely elements \( a, b \in J \), \( 0 < a < b \) existed, such that it holds \( na < b \) for all positive integers \( n \), then the set of all elements \( c \in J \) such that \( |c| \leq na \) for some positive integers \( n \), would be a convex 1-subgroup different from zero and \( J \), what is again a contradiction. Thus \( J \) is according to Hölder's Theorem isomorphic with a subgroup in \( R \).

6.3: **Theorem:** Let \( x \in \mathfrak{A}(K) \). Then the ultrafilter \( x \) is a zero 0-system, iff the following properties are fulfilled:
1. To arbitrary elements $g \in G$, $x \in x$ there exists an element $c \in x$ such that $0 \neq c$ and $-g + c + g \in x$.

2. The ultrafilter $x$ does not contain any convex 1-subgroup in $G$, which is isomorphic with a subgroup of the fully ordered additive group of real numbers.

**Proof:** It follows from 4.1, 5.2, 6.1 and 6.2.

7.

Let us specialize the considerations from topological 1-groups to $r$-groups, whose complete system of neighbourhoods of zero is formed by a filter in the set $\Gamma$, respectively $\Pi'$.

7.1: Let $G$ be an $r$-group, $x \in \mathcal{F}(\Gamma)$. Then $x$ is an 0-system.

**Proof:** According to [9], Lemma 3 or [10], I., Theorem 2.2, all polars in $G$ are 1-ideals and then $x$ is an 0-system.

7.2: Let $G$ be an $r$-group. Then $(G, \Pi')$ is a topological 1-group for which it holds:

1. $\Pi'$ is a zero 0-system in $G$.
2. $G$ is a discrete topological space, iff $G$ contains a weak unit element.

**Proof:** For an arbitrary element $g \in G$, $g'$ is a convex 1-subgroup in $G$. Further for arbitrary elements $g_1, g_2 \in G$ it holds $g_1' \cap g_2' = (|g_1| \lor |g_2|)'$. Elements of $\Pi'$ are according to [10], I., Theorem 2.2. normal 1-subgroups and by that it is proved $\Pi'$ to be an 0-system in $G$. Further for $0 \neq g \in G$ it holds $g \notin g'$ and thus, $\bigcap g' = 0$ and $\Pi'$ is a zero 0-system $\forall g' \in \Pi'$ in $G$. Finally, let $G$ be a discrete space. Then $\{0\} \in \Sigma = \Pi'$ i.e. $\{0\} = a'$, for a suitable element $a \in G$ and this is possible if, and only if, $a$ is a weak unit element in $G$.

Note: If it is $\Sigma = \Gamma$ or $\Sigma = \Pi$, then a topology in $G$ is discrete as $\{0\} = 0^\in G \subset \Gamma$.

Let $(G, \Sigma)$ be a topological 1-group and let $\Sigma \in \mathcal{F}(\Pi')$. Then we denote $G(\Sigma) = \{g \in G^+: g' \in \Sigma\}$.

7.3: Let $(G, \Sigma)$ be a topological 1-group, $\Sigma \in \mathcal{F}(\Pi')$. Then it holds:

1. $G(\Sigma)$ does not contain any weak unit element in $G$.
2. $G(\Sigma)$ is a convex $\land$-subsemilattice of the lattice $G$.

**Proof:** As $\Sigma$ is a filter in $\Pi'$, it is $\{0\} \notin \Sigma$. Let $a \in G(\Sigma)$ be a weak unit element in $G$. Then $\{0\} = a' \in \Sigma$, what is a contradiction. Further let $r, s \in G(\Sigma)$, $g \in G$, $0 \leq r < g < s$. Then $s' \subset g'$ and thus, $g' \in \Sigma$ i.e. $g \in G(\Sigma)$. Finally for $r, s \in G(\Sigma)$ it holds $r \land s' \supseteq r'$ and then $r \land s \in G(\Sigma)$.

7.4: Let $(G, \Sigma)$ be a topological $r$-group, $\Sigma \in \mathcal{F}(\Pi')$. If we identify the $r$-group $G$ with some of its complete regular realization, then for arbitrary elements $r, s \in G(\Sigma)$ there exists an element $t \in G(\Sigma)$ such that it holds $\Phi \neq Z(t) \subset Z(r) \cap Z(s)$. 


Proof: Let \( r, s \in G(\Sigma) \). Then there exists an element \( t \in G(\Sigma) \) such that \( t' = r' \cap s' \). Then it holds \( Z(t') = Z(r' \cap s') \supseteq Z(r') \cup Z(s') = cZ(r) \cup cZ(s) = c[Z(r) \cap Z(s)] \) according to (10), IV., 8.10. Moreover, \( t \) is not a weak unit element in \( G \) and thus according to [10], V., 12.11 it is \( \Phi \neq Z(t) = cZ(t') \subset Z(r) \cap Z(s) \).

7.5: Let \((G, \Sigma)\) be a topological r-group, \( \Sigma \in \mathfrak{F}(\Pi') \). Then \( \Sigma \) is a zero 0-system, iff to any element \( g \in G, g \neq 0 \) there exists an element \( s \in G(\Sigma) \) such that \( |g| \land s \neq 0 \).

Proof: \( \Sigma \) is a zero 0-system if, and only if, \( \bigcap_{U \in \Sigma} U = 0 \), what is equivalent with the fact, that to any element \( 0 \neq g \in G \) there exists an element \( s \in G(\Sigma) \) such that \( g \notin s' \).

Example: Let \( G \) be an r-group, which has a weak unit element. Let \( \Sigma \in \mathfrak{F}(\Pi') \) be a principal filter. Then \((G, \Sigma)\) is a topological r-group according to 7.1 and according to 6.1 it contains an atom in \( \Pi' \). Further it holds \( \bigcap_{g' = m'} \neq \{0\} \), what means \( G \) not to be a Kuratowski space.

7.6. Theorem: Let \( G \) be a complete regular realization of an r-group and let \( \Sigma \in \mathfrak{F}(\Pi') \). Then \( \Sigma \) is a zero 0-system in \( G \), iff to any element \( g \in G, g \neq 0 \) there exists an element \( s \in G(\Sigma) \) such that \( Z(g) \supseteq cZ(s) \).

Proof: Let \( g \in G, s \in G(\Sigma) \), \( g \neq 0 \). According to [10], IV., 8.10, \( g \in s' \) is equivalent with \( Z(g) \supseteq Z(s') = cZ(s) \). Hence it follows \( |g| \land s \neq 0 \) to be equivalent with \( Z(g) \supseteq cZ(s) \). It occurs according to 7.5 if, and only if, \( \Sigma \) is a zero 0-system in \( G \).

Note: A realization of an r-group \( G \), which is complete regular, is e.g. a \( \Pi' \)-realization, see [10], I.

Corollary: Let \( G \) be a complete regular realization of an r-group and let \( \Sigma \in \mathfrak{F}(\Pi') \). Then \((G, \Sigma)\) is a topological r-group, the space of which is a Kuratowski one, iff to any element \( g \neq 0, g \in G \) there exists an element \( s \in G(\Sigma) \) such that \( Z(g) \supseteq cZ(s) \).

Proof: Results follow from 4.1, 4.4 and 7.6.

REFERENCES:


